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UNIFORM ZEROS FOR BEADED STRINGS

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1. Introduction. Early attempts to model the fundamental vibration of a musical string focussed on three physical properties which were believed to underlie this phenomenon: isochronism, the pendulum principle and a "simultaneous crossing of the axis" [3]. With the discovery that the small vibrations of such strings are described by hyperbolic partial differential equations, interest in these physical concepts declined. Isochronism became embodied in the "small amplitude assumption" which underlies the linearity of the resulting equation, while the pendulum principle (asserting that restoring forces are proportional to displacements from equilibrium) turned out to be incorrect for the wave equation. The notion of a simultaneous crossing of the axis has become identified with separation of variables and does not seem to have been pursued in its own right.

However given a linear hyperbolic PDE of the form

$$u_{tt} - u_{xx} + p(x,t)u = 0, \tag{1.1}$$

with $p(x,t)$ continuous and positive for $0 \leq x \leq L$ and $t \geq 0$, and given boundary conditions such as

$$u_x(0,t) = u_x(L,t) = 0, \tag{1.2}$$

the question of a simultaneous crossing of the axis is an important one. Specifically, the question arises whether it is possible to assign Cauchy data of the form

$$u(x,0) = 0; \; u_t(x,0) = g(x) \tag{1.3}$$

for $0 \leq x \leq L$ such that the solution of (1.1)-(1.3) will satisfy

$$u(x,T) \equiv 0 \quad \text{for} \quad 0 \leq x \leq L$$

for some $T > 0$. We shall refer to such solutions as having a uniform zero at $t = T$.

Except in the case of separation of variables there seems to be no simple answer to this question. As such it is natural to consider a semi-discrete approximation to (1.1)-(1.3) corresponding to a beaded string. In this context one obtains [7] a system of ordinary differential equations of the form

$$\frac{d^2\underline{u}}{dt^2} + G(t)\underline{u} = 0 \tag{1.4}$$

subject to initial conditions of the form

$$\underline{u}(0) = 0; \; \underline{u}'(0) = \underline{g}. \tag{1.5}$$

The problem of choosing $\underline{g}$ in (1.5) so as to satisfy $\underline{u}(T) = \underline{0}$ for some $T > 0$ is now the classical problem of establishing the existence of conjugate points of zero relative to (1.4). Such problems have been studied by M. Morse [9] and W. T. Reid [10] in the more general context of Hamiltonian systems. More recently Ahmad and Lazer [1] have also studied conjugate points under the assumption the entries of $G(t)$ satisfy appropriate positivity conditions.

In the case at hand, the matrix function $G(t)$ is a Jacobi matrix given by

$$\begin{aligned} g_{ii}(t) &= 2 + p_i(t) && \text{for} \quad 1 \leq i \leq n, \\ g_{i,i-1} &= g_{i-1,i} = -1 && \text{for} \quad 2 \leq i \leq n, \\ g_{ij} &= 0 && \text{for} \quad |i-j| \geq 2. \end{aligned} \tag{1.6}$$

While this matrix function has the symmetry required in [9] and [10], the variational criteria for conjugate points established therein are based on positive definiteness and provide no information regarding the sign of the solution which realizes a particular conjugate point. Also, the essential indefiniteness of $G(t)$ prevents the techniques of [1] from being brought to bear in establishing uniform zeros for solutions of (1.4). Accordingly, criteria for the existence of uniform zeros of (1.1) would seem to require the development of novel techniques for establishing the existence of conjugate points for (1.4).

2. The Oppositional Mode of Vibration. A special case of interest in connection with (1.4) and (1.6) is that where the initial data $\underline{g} = \mathrm{col}(g_1,\ldots,g_n)$ in (1.5) satisfies

$$(-1)^j g_j < 0; \quad 1 \leq j \leq n.$$

In this case the solution of (1.4) and (1.5) also satisfies $(-1)^j u_j(t) < 0$ for sufficiently small values of t and is said to be (initially) in an *oppositional mode of vibration*.

The special Jacobi form of (1.6) makes tractable the problem of establishing the existence of the conjugate point T whose corresponding solution $\underline{u}(t) = \mathrm{col}(u_1(t),\ldots,u_n(t)$ is in an oppositional mode for $0 < t < T$. Indeed, if we define $\underline{v}(t) = \mathrm{col}(v_1(t),\ldots,v_n(t))$ by

$$v_j(t) = (-1)^j u_j(t)$$

then $\underline{v}(t)$ is a solution of

$$\underline{v}'' + F(t)\underline{v} = 0$$

(2.2)

$$\underline{v}(0) = \underline{0}; \; v'(0) = \underline{f}$$

where $f_{ij} = |g_{ij}|$ and $f_i = |g_i|$ for $1 \le i, j \le n$. Because of the positivity properties of $F(t)$ and $\underline{f}$ one can apply the techniques of Ahmad and Lazer [1] to establish the existence of a conjugate point T for (2.2) which is realized by solution $\underline{v}(t)$ whose components are positive for $0 < t < T$.

A nonlinear version of this problem has been considered by Duffin [4] in connection with the "plucked string" (corresponding to a right focal point). Indeed given appropriate positivity conditions on $\underline{h}(t,\underline{v})$ one can also use the techniques of Krasnoselskii [6; Ch. 7.4] to establish the existence of positive solutions of boundary value problems of the form

$$\begin{aligned} &\underline{v}'' + \underline{h}(t,\underline{v}) = 0 \\ &\underline{v}(0) = \underline{v}(T) = 0, \end{aligned} \tag{2.3}$$

leading to more general equations which allow for solutions in this oppositional mode.

While of interest, these results are of little help in establishing uniform zeros for (1.1). For as we seek to approximate (1.1) by systems such as (1.4) and let $n \to \infty$, solutions in the oppositional mode do not converge to solutions of (1.1).

For this reason one is led to the more difficult problem of establishing the existence of conjugate points for (1.4) which are realized by positive solutions.

<u>3. Positive Solutions.</u> In case the matrix $G(t)$ given by (1.6) is a <u>constant</u> Jacobi matrix, the existence of a conjugate point realized by a positive solution can be established by algebraic means. In this case (1.4) can be written as

$$G^{-1}\underline{u}'' + \underline{u} = 0 \tag{3.1}$$

where G^{-1} is totally positive in the sense of Gantmacher and Krein [5]. As shown in [5], it now follows that G^{-1} has n simple positive eigenvalues $\lambda_1 > \lambda_2 > \ldots > \lambda_n > 0$, where λ_1 corresponds to an eigenfunction $\underline{\xi}_1$ which may be taken to be positive. Accordingly the choice $\underline{g} = \underline{\xi}_1$ in (1.5) leads to $\underline{u}_1 = \underline{\xi}_1 \sin t/\sqrt{\lambda_1}$ and $T = \pi\sqrt{\lambda_1}$. (It also follows from [5] that the solution $\underline{u}_n = \xi_n \sin t/\sqrt{\lambda_n}$ corresponds to the oppositional mode of vibration considered in § 2).

In order to deal with non-constant $G(t)$ in (1.4) it will be necessary to give a non-algebraic argument for the existence of the above solution $\underline{u}_1(t)$. To that end we consider the case where

$$G(t) = \Gamma_0 + \Pi(t), \tag{3.2}$$

Γ_0 being a constant matrix with entries

$$\gamma_{ii} = p > 0; \quad \gamma_{i,i-1} = \gamma_{i-1,i} = -1; \quad \gamma_{ij} = 0 \quad \text{otherwise}$$

and $\Pi(t) = \mathrm{diag}(\pi_1(t), \ldots, \pi_n(t))$ playing the role of a perturbation of Γ_0. By [5] Γ_0 has positive eigenvalues $\mu_1 < \mu_2 < \ldots < \mu_n$, for which we establish the following property.

3.1 Lemma. For sufficiently large values of p the eigenvalues of Γ_0 satisfy

$$1 < \sqrt{\frac{\mu_i}{\mu_1}} < \frac{3}{2}\,; \; 2 \le i \le n. \tag{3.3}$$

Proof. The eigenvalues of $\frac{1}{p}\Gamma_0$ satisfy $\frac{1}{p}\mu_1 < \ldots < \frac{1}{p}\mu_n$ and tend to 1 as $p \to \infty$. Therefore (3.3) follows for the eigenvalues of $p\,\Gamma_0$ and for the eigenvalues of Γ_0 as well.

In order to establish topological criteria for the existence of uniform zeros it will be useful to regard solutions of

$$\underline{u}'' + [\Gamma_0 + \Pi(t)]\underline{u} = 0; \; \underline{u}(0) = 0, \; u'(0) = \underline{g} \tag{3.4}$$

as trajectories in $\mathbb{R}_n$ which emanate from the origin with initial velocity $\underline{g}$. We seek to show the existence of $\underline{g} > 0$ such that the corresponding trajectory exits the positive n-tant $\mathbb{R}_n^+$ through the origin. To that end we denote the normalized eigenvectors of Γ_0 (corresponding to the eigenvalues μ_i) by $\underline{\phi}_i$, requiring that $\underline{\phi}_1$ lie in $\mathbb{R}_n^+$ and, more generally, that the sum of the components of each $\underline{\phi}_i$ be nonnegative. This sign convention has the consequence that when we express any $\underline{g} \geq 0$ in the form $\underline{g} = c_1\underline{\phi}_1 + \ldots + c_n\underline{\phi}_n$, then $c_i \geq 0$ for $1 \leq i \leq n$.

As in [4] we define a <u>contact point</u> of a trajectory $\underline{u}(t)$ as its first point of intersection with a coordinate plane. An <u>exit point</u> is a contact point at which the trajectory also crosses that coordinate plane. In the oppositional mode one can readily show [4] that such first contact points are also exit points, but this need not be the case for trajectories in $\mathbb{R}_n^+$. However, the following theorem shows that under the condition of Lemma 3.1 such an equivalence also exists for trajectories in $\mathbb{R}_n^+$.

<u>3.2 Theorem.</u> <u>If</u> $\underline{v}(t)$ <u>is a trajectory of</u>

$$\underline{v}'' + \Gamma_0\underline{v} = 0; \ \underline{v}(0) = 0, \ \underline{v}'(0) = \underline{g} > 0,$$

<u>and if condition</u> (3.3) <u>is satisfied, then the point at which</u> $\underline{v}(t)$ <u>first intersects a coordinate plane bounding</u> $\mathbb{R}_n^+$ <u>is also an exit point.</u>

<u>Proof.</u> Suppose the contact point occurs at $t = t_0$ and lies in the coordinate plane $(\underline{v},\underline{e}_j) = 0$, where $\underline{e}_j$ is a unit vector along the positive v_j-axis. Because of (3.3) and the fact that the $\underline{\phi}_1$ component has maximal amplitude among the characteristic directions, it follows that we must have

$$(3.5) \qquad \frac{\pi}{2} < \sqrt{\mu_1}\, t_0 < \pi \quad \text{and} \quad \pi < \sqrt{\mu_i}\, t_0 < \frac{3\pi}{2}$$

for $2 \leq i \leq n$. Writing $\underline{v}(t_0)$ in terms of the eigenvectors of Γ_0 leads to the equation

(3.6) $$(c_1\underline{\phi}_1 - c_2\underline{\phi}_2 - \ldots - c_n\underline{\phi}_n, \underline{e}_j) = 0$$

for appropriate choice of positive constants $c_1, \ldots, c_n$. If now $\underline{v}(t_0)$ were not an exit point we would also have $(\underline{v}'(t_0), \underline{e}_j) = 0$ and, because of (3.5),

$$(d_1\underline{\phi}_1 + d_2\underline{\phi}_2 + \ldots + d_n\underline{\phi}_n, \underline{e}_j) = 0$$

for positive constants $d_1, \ldots, d_n$. This contradicts (3.6) and completes the proof.

Remarks

1. Given specific eigenvalues satisfying (3.3) the above argument remains valid under small perturbations of the trajectories. Accordingly Theorem 3.2 remains valid for (3.4) when $\Pi(t)$ is sufficiently small.
2. The fact that contact points are also exit points assures that contact points will vary continuously with initial data. This observation is crucial to the proof of Theorem 3.4 below.
3. In the case of oppositional vibrations the fact that the initial velocity vector $\underline{g}$ has $g_j = 0$ assures that the resulting trajectory exits the oppositional quadrant at $t = 0$ across $v_j = 0$. Lemma 3.3 shows that for (3.4) a very different situation exists.

3.3 Lemma (Crossover property). If in Theorem 3.2 the vector $\underline{g}$ has $g_j = 0$, then the trajectory $\underline{v}(t)$ does not exit $\mathbb{R}_n^+$ across the coordinate plane $(\underline{v}, \underline{e}_j) = 0$.

Proof. The proof is similar to that of Theorem 3.2 by writing $g = c_1\phi_1 + c_2\vec{\psi}$ where ψ lies in $\phi_1^{\perp}$. The fact that all components of ψ will satisfy $\Pi < \sqrt{\mu_i}\, t_0 < \frac{3\pi}{2}$ at the time of contact precludes an exit across the plane $(\underline{v}, \underline{e}_j) = 0$.

The above properties of trajectories of $\underline{v}'' + \Gamma_0 \underline{v} = 0$ lead to the existence of a conjugate point as follows. Among all initial velocity vectors g satisfying $\underline{g} > 0$ all $\|g\| = 1$ we define

$$T_j = \{\underline{g} : \underline{v}(t) \text{ exits } \mathbf{R}_n^+ \text{ across } (\underline{v}, e_j) = 0\}.$$

A well known corollary to Sperner's lemma then leads to the fact that $\bigcap_{j=1}^{n} T_j \neq \phi$ and the following result.

__3.4 Theorem__. __Under the hypotheses of Theorem__ 3.2, __and for sufficiently small perturbations__ $\Pi(t)$, __the system__ (3.4) __has a conjugate point of zero which is realized by a trajectory in__ $\mathbf{R}_n^+$.

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