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2. PARTIAL DIFFERENTIAL EQUATIONS

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A GENERAL METHOD OF MAJORATING OF DIRICHLET PROBLEM SOLUTIONS

1. Let $u(x)$ be a function in a domain G in the Euclidean n-space E_n. We say that $x_0 \in G$ is its convexity point if the surface $S : z = u(x)$ in $(n + 1)$-space has at the point x_0, $u(x_0)$ a supporting plane from below, i.e. $z = p_i x^i + q \leq u(x)$, $p_i x_0^i + q = u(x_0)$. To such a plane we make to correspond the point $(p_1, \ldots, p_n)$ in E_n. Let $\Psi_u(M)$, $M \subset E_n$, be the set of all such points corresponding to all points $x \in M$ (if M includes no convexity points of u, $\Psi_u(M)$ is empty). It is „the lower supporting image of M by u". mes $\Psi_u(M)$ is a totally additive set functions. One can obviously define the upper supporting image $\overline{\Psi}_u(M)$.

We consider functions u subject to the following conditions;

(A) u is continuous in $G + \partial G$,

(B) the set function mes $\Psi_u\,(M_u)$ is absolutely continuous: this is fulfilled, in particular, if $u \in W_n^2(D)$ for every D, $D + \partial D \subset G$.

Suppose that u satisfies at almost all its convexity points the inequality

$$w \leq X(x, u)\, \mathrm{U}(\nabla u), \qquad w = \det\,(u_{ij}), \qquad X, U \geq 0. \tag{1}$$

(Note: any function is twice approximatively differentiable at almost all its convexity points. Thus no special differentiability conditions are necessary as soon as we understand u_i, u_{ij} as the coefficients of the approximative differentials $\mathrm{d}u$, d^2u).

In order to formulate our basic theorem introduce the following notations: $h(x, \nu)$ be the distance from a point $x \in G$ to the supporting plane to ∂G with the external normal ν; Ω be the unite sphere — the set of all unite vectors ν; we put $\nabla u = p\nu$, $p = |\nabla u|$.

Theorem 1. *If a function u with above conditions* (A), (B) *satisfies* (1) *at almost all convexity points, then for any $x \in G$ where $u(x) < 0$ the following inequality takes place*

$$\int_{\Omega} \int_{0}^{\frac{|u(x)|}{h(x,\nu)}} U^{-1}(p\nu)\, p^{n-1}\, \mathrm{d}p\, \mathrm{d}\nu < \int_{G} X(x, u(x))\, \mathrm{d}x. \tag{2}$$

This implies an estimation for $|u(x)|$ provided $X(x, u(x))$ is summable and the left integral grows to the infinity with the upper limit of integration.

The proof of our theorem runs as follows. Let M be the set of convexity points of u. Owing to (1)

$$\int_{M} U^{-1} w\, \mathrm{d}x \leq \int_{M} X\, \mathrm{d}x. \tag{3}$$

But $w = \dfrac{\partial(u_1, \ldots, u_n)}{\partial(x^1, \ldots, x^n)}$ is the Jacobian of the supporting mapping $(x^1, \ldots, x^n) \to (p_1, \ldots, p_n)$ for almost everywhere $p_i = u_i$. Thus owing to the condition (B)

$$\int_{M} U^{-1} w\, \mathrm{d}x = \int_{\Psi u(M)} U^{-1}(p\nu)\, \mathrm{d}p_1 \ldots \mathrm{d}p_n. \tag{4}$$

Obviously $\Psi_u(M) = \Psi_u(G)$ and $\int_{M} X\, \mathrm{d}x \leq \int_{G} X\, \mathrm{d}x$. Therefore (3) and (4) imply

$$\int_{\Psi_u(G)} U^{-1}(p\nu)\, \mathrm{d}p_1 \ldots \mathrm{d}p_n \leq \int_{G} X(x, u(x))\, \mathrm{d}x. \tag{5}$$

Now take a point $x \in G$ where $u(x) < 0$ and construct in $(n + 1)$-space the cone C that projects ∂G from the point x, $u(x)$. One can easily observe, from direct geometrical consideration, that to every supporting plane to the cone C there corresponds a parallel supporting plane to the surface $S : z = u(x)$. It means that the supporting image of S includes that of C; i.e. $\Psi_u(G) \supset \Psi_C$; and moreover $\operatorname{mes} \Psi_u > \operatorname{mes} \Psi_C$. Hence (5) implies

$$\int_{\Psi_C} U^{-1} \mathrm{d}p_1 \ldots \mathrm{d}p_n < \int_{G} X\, \mathrm{d}x. \tag{6}$$

Now, elementary geometrical consideration show that the supporting image of the cone C is a convex domain bounded by the surface with the equation (in spherical coordinates p, ν)

$$p = \frac{|u(x)|}{h(x, \nu)}.$$

Thus, if we transform the left integral (6) to the spherical coordinates p, ν, we shall see that it is the left integral in (2). Hence (6) implies (2) and our theorem is proved.

2. Suppose u satisfies an equation

$$F(u_{ij}, u_i, u, x) = 0 \tag{7}$$

where F is such that (7) implies (1) at almost all convexity points of u. Then we can apply our Theorem 1 which will give the estimations of the values $u(x)$.

One can observe that the inequality $F \leq 0$ imlies $w \leq K(x, u, \nabla u)$, when $d^2u \geq 0$, for every strictly elliptic F and even for wider class of F. The estimation $K(x, u, \nabla u) \leq X(x, u)\, U(\nabla u)$ usually takes place. Thus Theorem 1 proves to be applicable to a very wide class of equations.

The simplest case is the linear equation

$$a^{ij}u_{ij} + b\,\nabla u = g, \qquad g = f - cu, \qquad a^{ij}\xi_i\xi_j \geq 0. \tag{8}$$

Because of $a^{ij}\xi_i\xi_j \geq 0$ we have at the point where $d^2u > 0$

$$a^{ij}u_{ij} \geq n(aw)^{\frac{1}{n}}, \qquad a = \det(a^{ij}). \tag{9}$$

Hence $n(aw)^{\frac{1}{n}} \leq g - b\,\nabla u$ which easily leads to the inequality of the form (1). The results got for linear equations will be given somewhat further.

3. Under certain conditions on the function U in (1) the inequality (2) can be transformed into a simpler form. Introduce the functions $h_K(x)$ — the mean values of the distances $h(x, \nu)$:

$$h_K(x) = \left[\frac{1}{\varkappa_n}\int\limits_\Omega h^{-K}(x, \nu)\, d\nu,\right]^{-\frac{1}{K}} \quad K \neq 0; \quad h_0(x) = \exp\frac{1}{\varkappa_n}\int\limits_\Omega \ln h(x, \nu)\, d\nu \tag{10}$$

where $\varkappa_n = \operatorname{mes}\Omega$.

Theorem 2. *If* $U(p\nu) \leq \overline{U}(p)$ *and* $\overline{U}(p)\, p^{K-n}$ *is a non-increasing function, then* (2) *implies*

$$\varkappa_n \int\limits_0^{\frac{|u(x)|}{h_K(x)}} \overline{U}^{-1}(p)\, p^{n-1}\, dp < \int\limits_G X(x, u(x))\, dx. \tag{11}$$

4. For the linear equation (8) we get the following results.

Theorem 3. *If in* (8) $\det(a^{ij}) = 1$ *then at every point* x *where* $u(x) < 0$

$$|u(x)| < \alpha_n \|g_+\|\, F_n(\|b\|)\, h_0(x) \tag{12}$$

where the norms are those in $L_n(G)$, $\alpha_n = n^{-1}\tau_n^{-\frac{1}{n}}$, $\tau_n = \varkappa_n n^{-1}$ *is the volume of the unite sphere,*

$$F_n(\xi) = e^{\frac{\xi^n}{n^n \varkappa_n}}, + \varphi_n(\xi), (\xi \geq 0), \tag{13}$$

$\varphi_n(\xi)$, *for* $n > 1$, *being a bounded increasing function,* $\varphi_n(0) = 0$, *and* $\varphi_1(\xi) \equiv 0$. *Presize definition of the function* F_n *can be given as a convers to an explicitely represented elementary function.*

Theorem 3 leads to the following corollaries.

Theorem 4. *The homogeneous equation* (8) *with* det $(a^{ij}) = 1$ *has no non-zero solution if* $||c_+|| < \infty$ *and*

$$\alpha_n ||c_+ h_0|| \; F_n(||b||) \leq 1. \tag{14}$$

If the strict inequality takes place here, then at every x where $u(x) < 0$

$$|u(x)| < \frac{||f_+|| \; h_0(x)}{\alpha_n^{-1} F_n^{-1}(||b||) - ||c_+ h_0||}. \tag{15}$$

5. The inequalities of Theorems 3,4 are presize and no general estimations nor general uniqueness conditions are possible in terms of norms weaker than those in $L_n(G)$. The presize meaning of this statement is given by the following theorems in which we speak on elliptic equations (8) with smooth coefficients, det $(a^{ij}) = 1$ and on theri smooth solutions u with $u/\partial G = 0$.

Theorem 5. *Let the domain G be convex.*

(1) *Consider in G equations with a given value of the magnitude* $\alpha_n ||g|| \; F_n(||b||) = H$. *The lower upper bound of the values* $|u(x)|$ *of their solutions, for every* x, *is* sup $|u(x)| = H h_0(x)$. *(If G is a sphere.* x_0 *is its center,* A, B, ε *positive numbers, there exist in G equations with* $||g|| = A$, $||b|| = B$ *and the solution, u for which,* $|u(x_0)|$ *differs from the right part of* (12) *less than by* ε.*)*

(2) *For any* $\varepsilon > 0$ *such a homogeneous equation can be given that*

$$\alpha_n ||c_+ h_0|| \; F_n(||b||) < 1 + \varepsilon,$$

but it has non-zero solution.

(3) *The estimation* (15) *is presize in the sense analogous to* (1).

Theorem 6. *Let G be a sphere; let* $\varphi(\xi)$ *be such a function,* $\xi \in [0, \infty)$, *that* $\varphi(\xi)\, \xi^{-1} \to 0$ *when* $\xi \to \infty$. *Put for a function g in G*

$$N(g) = \int_G \varphi(g^n) \, \mathrm{d}x. \tag{17}$$

(1) *Such a sequence of equations* $a^{ij} u_{ij} = f$ *can be given in G that* $N(f) \to 0$, *but* $|u(x)| \to \infty$ *for every* $x \in G$.

(2) *For any* $\varepsilon > 0$ *such equations*

$$a^{ij} u_{ij} + b \nabla u = 0, \qquad \bar{a}^{ij} u_{ij} + cu = 0 \tag{18}$$

can be given in G that $N(b) < \varepsilon$, $N(c) < \varepsilon$, *but the equations have non-zero solutions.*

6. Let $r = r(x)$ be the distance from $x \in G$ to the boundary of the convex hull of G in the direction of the vector $-b = -b(x)$. Put $\bar{c} = c + |b| \, r^{-1}$, $\bar{g} = f - \bar{c}u$.

Theorem 7. *Under the conditions of Theorem* 3

$$|u(x)| < \alpha_n ||\bar{g}_+|| \; h_n(x). \tag{19}$$

The condition of nonexistance of non-zero solution is

$$\alpha_n||\bar{c}_+h_n|| \leq 1, \qquad ||\bar{c}_+|| < \infty, \tag{20}$$

and if here the strict inequality takes place,

$$|u(x)| < \frac{||f_+||\, h_n(x)}{\alpha_n^{-1} - ||\bar{c}_+h_n||}. \tag{21}$$

These inequalities are presize in a sense analogous to that of Theorem 5; we have but to consider in this Theorem the equations with $b \equiv 0$.

The estimation (19) is formally always true but it has a meaning if $||\bar{g}_+|| < < \infty$ which is ensured if $||br^{-1}|| < \infty$. This implies certain conditions on b. Let G be convex and $x \to \partial G$. Then, if roughly speaking $b(x)$ is directed from ∂G, $r(x) \to 0$ and the condition $||br^{-1}|| < \infty$ gives a comparatively strong limitation on $|b(x)|$; but if $b(x)$ is directed towards ∂G, $r(x) >$ const > 0, and $||br^{-1}|| < \infty$ if $||b|| < \infty$.

The advantage of the inequalities of Theorem 7 in comparison to those of Theorems 3,4 consists in the properties of the function $h_n(x)$. Owing to well known properties of meanvalues, $h_n(x) < h_0(x)$ with the only exception when G is a sphere and x is its center. Moreover, if G is convex and $\varrho(x)$ denotes the distance of x from ∂G, we have the estimation $h_n(x) <$ Const $\varrho^{\frac{1}{n}}(x)$. On the contrary, at every point $x \in \partial G$ which is the vertex of a paraboloid (of any degree > 1) included in G, $h_0(x) > 0$.

7. All above results allow of an essential generalization which, shortly speaking, consists in application of the some considerations to the projections of the solution u on various planes E of any dimensionality m, $1 \leq m \leq n$. We may suppose that E is $(x^1, \ldots, x^m)$ — plane. Then the lower projection of a function $\varphi(x) \equiv \varphi(x^1, \ldots, x^n)$, $x \in G$, is

$$\varphi_E(x^1, \ldots, x^m) = \inf_{(x^{m+1}, \ldots, x^n)} \varphi(x^n, \ldots, x^n), \tag{22}$$

and the upper projection is $\varphi_E(x^1, \ldots, x^m) = \sup \varphi(x^1, \ldots, x^n)$; they are defined in the projection G_E of G.

The results for linear equation (8) imply the norms $||\varphi||_E$ defined as follows. Let $a_E = \det(a^{ij})$, $i, j \leq m$, provided E is $(x^1, \ldots, x^m)$-plane. We define

$$||\varphi||_E = ||a_E^{-\frac{1}{m}} \cdot |\varphi|^E||_{L_m(G_E)}. \tag{23}$$

We define the functions $h_{KE}(x)$ by the same formula (10) with the only difference that we integrate over the set Ω_E of the unite vectors in E and pivide by $\varkappa_m = \text{mes}\, \Omega_E$.

Theorem 8. *Under the conditions of the Theorem* 3, *for almost all planes* E *of any bundle there takes place the inequalities*

$$|u(x)| < \alpha_m \|g_+\|_E F_m(\|b\|_E)\, h_{0E}(x). \tag{24}$$

Theorems 4, 5 admit corresponding generalizations, too.

8. The methods and results given here are expounded with proofs in a series of my papers published in

Сибирский математический журнал. 1966, № 3; *Вестник Ленинградского универзитета* 1966, NN№ 1, 7, 13; *Доклады Академии наук СССР*, 1966, v. 169, № 4,

and partly in a course of lectures "*The method of normal map in uniqueness problems and estimations for elliptic equations*", *Seminari dell' Instituto Nazionale di Alta Matematica* 1962—1963, *vol.* 2, *Roma* 1965.

By a different method under different conditions the problem of majorating the Dirichlet problem solutions has been studied by C. Pucci and M. Fraxa; cf. in particular C. Pucci, *Operatori ellittici estremanti*, *Annali di Mat.*, *vol.* 72, *pp.* 141—170 (1966).