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ON GEL'FAND'S METHOD OF CHASING FOR SILVING MULTIPOINT BOUNDARY VALUE PROBLEMS

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Recently, for multipoint boundary value problems for ordinary differential equations several constructive methods have been suggested, e.g. the method of complementary functions and the method of adjoints [1,2], the integral equations method [3,4], initial adjusting method [12,16], the method of quasilinearization [5,8] etc. Here, we shall report the formulation of another practical shooting method, namely the method of chasing for nth order ordinary linear differential equation

$$x^{(n)} + \sum_{i=1}^{n} p_i(t)\, x^{(n-i)} = f(t) \tag{1}$$

subject to linearly independent multipoint boundary conditions

$$\sum_{k=0}^{n-1} c_{ik}\, x^{(k)}(a_i) = A_i, \qquad 1 < i < n \tag{2}$$

where $a_1 < a_2 < \ldots < a_n$ $(a_1 < a_n)$. This method is originally developed for second order differential equations by Gel'fand and Lokutsiyevskii and first appeared in english literature only recently [9]. Na [11] has briefly described the method and given different formulations for the different particular cases of (1), (2). The general systems derived here include the systems given by Na [11] as special cases. The power of the method is illustrated by solving known Holt's problem.

Since the boundary conditions (2) are assumed to be linearly independent, at the point a_i at least one of the c_{ik}, $0 < k < n-1$ is not zero. Let $c_{ij} \neq 0$ then, at this point a_i the boundary condition (2) can be rewritten as

$$x^{(j)}(a_i) = \sum_{\substack{k=0 \\ k\neq j}}^{n-1} d_{ik}\, x^{(k)}(a_i) + \alpha_i, \quad i < i < n \tag{3}$$

where $d_{ik} = -\dfrac{c_{ik}}{c_{ij}}$; $0 < k < n-1$, $k = j$ and $\alpha_i = \dfrac{A_i}{c_{ij}}$

In the differential equation (1), we begin with the assumption that $p_1(t) \equiv 0$, so that

$$x^{(n)} = - \sum_{i=2}^{n} p_i(t)x^{(n-i)} + f(t). \tag{4}$$

Now, for the boundary condition (3) we assume that the solution x(t) of (4) satisfies (n-1)th order linear differential equation

$$x^{(j)}(t) = \sum_{\substack{k=0 \\ k\neq j}}^{n-1} d_{ik}(t)\, x^{(k)}(t) + \alpha_i(t) \tag{5}$$

where the n functions $d_{ik}(t)$; $0 \leq k \leq n-1$, $k \neq j$ and $\alpha_i(t)$ are to be determined.

Differentiating (5) once, we get

$$x^{(j+1)}(t) = \sum_{\substack{k=0 \\ k\neq j}}^{n-1} [d_{ik}(t)x^{(k+1)}(t) + d'_{ik}(t)x^{(k)}(t)] + \alpha'_i(t). \tag{6}$$

Next, we shall use (5) to eliminate the term $x^{(n-1)}(t)$ from (6), however it depends on a particular value of j and we need to consider four different cases :

(i) $j = 0$, $n > 3$: From (5), we have

$$x^{(n-1)}(t) = \frac{1}{d_{i,n-1}(t)}[x(t) - \sum_{k=1}^{n-2} d_{ik}(t)x^{(k)}(t) - \alpha_i(t)]. \tag{7}$$

Using (7) in (6) and rearranging the terms, we get

$$\begin{aligned} x^{(n)}(t) = &- \frac{[d_{i,n-2}(t) + d'_{i,n-1}(t)]}{d^2_{i,n-1}(t)}\, x(t) \\ &+ [\frac{1}{d_{i,n-1}(t)} + \frac{d_{i,n-2}(t) + d'_{i,n-1}(t)}{d^2_{i,n-1}(t)}\, d_{i1}(t) - \frac{d'_{i1}(t)}{d_{i,n-1}(t)}]x'(t) \\ &+ \sum_{k=2}^{n-2} [\frac{d_{i,n-2}(t) + d'_{i,n-1}(t)}{d^2_{i,n-1}(t)}\, d_{ik}(t) - \frac{d_{i,k-1}(t) + d'_{ik}(t)}{d_{i,n-1}(t)}]x^{(k)}(t) \\ &+ [\frac{d_{i,n-2}(t) + d'_{i,n-1}(t)}{d^2_{i,n-1}(t)}\, \alpha_i(t) - \frac{\alpha'_i(t)}{d_{i,n-1}(t)}]. \end{aligned} \tag{8}$$

Comparing (4) and (8), we find the system of n differential equations

$$d'_{i,n-1}(t) = -d_{i,n-2}(t) + p_n(t)d^2_{i,n-1}(t)$$
$$d'_{ki}(t) = p_{n-k}(t)d_{i,n-1}(t) - d_{i,k-1}(t) + p_n(t)d_{i,n-1}(t)d_{ik}(t);\ k=n-2,n-3,\dots,2$$
$$d'_{i1}(t) = 1 + p_n(t)d_{i,n-1}(t)d_{i1}(t) + p_{n-1}(t)d_{i,n-1}(t) \tag{9}$$
$$\alpha'_i(t) = -f(t)\ d_{i,n-1}(t) + p_n(t)\ d_{i,n-1}(t)\alpha_i(t).$$

We also desire that this solution x(t) must satisfy the boundary condition (3). For this, we compare (3) and (5) at the point a_i and find

$$d_{ik}(a_i) = d_{ik}, \quad 1 \leqslant k \leqslant n-1 \tag{10}$$
$$\alpha_i(a_i) = \alpha_i.$$

In the rest we proceed as for the case j = 0 and obtain the following systems

(ii) $1 \leqslant j \leqslant n-3$

$$d'_{i,n-1}(t) = -d_{i,n-2}(t) - d_{i,j-1}(t)\ d_{i,n-1}(t) + p_{n-j}(t)d^2_{i,n-1}(t)$$
$$d'_{ik}(t) = -d_{i,k-1}(t) - d_{i,j-1}(t)\ d_{ik}(t) + (p_{n-k}(t) + p_{n-j}(t)d_{ik}(t))d_{i,n-1}(t)$$
$$k = n-2,\ n-3,\dots,\ 1;\ k \neq j,\ j+1$$
$$d'_{i,j+1}(t) = 1 - d_{i,j-1}(t)d_{i,j+1}(t) + (p_{n-j-1}(t) + p_{n-j}(t)d_{i,j+1}(t))d_{i,n-1}(t)$$
$$d'_{i0}(t) = -d_{i,j-1}(t)d_{io}(t) + (p_n(t) + p_{n-j}(t)d_{io}(t))d_{i,n-1}(t) \tag{11}$$
$$\alpha'_i(t) = -d_{i,j-1}(t)\alpha_i(t) + (p_{n-j}(t)\alpha_i(t) - f(t))d_{i,n-1}(t)$$

$$d_{ik}(a_i) = d_{ik};\ 0 \leqslant k \leqslant n-1,\ k = j \tag{12}$$
$$\alpha_i(a_i) = \alpha.$$

(iii) $j = n-2$

$$d'_{i,n-1}(t) = 1 - d_{i,n-1}(t)d_{i,n-3}(t) + p_2(t)d^2_{i,n-1}(t)$$

$$d'_{ik}(t) = -d_{i,k-1}(t) + (p_{n-k}(t) + p_2(t)d_{ik}(t))d_{i,n-1}(t) - d_{i,n-3}(t)d_{ik}(t),$$

$$1 \le k \le n-3$$

$$d'_{i0}(t) = -d_{i,n-3}(t)d_{i,0}(t) + (p_n(t) + p_2(t)d_{i,0}(t))d_{i,n-1}(t) \tag{13}$$

$$\alpha'_i(t) = -d_{i,n-3}(t)\alpha_i(t) + (-f(t) + p_2(t)\alpha_i(t))d_{i,n-1}(t)$$

$$d_{ik}(a_i) = d_{ik}\ ;\ 0 \le k \le n-1,\ k \ne n-2$$

$$\alpha_i(a_i) = \alpha_i. \tag{14}$$

(iv) $j = n-1$

$$d'_{ik}(t) = -d_{i,k-1}(t) - d_{i,n-2}(t)d_{ik}(t) - p_{n-k}(t), \quad 1 \le k \le n-2$$

$$d'_{i0}(t) = -d_{i,n-2}(t)d_{i0}(t) - p_0(t) \tag{15}$$

$$\alpha'_i(t) = -d_{i,n-2}(t)\alpha_i(t) + f(t)$$

$$d_{ik}(a_i) = d_{ik};\quad 0 \le k \le n-2 \tag{16}$$

$$\alpha_i(a_i) = \alpha_i.$$

For the particular value of j, we integrate the above appropriate system from the point a_i to a_n and collect the values of $d_{ik}(a_n)$; $0 \le k \le n-1$, $k \ne j$ and $\alpha_i(a_n)$. Thus, (5) provides a new boundary relation at the point a_n

$$x^{(j)}(a_n) = \sum_{\substack{k=0\\k\ne j}}^{n-1} d_{ik}(a_n)x^{(k)}(a_n) + \alpha_i(a_n). \tag{17}$$

Let N be the number of different boundary points i.e. $a_1 < a_2 < \ldots < a_N = a_n$ ($n \ge N \ge 2$) and $m(a_j)$ represents the number of boundary relations (3) prescribed at the point a_j and hence $\sum_{j=1}^{N} m(a_j) = n$. Thus, in (3) we have $m(a_n)$ boundary relations at the point a_n and to find $x^{(j)}(a_n)$, $0 \le j \le n-1$ we need $n-m(a_n)$ more new relations (17) i.e. we need to integrate $n-m(a_n)$ appropriate differential systems.

Finally, from the obtained values of $x^{(j)}(a_n)$, $0 \le j \le n-1$ we integrate backward differential equation (4) and obtain the required solution.

With the help of the following guidelines unnecessary computation can be avoided : (a) $m(a_n) = \max_{1<j<N} m(a_j)$, otherwise the role of the point a_n with the point a_j where $m(a_j)$ is maximum can be interchanged. (b) We need to integrate $n-m(a_n)$ times but not necessarily different differential systems, specially because differential system does not change as long as in (3) j is same. In fact, we can have at most n different differential systems.

For the case $p_1(t) \not\equiv 0$, we rewrite the differential equation (1) as

$$[P(t)x^{(n-1)}]' = -\sum_{i=2}^{n} P(t)p_i(t)x^{(n-i)} + P(t)f(t) \tag{18}$$

where $P(t) = \exp(\int_{a_1}^{t} p_1(s)ds)$.

Assumption that the solution of (18) should satisfy (n-1)th order linear differential equation

$$d_{ij}(t)x^{(j)}(t) = \sum_{\substack{k=0\\k\neq j}}^{n-1} d_{ik}(t)x^{(k)}(t) + \alpha_i(t) \tag{19}$$

with $d_{i,n-1}(t) = P(t)$ brings the problem in the realm of the foregoing analysis.

Example. The two point boundary value problem

$$x'' = (2m + 1 + t^2)x \tag{20}$$

$$x(0) = \beta,\ x(\infty) = 0 \tag{21}$$

where m and β are specified constants, known as Holt's problem [10] is a typical example where usual shooting methods fail [10,13,14,15]. Faced with this difficulty Holt [10] used a finite difference method, whereas Osborne [13] used multiple shooting method and Roberts and Shipman [14,15] used a multipoint approach.

For this problem the solution representation (5) reduces to

$$x(t) = d_{01}(t)x'(t) + \alpha_0(t) \tag{22}$$

and the case (iii) provides the differential system to be integrated

$$\begin{aligned} d_0'(t) &= 1 - (2m + 1 + t^2)d_{01}^2(t) \\ \alpha_0'(t) &= -(2m + 1 + t^2)d_{01}(t)\alpha_0(t) \end{aligned} \tag{23}$$

together with the initial conditions

$$d_{01}(0) = 0, \ \alpha_0(0) = \beta. \tag{24}$$

We use fourth order Runge-Kutta method with step size 0.01 and obtain $d_{01}(t)$, $\alpha_0(t)$ at $t = 18.01$. These values are used to calculate $x'(18.01)$ from (22). The differential equation (20) is integrated backward with the given $x(18.01) = 0$ and the obtained value of $x'(18.01)$ using fourth order Runge-Kutta method with the same step size. The value $t = 18.01$ has been chosen in view of restricted Computer capabilities.

The solution thus obtained has been presented in Tables 1-3 for different choices of m and β. These tables also contain solutions of the problem obtained earlier in [10,13,14,15]. For further details of the method and its applications see [6,7].

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Table 1.m = 0,β = 1

t	Present Solution	Complementary Functions [8]	Finite Difference	Solution by Osborne [6]	Roberts and Shipman [7]
0	0.9999876 E 00	0.10000000 E 01	0.100000 E 01	0.1000 E 01	0.10000000 E 01
1	0.2593404 E 00	0.15729920 E 00	0.157300 E 00	0.2593 E 00	0.15729921 E 00
2	0.3456397 E-01	0.46777349 E-02	0.467778 E-02	0.3455 E-01	0.46777350 E-02
3	0.1988532 E-02	0.22090497 E-04	0.220908 E-04	0.1987 E-02	0.22090497 E-04
4	0.4595871 E-04	0.15417257 E-07	0.154175 E-07	0.4590 E-04	0.15417259 E-07
5	0.4125652 E-06	0.15366706 E-11	0.153749 E-11	0.4188 E-06	0.15374602 E-11
6	0.1413020 E-08	-0.73163560 E-15	0.215201 E-16	0.1409 E-08	0.21519753 E-16
7	0.1827268 E-11	-0.75311525 E-15	0.418390 E-22	0.1821 E-11	0.41838334 E-22
8	0.8863389 E-15	-0.75315520 E-15	0.112244 E-28	0.8825 E-15	0.11224343 E-28
9	0.1605597 E-18		0.413703 E-36	0.1597 E-18	0.41370659 E-36
10	0.1082885 E-22		0.208844 E-44	0.1058 E-22	0.20895932 E-44
11	0.2713141 E-27		0.144078 E-53		0.12279100 E-49
12	0.2521085 E-32		0.135609 E-63		0.13487374 E-49
13	0.8677126 E-38				0.17299316 E-60
14	0.1105113 E-43				-0.25496486 E-65
15	0.5203999 E-50				
16	0.9055032 E-50				
17	0.5818867 E-64				
18	0.4179442 E-72				

Table 2. m = 1, = $\pi^{-1/2}$

t	Present Solution	Complementary Function [8]	Finite Differences [5]
0	0.5641878 E 00	0.56418960 E 00	0.5642 E 00
1	0.8285570 E-01	0.50254543 E-01	0.5026 E-01
2	0.7226698 E-02	0.97802274 E-03	0.9782 E-03
3	0.3020138 E-03	0.33550350 E-05	0.3356 E-05
4	0.5431819 E-05	0.18221222 E-08	0.1823 E-08
5	0.3975088 E-07	0.12367523 E-12	0.1482 E-12
6	0.1146879 E-09	-0.29349128 E-13	0.1747 E-17
7	0.1279827 E-12	-0.34242684 E-13	0.2931 E-23
8	0.5456289 E-16	-0.39134491 E-13	0.6912 E-30
9	0.8813160 E-20		
10	0.5361614 E-24		
11	0.1223266 E-28		
12	0.1043287 E-33		
13	0.3317918 E-39		
14	0.3926980 E-45		
15	0.1727057 E-51		
16	0.2818780 E-58		
17	0.1705581 E-65		
18	0.1160366 E-73		

Table 3. m = 2, β = 1/4

	Present Solution	Complementary Functions [8]	Finite Differences [5]
0	0.2500006 E 00	0.25000000 E 00	0.2500 E 00
1	0.2340787 E-01	0.14197530 E-01	0.1420 E-01
2	0.1414359 E-02	0.19141103 E-03	0.1914 E-03
3	0.4411547 E-04	0.49007176 E-06	0.4901 E-06
4	0.6261059 E-06	0.20999802 E-08	0.2101 E-08
5	0.3764660 E-08	-0.36865462 E-13	0.1403 E-13
6	0.9193294 E-11	-0.72849101 E-13	0.1400 E-18
7	0.8879995 E-14	-0.98795539 E-13	0.2034 E-24
8	0.3334327 E-17	-0.12873356 E-12	0.4224 E-31
9	0.4809239 E-21		
10	0.2641945 E-25		
11	0.5493305 E-30		
12	0.4302831 E-35		
13	0.1265030 E-40		
14	0.1391954 E-46		
15	0.5719099 E-53		
16	0.8757835 E-60		
17	0.4990734 E-67		
18	0.3216635 E-75		