Alberto Valli

Qualitative properties of the solutions to the Navier-Stokes equations for compressible fluids

In: Jaromír Vosmanský and Miloš Zlámal (eds.): Equadiff 6, Proceedings of the International Conference on Differential Equations and Their Applications held in Brno, Czechoslovakia, Aug. 26 - 30, 1985. J. E. Purkyně University, Department of Mathematics, Brno, 1986. pp. [259]--264.

Persistent URL: http://dml.cz/dmlcz/700187

QUALITATIVE PROPERTIES OF THE SOLUTIONS TO THE NAVIER-STOKES EQUATIONS FOR COMPRESSIBLE FLUIDS

A. VALLI
Dipartimento di Matematica, Università di Trento
38050 Povo (Trento), Italy

1. Introduction.

We want to present a new method for showing the existence of a stationary solution to the equations which describe the motion of a viscous compressible barotropic fluid.

At first it is useful to recall some known results concerning the non-stationary case. The equations of motion are

$$
(NS)\quad \begin{cases}
\rho\left[\dfrac{\partial v}{\partial t}+(v\cdot\nabla)v-f\right]=-\nabla[p(\rho)]+\mu\Delta v+(\zeta+\mu/3)\nabla \operatorname{div} v & \text{in }]0,T[\times\Omega\,,\\
\dfrac{\partial\rho}{\partial t}+\operatorname{div}(\rho v)=0 & \text{in }]0,T[\times\Omega\,,\\
v_{|\partial\Omega}=0 & \text{on }]0,T[\times\partial\Omega\,,\\
\int_\Omega \rho=\bar\rho|\Omega|>0 & (|\Omega|\equiv \operatorname{meas}(\Omega))\,,\\
v_{|t=0}=v_o & \text{in } \Omega\,,\\
\rho_{|t=0}=\rho_o & \text{in } \Omega\,,
\end{cases}
$$

where $\Omega\subset R^3$ is a bounded domain, with smooth boundary $\partial\Omega$; v and ρ are the velocity and the density of the fluid; p is the pressure, which is assumed to be a known function of ρ; f is the (assigned) external force field; the constants $\mu>0$ and $\zeta\geq 0$ are the viscosity coefficients; $\bar\rho>0$ is the mean density of the fluid, i.e. the total mass of fluid divided by $|\Omega|$; v_o and ρ_o are the initial velocity and density.

In the last years it has been proved that:

(i) if v_o and $\rho_o-\bar\rho$ are small enough and $f=0$, then problem (NS) has a unique global (in time) solution (Matsumura-Nishida [1]);

(ii) the preceeding result also holds for a sufficiently small $f \neq 0$; moreover, two small solutions are asymptotically equivalent as $t \to +\infty$, and consequently if f is periodic (independent of t) then there exists a periodic (stationary) solution (Valli [3]).

It must be underlined that no other method is known for showing the existence of a stationary solution, excepting when the viscosity coefficients satisfy $\zeta >> \mu$. In this case Padula [2] proved that, if f is small enough, then there exists a stationary solution. Remark, however, that in general the shear viscosity coefficient μ is larger than the bulk viscosity coefficient ζ. Moreover, from the mathematical point of view it would seem only necessary to require that μ is positive, without assumptions on the largeness of ζ.

The method that we want to present here is based on a "natural" linearization of the problem, followed by a fixed point argument. The viscosity coefficients are only required to satisfy the thermodynamic restrictions $\mu > 0$, $\zeta \geq 0$.

2. The linear problem (L).

Since we are searching for a solution in a neighbourhood of the equilibrium solution $\tilde{\rho} = \bar{\rho}$, $\tilde{v} = 0$, it is useful to introduce the new unknown

$$\sigma = \rho - \bar{\rho} .$$

The equations of motion in the stationary case thus become

$$(S) \begin{cases} -\mu\,\Delta v - (\zeta+\mu/3)\,\nabla \mathrm{div}\, v + p_1 \nabla\sigma = (\sigma+\bar{\rho})[f - (v\cdot\nabla)v] + \\ \qquad + [p_1 - p'(\sigma+\bar{\rho})]\nabla\sigma & \text{in } \Omega, \\ \bar{\rho}\,\mathrm{div}\, v + \mathrm{div}(v\sigma) = 0 & \text{in } \Omega, \\ v_{|\partial\Omega} = 0 & \text{on } \partial\Omega, \\ \int_\Omega \sigma = 0 & , \end{cases}$$

where it is assumed that $p_1 \equiv p'(\bar{\rho}) > 0$.

It is easily verified that a solution of (S) exists if we find a fixed point of the map

$$\Phi : (v,\sigma) \longrightarrow (w,\eta) ,$$

defined by means of the solutions of the following linear problem

$$
(L)\begin{cases}
-\mu\,\Delta w-(\zeta+\mu/3)\nabla\mathrm{div}\,w+p_1\nabla\eta=(\sigma+\bar\rho)[f-(v\cdot\nabla)v]+\\
\qquad\qquad+[p_1-p'(\sigma+\bar\rho)]\nabla\sigma\equiv F & \text{in } \Omega,\\
\bar\rho\,\mathrm{div}\,w+\mathrm{div}(v\eta)=0 & \text{in } \Omega,\\
w_{|\partial\Omega}=0 & \text{on } \partial\Omega,\\
\int_\Omega \eta=0 & .
\end{cases}
$$

3. A-priori estimates for the solution of (L).

We want to obtain a-priori estimates in Sobolev spaces of sufficiently large order, in such a way that we can control the behaviour of the non-linear terms which appear in F. We shall prove that a solution (w,η) of (L) satisfies

(3.1) $$\|w\|_3+\|\eta\|_2\leq c_1\|F\|_1$$

for $v_{|\partial\Omega}=0$ and $\|v\|_3\leq A$ small enough. Here $\|\cdot\|_k$ is the norm in the Sobolev space $H^k(\Omega)$, and c_1 depends in a continuous way on μ, ζ and A (but it is independent of v).

(a) At first, from well-known results on Stokes problem we get

(3.2) $$\|w\|_3+\|\eta\|_2\leq c(\|F\|_1+\|\mathrm{div}\,w\|_2) \quad .$$

Hence our aim is to estimate $\|\mathrm{div}\,w\|_2$.

(b) Multiplying $(L)_1$ by w and $(L)_2$ by $(p_1/\bar\rho)\eta$ and integrating in Ω one has

(3.3) $$\|w\|_1+\|\mathrm{div}\,w\|_0\leq c(\|F\|_{-1}+\|v\|_3^{1/2}\|\eta\|_0).$$

The same argument can be used for estimating all the successive derivatives in the interior of Ω, and the tangential derivative $D_\tau\mathrm{div}\,w$ near the boundary $\partial\Omega$, obtaining in this way (in local coordinates near $\partial\Omega$)

(3.4) $$\|D_\tau w\|_1+\|D_\tau\mathrm{div}\,w\|_0\leq c(\|F\|_0+\|v\|_3^{1/2}\|\eta\|_1).$$

(c) The estimate for the normal derivative $D_n\mathrm{div}\,w$ is obtained by observing that on $\partial\Omega$

$$\Delta w\cdot n\cong\nabla\mathrm{div}\,w\cdot n\ ,$$

in the sense that their difference does not contain D_n^2w.

Hence by taking the normal derivative of $(L)_2$, multiplied by $(\bar{\rho})^{-1}\cdot$ $(\zeta+4\mu/3)$, and adding it to the normal component of $(L)_1$ we get (in local coordinates near $\partial\Omega$)

$$(3.5) \qquad p_1 D_n \eta + (\zeta+4\mu/3)/\bar{\rho}\, D_n \operatorname{div}(v\eta) \cong F\cdot n \ .$$

From this equation one easily gets

$$(3.6) \qquad \|D_n \eta\|_0 \leq c(\|F\|_0 + \|v\|_3^{1/2}\|\eta\|_1) .$$

Moreover, going back to $(L)_1$, one has

$$p_1 D_n \eta = \mu\, \Delta w\cdot n + (\zeta+\mu/3)\nabla \operatorname{div} w\cdot n + F\cdot n \cong (\zeta+4\mu/3) D_n \operatorname{div} w + F\cdot n \ ,$$

hence from (3.6)

$$(3.7) \qquad \|D_n \operatorname{div} w\|_0 \leq c(\|F\|_0 + \|v\|_3^{1/2}\|\eta\|_1) .$$

By repeating the same argument for the second order derivatives one gets

$$(3.8) \qquad \|\operatorname{div} w\|_2 \leq c(\|F\|_1 + \|v\|_3^{1/2}\|\eta\|_2) ,$$

hence (3.1) holds if $\|v\|_3 \leq A$ small enough.

4. Existence of the solution of (L).

Though problem (L) is linear, and we know that the a-priori estimate (3.1) holds, the existence of a solution $w\in H^3(\Omega)$, $\eta\in H^2(\Omega)$ is not obvious.

In fact, the usual elliptic approximation cannot work in this case. More precisely, if we add $-\varepsilon\,\Delta\eta_\varepsilon$ to $(L)_2$, we must also require a boundary condition (say, Dirichlet or Neumann) on η_ε. But the limit function η is free on $\partial\Omega$. Hence the sequence η_ε can only converge in $L^2(\Omega)$ (Dirichlet condition), or in $H^1(\Omega)$ (Neumann condition), and cannot converge in $H^2(\Omega)$!

Moreover, if $v \neq 0$ (L) is not an elliptic system in the sense of Agmon-Douglis-Nirenberg (if $v = 0$ (L) is the Stokes system). Hence the usual regularization procedures do not work.

One can proceed in the following way. By adapting the method of Padula [2] to problem (L), one defines

$$\pi \equiv (p_1/\mu)\eta - (\zeta/\mu + 1/3)\,\mathrm{div}\, w \tag{4.1}$$

and (L) is transformed into

$$(L')\qquad \begin{cases} -\Delta w + \nabla\pi = F/\mu & \text{in } \Omega, \\ \mathrm{div}\, w = (\zeta/\mu + 1/3)^{-1}(p_1\eta/\mu - \pi) & \text{in } \Omega, \\ w_{|\partial\Omega} = 0 & \text{on } \partial\Omega, \end{cases}$$

$$(L'')\qquad \begin{cases} \bar\rho(\zeta/\mu + 1/3)^{-1} p_1\eta/\mu + \mathrm{div}(v\eta) = \bar\rho(\zeta/\mu + 1/3)^{-1}\pi & \text{in } \Omega, \\ \int_\Omega \eta = 0 \end{cases}$$

These equations can be solved via a fixed point argument if $\zeta >> \mu$. Hence the a-priori estimates (3.1) and the continuity method give the result for any pair of viscosity coefficients satisfying $\mu > 0$ and $\zeta \geqq 0$.

5. Existence of a solution of (S).

We prove at last the existence of a fixed point for the map

$$\Phi : (v,\sigma) \longrightarrow (w,\eta).$$

Taking

$$K \equiv \{ (v,\sigma) \in H^3(\Omega) \times H^2(\Omega) \mid \ v_{|\partial\Omega} = 0,\ \int_\Omega \sigma = 0,\ \| v \|_3 + \| \sigma \|_2 \leqq A \},$$

by using (3.1) one sees that

$$\| w \|_3 + \| \eta \|_2 \leqq c_1 \| F \|_1 \leqq c\, [(\| \sigma \|_2 + 1)(\| f \|_1 + \| v \|_2^2) + \| \sigma \|_2^2] \leqq c(A+1)(\| f \|_1 + A^2) \quad .$$

Choosing $A^2 \equiv \| f \|_1 << 1$, one has

$$\| w \|_3 + \| \eta \|_2 \leqq A,$$

hence $\Phi(K) \subset K$. The set K is convex and compact in $X \equiv H^2(\Omega) \times H^1(\Omega)$, and it is easily seen that the map Φ is continuous in X. The existence of a fixed point is now a consequence of Schauder's theorem.

References.

[1] A. Matsumura - T. Nishida, Initial boundary value problems for the equations of motion of general fluids, in "Computing methods in applied sciences and engineering, V", ed. R. Glowinski - J.L. Lions, North-Holland Publishing Company, Amsterdam-New York-Oxford, 1982. (See also Preprint University of Wisconsin, MRC Technical Summary Report n° 2237 (1981)).

[2] M. Padula, Existence and uniqueness for viscous steady compressible motions, Arch. Rational Mech. Anal., to appear.

[3] A. Valli, Periodic and stationary solutions for compressible Navier-Stokes equations via a stability method, Ann. Scuola Norm. Sup. Pisa, (4) 10 (1983), 607-647.