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ON NONPARASITE SOLUTIONS

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1. Introduction

We shall investigate the differential relation

(1) $\dot{x} \in F(t,x), \quad x(0) = x_0$

where $F : U \to K$, $U = \langle 0,1\rangle \times B_1$, K is the set comprising nonempty, compact subsets of some ball in R^n, B_1 is the unit ball in R^n. Jarník and Kurzweil [2] proved that if $F(t,x)$ is convex then we can suppose F to be Scorza-Dragonian. These authors and many others (see e.g. [1], [2], [3], [10], [12]) have studied the convex case very thoroughly. The nonconvex r.h.s. has been attacked too, certain very strong results being obtained e.g. by Olech [7], Tolstonogov [10], [11], Vrkoč [12]. It is easy to see that to obtain some reasonable existence theorem in nonconvex case it is necessary to suppose F to be continuous. It is a well known fact that the solutions of $\dot{x} \in F$ are then dense in the set of all solutions of $\dot{x} \in \text{conv } F$, see e.g. Tolstonogov [9].

It is tempting then to use the Filipov respectively Krasovskij operation to define generalized solutions of $\dot{x} \in F(t,x)$, F being possibly nonconvex. To be more specific, we can define the solution of $\dot{x} \in F(t,x)$ through the relation $\dot{x} \in G(t,x)$ where

$$G(t,x) = \bigcap_{\delta>0} \bigcap_{\mu(N)=0} \overline{\text{conv}}\, F(t,B_\delta(x) - N) \quad \text{or}$$

$$G(t,x) = \bigcap_{\delta>0} \overline{\text{conv}}\, F(t,B_\delta(x)) .$$

The main problem is that introducing even the solution of $\dot{x} = f(x)$, f discontinuous real valued function, through Filippov or even Krasovskij operation we can obtain certain meaningless solutions.

2. Example 1. (Sentis [8])

Let $f : R \to R$, $f(x) = -1$ for $x \geq 0$, $f(x) = +1$ for $x < 0$. Then $x(t) = 0$ is a (unique) Filippov solution of the Cauchy problem $\dot{x} = f(x)$, $x(0) = 0$, $t \in \langle 0,1\rangle$. This type of solution is called sliding motion and there are good reasons to consider it to be the solution.

On the other hand let $f(x) = 1$ for $x \geq 0$, $f(x) = -1$ for $x < 0$. Then the Cauchy problem $\dot{x} = f(x)$, $x(0) = 0$ has the Filippov solution $x_+(t) = t$, $x_-(t) = -t$ and $x_a(t) = 0$ for $t \in \langle 0, |a| \rangle$, $x_a(t) = \operatorname{sgn} a.(t - |a|)$ for $t \geq |a|$. All the $x_a(.)$ solutions are physically meaningless, they are called parasite solution. For the exact definition of sliding and parasite solution see [4] or Sentis [8].

3. Generalized solutions

Our aim is to define the solution of $\dot{x} \in F(t,x)$ in such a manner that all the sliding solutions are retained and all parasite are expelled. The first definition of this type was given by Sentis [8] in 1976 and it was as follows:

Definition 1. Function $y(.) : \langle 0,1 \rangle \to R^n$ is a g-solution of the differential relation $\dot{x} \in F(t,x)$, $x(0) = x_0$ on $\langle 0,1 \rangle$ iff there exists a sequence $\{y_n\}_{n=1}^{\infty}$ of piecewise linear functions and a sequence $\{h_n\}_{n=1}^{\infty}$ of divisions such that (denote $y_n(h_n^k)$ by x_n^k and $\nu(h_n)$ by ν_n)

i) $\lim_{n\to\infty} |h_n| = 0$,

ii) $x_n^0 = x_0$

iii) for every positive integer n and $k = 0,1,\dots,\nu_n$ there are $a_n^k \in F(h_n^k, x_n^k)$ and $\varepsilon_n^k \in R^n$ such that $x_n^{k+1} = x_n^k + a_n^k(h_n^{k+1} - h_n^k) + \varepsilon_n^k$
and $y_n(.)$ is linear on every $\langle h_n^k, h_n^{k+1} \rangle$, $k = 0,1,\dots,\nu_n$

iv) $\lim_{n\to\infty} \sum_{k=1}^{\nu_n} \|\varepsilon_n^k\| = 0$

v) $\lim_n y_n = y$ uniformly on $\langle 0,1 \rangle$.

Sentis introduced this definition to cover the case (cl stands for closure)

$F(t,x) = \bigcap_{\delta>0} \bigcap_{\substack{N\subset R^{n+1} \\ \mu(N)=0}} \operatorname{cl} f(B_\delta(t,x) - N)$ and his definition works well for such right-hand sides. He proved that any classic solution of $\dot{x} \in F(t,x)$ (i.e. any absolutely continuous function $x(.)$ such that $\dot{x}(t) \in F(t,x(t))$ a.e.) is a g-solution, any g-solution of $\dot{x} \in F(t,x)$ is a classic solution of $\dot{x} \in \operatorname{conv} F(t,x)$ and there are no parasite solutions.

4. Example 2.

For $R^n = R$ set $F_1(t,x) = \{-1\}$ for $x < 0$ and every t, $F_1(t,x) =$

$= \{-1,1\}$ for $x = 0$ and every t and $F_1(t,x) = \{1\}$ for $x > 0$ and every t, $F_2(t,x) = F_1(t,x)$ for t dyadically irrational and every x. For $t = (k/2^m)$, k odd, set $F_2(t,x) = F_1(t,x)$ for $x \notin \langle -1/2^m, 1/2^m \rangle$ and $F_2(t,x) = \{-1,1\}$ for $x \in \langle -1/2^m, 1/2^m \rangle$. Then both F_1 and F_2 are u.s.c. mappings and $\mu\{t \in \langle 0,1 \rangle \mid \exists_x (F_1(t,x) \neq F_2(t,x))\} = 0$.

The function y(.), identically equal to zero on $\langle 0,1 \rangle$ is not a g-solution of $\dot{x} \in F_1(t,x)$, $x(0) = 0$ but it is a g-solution of the relation $\dot{x} \in F_2(t,x)$, $x(0) = 0$ on $\langle 0,1 \rangle$.

This example shows that even for F u.s.c. the solution does depend on values which F obtaines on a set whose projection on t-axis is of measure zero. In the sequel we shall modify the definition of the g-solution to avoid this discrepancy.

5. Regular Generalized Solutions

Let F be Scorza-Dragonian. Denote $G_M F = \{(t,x,y) \mid y \in F(t,x), t \notin M\}$ i.e. $G_M F$ is the graph of the partial mapping $F|_{(\langle 0,1 \rangle - M) \times B}$. We set $G^*F = \bigcap_{\substack{\mu(M)=0 \\ M \subset \langle 0,1 \rangle}} \mathrm{cl}\, G_M F$ and define a multivalued mapping F^* through its graph i.e. we set graph $F^* = G^*F$. It is possible to prove that there exists a set $M_0 \subset \langle 0,1 \rangle$, $\mu(M_0) = 0$ and $G^*F = \mathrm{cl}\, G_{M_0} F$, so our definition is meaningfull. The set G^*F is closed hence F^* is u.s.c. If the mapping F is u.s.c. too then $F^* \subset F$ because graph $F^* = \mathrm{cl}\, G_{M_0} F \subset \mathrm{cl}\, GF = GF$ and $\{t \in \langle 0,1 \rangle \mid \exists_x (F^*(t,x) \neq F(t,x))\} \subset M_0$ i.e. its measure is zero. We define the solution of $\dot{x} \in F(t,x)$ through the Sentis g-solution of $\dot{x} \in F^*(t,x)$; resulting type of solution being called rg-solution. It retains all the nice properties of Sentis g-solution and is independent on behaviour of F on a set of measure zero (in t). If the mapping F is supposed to be only Scorza-Dragonian we have only graph $F^* \subset \mathrm{cl}\, GF$ and $F^*(t,x) \supset F(t,x)$ for $t \notin M_0$, nonetheless the rg-solution can be defined too. There is following characterisation of rg-solution:

Theorem 1. Let F be a Scorza-Dragonian mapping. Then a function y(.) is an rg-solution of $\dot{x} \in F(t,x)$ iff for every $M \subset \langle 0,1 \rangle$, $\mu(M) = 0$ there are sequences $\{y_n\}_{n=1}^{\infty}$ and $\{h_n\}_{n=1}^{\infty}$ such that all conditions of Definition 1 are fulfilled and $\bigcup_{n=1} h_n \cap M = \phi$.

To prove the theorem we will use the following trivial lemma.

Lemma. Let us suppose $a \in F^*(t,x)$, $M \subset [0,1]$, $\mu(M) = 0$. Then there are sequences $\{(t_n,x_n)\}_{n=1}^{\infty}$ and $\{a_n\}_{n=1}^{\infty}$ such that $a_n \in F^*(t_n,x_n)$,

$t_n \notin M$, $\lim_{n\to\infty} (t_n, x_n, a_n) = (t,x,a)$.

Proof. From $a \in F^*(t,x)$ we obtain as a consequence of the identity $GF^* = G^*F$ and of Lemma 1 that $(t,x,a) \in GF^* = \mathrm{cl}\, G_{M_0 \cup M}F$, $\mu(M_0 \cup M) = 0$. Hence there exists a sequence $\{t_n, x_n, a_n\} \to (t,x,a)$ such that $t_n \notin M_0 \cup M$ and $a_n \in F(t_n, x_n)$. Since $F^*(\tau,\xi) = F(\tau,\xi)$ for $\tau \notin M_0$ the proof is complete.

Proof of the theorem: Since $\{t \in [0,1] \mid \exists_{x\in R^n} F^*(t,x) = F(t,x)\} \subset M_0$, $\mu(M_0) = 0$, the "only if" part of the theorem follows immediately. To prove the "if" part let $y(.)$ be an rg-solution and $M \subset [0,1]$, $\mu(M) = 0$. Then there is a sequence $\{y_n\} \to y$ and the sequence $\{h_n\}$ such that the conditions (i),...,(v) from Definition 1 are fulfilled with F^* instead of F. Condition (iii) written explicitly has the following form:

$$y_n(h_n^{k+1}) = y_n(h_n^k) + a_n^k(h_n^{k+1} - h_n^k) + \varepsilon_n^k, \quad a_n^k \in F^*(h_n^k, y_n(h_n^k)).$$

As a consequence of Lemma we obtain that y_n, h_n^k, a_n^k and ε_n^k can be replaced by $\bar{y}_n, \bar{h}_n^k, \bar{a}_n^k, \bar{\varepsilon}_n^k$ such that

$$(2) \qquad \bar{h}_n = \{0 = \bar{h}_n^0 < \bar{h}_n^1 < \ldots < \bar{h}_n^{\nu_n+1} = 1\} \cap M = \phi$$

for every $n = 1,2,3,\ldots, \bar{h}_n^k < h_n^{k+1}$, $(\bar{h}_n^k - h_n^k) < 1/(n \cdot \nu_n)$, $\sum_{k=1}^{\nu_n} \|\bar{\varepsilon}_n^k\| \to 0$ as $n \to \infty$ and

$$(3) \qquad y_n(\bar{h}_n^{k+1}) = y_n(\bar{h}_n^k) + \bar{a}_n^k(\bar{h}_n^{k+1} - \bar{h}_n^k) + \bar{\varepsilon}_n^k, \quad \bar{a}_n^k \in F^*(\bar{h}_n^k, y_n(\bar{h}_n^k))$$

for $n = 1,2,\ldots$ and $k = 0,1,2,\ldots,\nu_n$.

We can proceed for example as follows. For every $n = 1,2,\ldots$ we set $\bar{h}_n^0 = h_n^0 = 0$, $y_n(\bar{h}_n^0) = x_0$, $\bar{h}_n^{\nu_n+1} = 1$, $\bar{y}_n(1) = y_n(1)$, $\bar{a}_n^0 = a_n^0$. Let us denote $1/(n\nu_n)$ by ρ. As a consequence of Lemma we can choose $\bar{h}_n^k$, $\bar{a}_n^k$ and ψ_n^k, such that (2) is fulfilled and $|\bar{h}_n^k - h_n^k| < \rho$, $\psi_n^k \in B_\rho(y_n(h_n^k))$ $\bar{a}_n^k \in F^*(\bar{h}_n^k, \psi_n^k)$, $\bar{a}_n^k \in B_\rho(a_n^k)$ holds for $k = 1,2,\ldots,\nu_n$. We set $\bar{y}_n(\bar{h}_n^k) = \psi_n^k$ and choose such $\bar{\varepsilon}_n^k$ that (3) is fulfilled. Then

$$\bar{\varepsilon}_n^k = \bar{y}_n(\bar{h}_n^{k+1}) - \bar{y}_n(\bar{h}_n^k) - \bar{a}_n^k(\bar{h}_n^{k+1} - \bar{h}_n^k)$$

and

$$\|\bar{\varepsilon}_n^k\| \le \|\bar{y}_n(\bar{h}_n^{k+1}) - y_n(h_n^{k+1})\| + \|y_n(h_n^k) - \bar{y}_n(\bar{h}_n^k)\| + \|\bar{a}_n^k - a_n^k\| \cdot$$
$$\cdot \|\bar{h}_n^{k+1} - \bar{h}_n^k\| + \|a_n^k\|(|\bar{h}_n^{k+1} - h_n^{k+1}| + |\bar{h}_n^k - h_n^k|) +$$
$$+ \|y_n(h_n^{k+1}) - y_n(h_n^k) - a_n^k(h_n^{k+1} - h_n^k)\| \le 3\rho + 2\rho + \|\varepsilon_n^k\| .$$

Hence $\lim_{n\to\infty} \Sigma\|\bar{\varepsilon}_n^k\| = 0$. Similarly we obtain $\lim \bar{y}_n = y$ uniformly on $[0,1]$

and the proof is complete.
It means that using division to construct a solution we can avoid any set of measure zero.

6. Gauge approach

To define rg-solution we need F to be Scorza-Dragonian (due to the definition of F^*) but by means of avoiding the sets of measure zero we can define the rg-solution for quite a general system. In the sequel, using gauge approach, we introduce another procedure to define solutions. Let us remind that a gauge is an arbitrary real valued positive function and a division $\Delta = \{t_\gamma\}$ is subordinated to a gauge δ (or Δ is δ-fine, $\Delta < \delta$) iff $t_{i+1} - t_i < \delta(t_i)$. We shall say that a set Ω is a gauge set iff for every positive constant c there exists a $\delta \in \Omega$ such that $\sup \delta(t) < c$ and for every $\delta_1,\dots,\delta_n \in \Omega$ there exists a $\delta \in \Omega$ such that $\delta \le \min(\delta_1,\dots,\delta_n)$.

There is a well known theorem about δ-fine divisions saying that for every δ there is a δ-fine division which is finite, see Kurzweil [6]. In our case this theorem doesn't hold because we operate with so called left divisions. But a similar theorem holds with a countable divisions. Let us note that using general division instead of left one we don't succeed in rejecting parasite solutions.

Let Ω be a gauge set. We shall say that y is an Ω-solution of $\dot{x} \in F(t,x)$, $x(0) = x_0$ iff all items of Definition 1 are fullfiled with δ-fine division, $\delta \in \Omega$ i.e.

$$\forall_{\varepsilon>0} \ \forall_{\delta\in\Omega} \ \exists_{\Delta<\delta} \ \exists_{\varepsilon_\Delta} \ \exists_{\xi_\Delta} \ \exists_{x_\Delta} \ (|\varepsilon_\Delta| < \varepsilon, \ |y - x_\Delta| < \varepsilon) .$$

The following theorem can be proved.

Theorem 2. Let F be bounded and let Ω be a gauge set. Then there exists an Ω-solution.

Proof: Let $\rho > 0$ be such that $\|y\| \le \rho$ for all $y \in F(t,x)$, $(t,x) \in [0,1] \times R^n$ and let K be the set of all $x(.) \in C(\langle 0,1\rangle)$ such that

a) $|x(t)| \le \rho$ for every $t \in [0,1]$

and

b) $|x(t_1) - x(t_2)| \le \rho|t_1 - t_2|$ for every $t_1,t_2 \in [0,1]$.

The K with the norm max is the compact metric space. Let $\delta \in \Omega$. We shall construct a set $S_\delta \subset K$. Let S_δ^J be the set of all functions fulfilling all the conditions of Definition 1 and such that (see condition iv) $\sum_i \|\varepsilon_i\| \leq \sup \delta(t)$. It can be proved, by the method of transfinite sequences (see [13]), that S_δ^J is non-empty. Every function $x(.) \in S_\delta^J$ can be modified, by subtracting jumps ε_i in points t_i of division Δ, to obtain a function $y(.) \in K$. This procedure results in a set $S_\delta \subset K$. The set K is compact, hence $\bigcap_{\delta\in\Omega} \overline{S}_\delta \neq \phi$. It is easy to see that every function $x(.)$, $X \in \bigcap_{\delta\in\Omega} \overline{S}_\delta$ is an Ω-solution, which completes the proof.

Let us denote $\Omega_0 = \{\delta(.) | \delta \geq a(\delta) > 0\}$, $\Omega_r = \{\delta(.) | \delta(t) \geq a(\delta)$ a.e., $a(\delta) > 0\}$. Then it is possible to prove that Ω_0-solutions are exactly the Sentis g-solutions and Ω_r-solutions are precisely the rg-solutions. Using the results mentioned above we can say that for F u.s.c. the gauge set Ω_r is the good one to define a solution. But this is not true for F Scorza-Dragonian because Ω_r-solutions are the solutions of $\dot{x} \in F^*$, F^* being u.s.c., $F^* \supset F$ a.e. Hence we cannot expect Ω_r-solutions to be solutions of $\dot{x} \in \text{conv } F$. So a natural problem arises:
What is the smallest but sufficient gauge set for Scorza-Dragonian right-hand side?

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