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ON THE ZEROS OF SOME SPECIAL FUNCTIONS: DIFFERENTIAL EQUATIONS AND NICHOLSON-TYPE FORMULAS

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1. Introduction. There are many results in the literature on special functions concerning the way in which a zero of a function changes with respect to one of the parameters on which the function depends. Methods based on differential equations, in particular Sturmian methods, are often useful in these discussions. Other methods are related to integral representations for the functions and seem to be provable, though not easily discoverable, by differential equations methods. Among these are methods based on Nicholson's formula [13, p.444]

$$J_\nu^2(z) + Y_\nu^2(z) = \frac{8}{\pi^2}\int_0^\infty K_0(2z \sinh t) \cosh 2\nu t \, dt \ , \quad \operatorname{Re} z > 0 \ , \tag{1}$$

and a companion formula

$$J_\nu(z)\partial Y_\nu(z)/\partial\nu - Y_\nu(z)\, \partial J_\nu(z)/\partial\nu = -\frac{4}{\pi}\int_0^\infty K_0(2z \sinh t)e^{-2\nu t} \, dt \ , \quad \operatorname{Re} z > 0 \ , \tag{2}$$

from which it follows [13, p.508] that

$$dc/d\nu = 2c \int_0^\infty K_0(2c \sinh t)e^{-2\nu t} \, dt \ . \tag{3}$$

Here J_ν and Y_ν are the usual Bessel functions, K_0 is the modified Bessel function and, in (3), $c = c(\nu,k,\alpha)$ is an x-zero of the linear combination

$$C_\nu(x) = \cos\alpha\, J_\nu(x) - \sin\alpha\, Y_\nu(x) \ .$$

Formula (1) was used by L. Lorch and P. Szego [9] to show some remarkable sign-regularity properties of the higher k-differences of the sequence $\{c(\nu,k,\alpha)\}$ in the case $|\nu| \ge \frac{1}{2}$. Beyond its obvious use to show that c increases with ν, (3) has been used to get further information about these zeros; see [10,11] for references. Á. Elbert has used (3) to show that $j_{\nu k}(=c(\nu,k,0))$ is a concave function on ν on $(-k,\infty)$. Elbert and A. Laforgia have used (3) in several recent papers. They proved, for example, that $j_{\nu k}^2$ is a convex function of ν on $(0,\infty)$ [6] and they have shown recently (personal communication) that $d^3 j_{\nu k}/d\nu^3 > 0$, $0 < \nu < \infty$.

2. Other Nicholson-type formulas. The usefulness of (1), (2) and (3) suggests the desirability of having similar formulas for other special functions. L. Durand [3,4] has given results analogous to (1) for some of the classical orthogonal polynomials. The simplest of these, for Hermite functions, is [3, p.371]

$$(4)\quad e^{-x^2}[H_\lambda^2(x) + G_\lambda^2(x)] = \frac{2^{\lambda+1}\Gamma(\lambda+1)}{\pi}\int_0^\infty e^{-(2\lambda+1)t + x^2\tanh t}(\cosh t \sinh t)^{-1/2}\,dt.$$

Durand does not use differential equations but points out [3, p.355] that, once the results are known, they can be checked by differential equations methods. In fact J.E. Wilkins, Jr. [14] (see also [12, pp.340-341]) proved (1) by showing that both sides satisfy the same third order differential equation and have the same asymptotic behaviour as $z \to +\infty$. I [10] did the same for equation (2) using a third-order nonhomogeneous equation.

More recently, I have tried to discover whether there is a natural way in which these formulas arise in a differential equations setting. I present such a setting here for Bessel functions but it is not clear to me yet whether the method applies to a general situation of which the Bessel function case would be a particular example. It turns out to be convenient to consider the more general formulas [2,7,13]

$$(5)\quad J_\mu(z)J_\nu(z) + Y_\mu(z)Y_\nu(z) = \frac{4}{\pi^2}\int_0^\infty K_{\nu-\mu}(2z\sinh t)[e^{(\mu+\nu)t} + e^{-(\mu+\nu)t}\cos(\mu-\nu)\pi]dt\ ,$$

$$(6)\quad J_\mu(z)J_\nu(z)+Y_\mu(z)Y_\nu(z) = \frac{4}{\pi^2}\int_0^\infty K_{\mu+\nu}(2z\sinh t)[e^{(\mu-\nu)t}\cos\nu\pi + e^{-(\mu-\nu)t}\cos\mu\pi]dt,$$

$$(7)\quad J_\mu(z)Y_\nu(z) - J_\nu(z)Y_\mu(z) = \frac{4}{\pi^2}\sin(\mu-\nu)\pi\int_0^\infty K_{\nu-\mu}(2z\sinh t)e^{-(\nu+\mu)t}dt$$

$$(8)\quad J_\mu(z)Y_\nu(z)-J_\nu(z)Y_\mu(z) = \frac{4}{\pi^2}\int_0^\infty K_{\nu+\mu}(2z\sinh t)[e^{(\nu-\mu)t}\sin\mu\pi - e^{(\mu-\nu)t}\sin\nu\pi]dt.$$

These are all valid for $\operatorname{Re} z > 0$ with $|\operatorname{Re}(\mu+\nu)| < 1$ in (6) and (8) and $|\operatorname{Re}(\nu-\mu)| < 1$ in (5) and (7).

Clearly (1) is got from (5) by setting $\mu = \nu$ while, as pointed out in [2], (2) (and hence (3)) is got from (7) by dividing by $\mu - \nu$ and letting $\mu \to \nu$. Dixon and Ferrar [2, p.142] find an analogue of (2) based on a similar treatment of (8).

The corresponding analogue of (3) is

$$dc/d\nu = -(2c/\pi)\int_0^\infty K_{2\nu}(2c\sinh t)[2t\sin\nu\pi - \pi\cos\nu\pi]dt\ ,\ c > 0\ ,\ |\nu| < \frac{1}{2}\ ,$$

but this is both more complicated and has a smaller range of validity then (3).

3. <u>A differential equations proof of (6)</u>. The proofs of (5), (7) and (8) are quite similar to that which we will give for (6). We may clearly suppose that μ and ν are real and that z is real and positive and we write $z = e^\theta$ so that (6) becomes

$$(9)\quad J_\mu(e^\theta)J_\nu(e^\theta) + Y_\mu(e^\theta)Y_\nu(e^\theta)$$

$$= \frac{4}{\pi^2}\int_0^\infty K_{\mu+\nu}(2e^\theta\sinh t)[e^{(\mu-\nu)t}\cos\nu\pi + e^{-(\mu-\nu)t}\cos\mu\pi]dt\ .$$

The functions $J_\nu(e^\theta)$, $Y_\nu(e^\theta)$ satisfy [13, p.99]

$$d^2y/d\theta^2 + (e^{2\theta} - \nu^2)y = 0$$

so that the left-hand side of (9) is a solution of [13, p.146]

$$L_\theta u \equiv (D_\theta^2 - b^2)(D_\theta^2 - a^2)u + 4e^{2\theta}(D_\theta + 1)(D_\theta + 2)u = 0 \tag{10}$$

where $a = \mu + \nu$, $b = \mu - \nu$. There is a standard method [1; 8, Ch. 8] for finding an integral representation

$$u(\theta) = \int_\alpha^\beta k(\theta,t)\ v(t)\ dt \tag{11}$$

for a solution of (10). We try to find a linear differential operator

$$M_t = \sum_{k=0}^{m} m_k(t)\ D_t^k$$

and a function $\kappa(\theta,t)$ such that

$$L_\theta\ k(\theta,t) = M_t\ \kappa(\theta,t)\ . \tag{12}$$

We then determine $v(t)$ as a solution of $\bar{M}_t\ v = 0$ where $\bar{M}_t$ is the adjoint of M_t , i.e.

$$\bar{M}_t\ v = \sum_{k=0}^{m} (-1)^k\ D_t^k[m_k(t)v]\ .$$

Then (11) is a solution of $L_\theta\ u = 0$ provided α and β are chosen so that

$$\left[\sum_{k=1}^{m} \sum_{\ell=0}^{k-1} (-1)^\ell\ (m_k\ v)^{(\ell)}\ \kappa^{(k-\ell-1)}\right]_\alpha^\beta = 0\ . \tag{13}$$

(The differentiations in (13) are with respect to t .) Most of the standard applications of the method are to second order equations and with $\kappa = k$ and its success depends on being able to solve the equation $\bar{M}_t\ v = 0$. In the present case, if we choose

$$k(\theta,t) = K_a(2e^\theta \sinh t) \tag{14}$$

we have the convenient "factorization"

$$L_\theta\ k(\theta,t) = (D_t^2 - b^2)(D_\theta^2 - a^2)\ k(\theta,t) \tag{15}$$

which is of the form (12) with

$$\kappa(\theta,t) = (D_\theta^2 - a^2)\ K_a(2e^\theta \sinh t) = 4e^{2\theta} \sinh^2 t\ K_a(2e^\theta \sinh t)$$

and $M_t = \bar{M}_t = D_t^2 - b^2$. Thus we get $v(t) = c_1\ e^{(\mu-\nu)t} + c_2\ e^{-(\mu-\nu)t}$ and it is easily shown that (13) holds if we choose $\alpha = 0$, $\beta = \infty$. To determine c_1 and c_2 we use [13, Ch.7]

$$J_\mu(x)\ J_\nu(x) + Y_\mu(x)\ Y_\nu(x) = \frac{2}{\pi x} \cos\frac{(\mu-\nu)\pi}{2} + \frac{1}{\pi x^2}(\mu^2-\nu^2) \sin\frac{(\mu-\nu)\pi}{2} + O(x^{-3})\ ,\ x \to \infty\ ,$$

and, using [12, Ch.9] ,

$$\int_0^\infty K_a(2x \sinh t)e^{\lambda t}\, dt = \frac{\pi}{4 \cos(\pi a/2)}\, x^{-1} + \frac{a\lambda\pi}{8 \sin(\pi a/2)}\, x^{-2} + O(x^{-3}) \ , \ x \to \infty \ .$$

Then, by comparing coefficients, we get $c_1 = \cos \nu\pi$, $c_2 = \cos \mu\pi$ so (9), and hence (6), is proved.

The key to the success of the method in the present case is the factorization (15) arising from the choice (14) for the kernel $k(\theta,t)$. The choice of a function of the form $f(2e^\theta \sinh t)$ may be motivated by the fact that for a polynomial P we have $P(D_\theta)\, f(2e^\theta \sinh t) = P(\tanh t\, D_t)\, f(2e^\theta \sinh t)$. We see from (10) that $L_\theta\, f(2e^\theta \sinh t)$ can be expected to take on a relatively simple form if we choose f to satisfy

$$(D_\theta^2 - a^2)\, f(2e^\theta \sinh t) = 4e^{2\theta} \sinh^2 t\, f(2e^\theta \sinh t)$$

But this is the modified Bessel equation satisfied by $f = K_a$.

4. <u>Another Nicholson-type formula</u>. Here we give a differential equations proof of

$$(16)\quad J_\nu^2(x) + Y_\nu^2(x) = \frac{4}{\pi^2} \frac{\Gamma(\nu)}{\Gamma(2\nu)} (4x)^\nu \int_0^\infty K_\nu(2x \sinh t)(\cosh t)^{2\nu} (\sinh t)^\nu\, dt$$

$$= \frac{4}{\pi^2} \frac{\Gamma(\nu)}{\Gamma(2\nu)} \left(\frac{x}{2}\right)^\nu \int_0^\infty K_\nu(xu)(u^2 + 4)^{\nu-1/2}\, u^\nu\, du \ , \ x > 0 \ , \ \nu > -\frac{1}{2} \ ,$$

the special case $n = 0$ of [3, p.368, (42)]. It is convenient to write this formula in the form

$$(17)\quad j_\alpha^2(x) + y_\alpha^2(x) = \int_0^\infty k_\alpha(xt)\ (t^{\alpha/2} + 4)^{-(\alpha+4)/(2\alpha+4)}\ t^{\alpha/2-1}\, dt \ , \ x > 0 \ , \ \alpha > 0$$

where we have adopted an *ad hoc* notation for the generalized Airy functions: j_α , y_α are appropriately normalized solutions of

$$(18)\qquad y'' + x^\alpha y = 0$$

where $\alpha = -2 - 1/\nu$, while k_α is a suitable solution vanishing at $+\infty$ of

$$(19)\qquad y'' - x^\alpha y = 0 \ .$$

In the special case $\alpha = 1$ $(\nu = -1/3)$, (17) becomes

$$Ai^2(-x) + Bi^2(-x) = \frac{24(2/3)^{1/6}}{\sqrt{\pi}\, \Gamma(1/6)} \int_0^\infty t^{-1/2}(t^3 + 4)^{-5/6}\, Ai(xt)\, dt \ , \ x > 0 \ .$$

In order to prove (17) we note that, using (18) and [13, p.145] its left-hand side satisfies

$$L_x\, u \equiv (D_x^3 + 4x^\alpha D_x + 2\alpha x^{\alpha-1})u = 0$$

and, using (19), we find that

$$L_x\, k_\alpha(xt) = M_t\, x^{\alpha-1}\, k_\alpha(xt)$$

where

$$M_t = (t^{\alpha+3} + 4t)D_t + \alpha(t^{\alpha+2} + 2)$$

and

$$\bar{M}_t = -(t^{\alpha+3} + 4t)D_t + (2\alpha - 4 - 3t^{\alpha+2}) .$$

Now $\bar{M}_t\, v = 0$ has the general solution

$$v(t) = ct^{\alpha/2-1}\,(t^{\alpha+2} + 4)^{-(\alpha+4)/(2\alpha+4)} .$$

We note that the condition (13) is also satisfied with $\alpha = 0$, $\beta = \infty$ leading to (17) apart from a constant factor. To evaluate the constant we return to the form (16) and use

$$J_\nu^2(x) + Y_\nu^2(x) = \frac{2}{\pi x} + O(x^{-2}) \ , \ x \to \infty$$

and, using [12, Ch.7],

$$x^\nu \int_0^\infty K_\nu(xu)(u^2 + 4)^{\nu-1/2}\, u^\nu\, au = \frac{1}{x}\, 2^{\nu-1}\, \pi\Gamma(2\nu)/\Gamma(\nu) + O(x^{-2}) \ , \ x \to \infty .$$

5. Zeros of generalized Airy functions. M.S.P. Eastham (private communication) raised the question of showing that the smallest positive zero x_α of a solution of (18), satisfying $y(0) = 0$, decreases as α increases, $0 < \alpha < \infty$. This, and more, has been proved by A. Laforgia and the author (to be published) using results (due to Elbert and Laforgia) based on (3) and the well-known connection between (18) and the Bessel equation. It would be nice to show this using (18) directly. The Sturm comparison theorem is not applicable in any obvious way because x^α is not monotonically increasing in α for each x in an interval $(0,b)$, $b > 1$. This raises the question of whether one can find an analogue of (3) (other than the awkward formula got by transforming (3) itself) for $dx_\alpha/d\alpha$. What we need in effect is a result that bears the same relation to (17) as (3) does to (1). One way to approach this problem would be to find an integral representation for $j_\alpha\, y_\beta - j_\beta\, y_\alpha$ which satisfies a known fourth order differential equation.

A perhaps more tractable problem would be to find the appropriate generalization of (4) for

$$e^{-x^2}[H_\lambda(x)\, G_\mu(x) - G_\lambda(x)\, H_\mu(x)] .$$

This would give, in particular, a formula for the derivative with respect to λ of a zero of a Hermite function.

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