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ON THE ASYMPTOTIC BEHAVIOR OF SOLUTIONS OF NONLINEAR ORDINARY DIFFERENTIAL EQUATIONS

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Section A

We are interested in the asymptotic behavior of solutions of the nonlinear differential equation

$$(1) \qquad y^{(n)} + f(t, y) = 0, \quad t > a,$$

subject to the hypotheses:

(A_1) $f : [a,\infty) \times \mathbb{R} \to (0,\infty)$ is continuous;

(A_2) $f(t, y)$ is nondecreasing in y for each fixed $t \in [a,\infty)$;

(A_3) $\lim_{y\to-\infty} f(t, y) = 0$ for each fixed $t \in [a,\infty)$.

A prototype of (1) satisfying (A_1) - (A_3) is

$$(2) \qquad y^{(n)} + \varphi(t)\, e^y = 0, \quad t > a,$$

where $\varphi : [a,\infty) \to (0,\infty)$ is continuous.

We note that all solutions of (1) can be indefinitely continued to the right, that is, for any $(\alpha_0,\alpha_1,\ldots,\alpha_{n-1}) \in \mathbb{R}^n$, the solution $y(t)$ of (1) satisfying $y^{(i)}(a) = \alpha_i$, $0 \leq i \leq n-1$, exists throughout $[a,\infty)$. Denoting by S the set of all solutions of (1) existing on $[a,\infty)$, we introduce the following notation:

(I) $S_+^{n-1} = \{y \in S : y^{(n-1)}(\infty) > 0\}$, $S_-^{n-1} = \{y \in S : y^{(n-1)}(\infty) < 0\}$,
$S_0^{n-1} = \{y \in S : y^{(n-1)}(\infty) = 0\}$;

(II) for $k = 1,2,\ldots,n-2$,
$S_+^k = \{y \in S : y^{(n-1)}(\infty) = \cdots = y^{(k+1)}(\infty) = 0,\ y^{(k)}(\infty) > 0\}$,
$S_-^k = \{y \in S : y^{(n-1)}(\infty) = \cdots = y^{(k+1)}(\infty) = 0,\ y^{(k)}(\infty) < 0\}$,
$S_0^k = \{y \in S : y^{(n-1)}(\infty) = \cdots = y^{(k+1)}(\infty) = y^{(k)}(\infty) = 0\}$;

(III) for $k = 1,2,\ldots,n-2$,
$S_{+b}^k = \{y \in S_+^k : y^{(k)}(\infty) < \infty\}$, $S_{+u}^k = \{y \in S_+^k : y^{(k)}(\infty) = \infty\}$,
$S_{-b}^k = \{y \in S_-^k : y^{(k)}(\infty) > -\infty\}$, $S_{-u}^k = \{y \in S_-^k : y^{(k)}(\infty) = -\infty\}$;

(IV) $S_+^0 = S_{+u}^0 = \{y \in S_0^1 : y(\infty) = \infty\}$, $S_-^0 = S_{-u}^0 = \{y \in S_0^1 : y(\infty) = -\infty\}$,
$S_b^0 = \{y \in S_0^1 : -\infty < y(\infty) < \infty\}$.

We then have a classification of S:

$$S = (S_+^{n-1} \cup S_+^{n-2} \cup \dots \cup S_+^1 \cup S_+^0) \cup S_b^0 \cup \cup (S_-^{n-1} \cup S_-^{n-2} \cup \dots \cup S_-^1) \quad \text{for } n \text{ even};$$

(3)

$$S = (S_+^{n-1} \cup S_+^{n-2} \cup \dots \cup S_+^1) \cup S_b^0 \cup \cup (S_-^{n-1} \cup S_-^{n-2} \cup \dots \cup S_-^1 \cup S_-^0) \quad \text{for } n \text{ odd}.$$

Below criteria are given for the existence (or nonexistence) of members of the subclasses of S appearing in (3).

THEOREM 1. $S_-^{n-1} \neq \phi$.

THEOREM 2. $S_{-u}^{n-1} \neq \phi$ if and only if

$$\int_a^\infty f(t, -ct^{n-1})dt = \infty \quad \text{for all } c > 0.$$

THEOREM 3. Let $1 \leq k \leq n-1$. Then, $S_{-b}^k \neq \phi$ if and only if

$$\int_a^\infty t^{n-k-1} f(t, -ct^k)dt < \infty \quad \text{for some } c > 0.$$

THEOREM 4. Let $1 \leq k \leq n-1$. Then, $S_{+b}^k \neq \phi$ if and only if

$$\int_a^\infty t^{n-k-1} f(t, ct^k)dt < \infty \quad \text{for some } c > 0.$$

THEOREM 5. Let $1 \leq k \leq n-2$. If $S_{-u}^k \neq \phi$, then $n \not\equiv k \pmod 2$ and

$$\int_a^\infty t^{n-k-1} f(t, -ct^k)dt = \infty \quad \text{for all } c > 0;$$

$$\int_a^\infty t^{n-k-2} f(t, -ct^{k+1})dt < \infty \quad \text{for all } c > 0.$$

THEOREM 6. Let $1 \leq k \leq n-2$. If $S_{+u}^k \neq \phi$, then $n \equiv k \pmod 2$ and

$$\int_a^\infty t^{n-k-2} f(t, ct^k)dt < \infty \quad \text{for all } c > 0;$$

$$\int_a^\infty t^{n-k-1} f(t, ct^{k+1})dt = \infty \quad \text{for all } c > 0.$$

Similar results hold for the subclasses S_+^0, S_-^0 and S_b^0.

Equation (1) is said to be superlinear [resp. sublinear] for $y > 0$ if $f(t, y)/y$ is nondecreasing [resp. nonincreasing] in $y > 0$ for each fixed $t \in [a,\infty)$.

THEOREM 7. Let (1) be superlinear for $y > 0$.

(i) $S_{+b}^1 = \dots = S_{+b}^{n-1} = \phi$ if

$$\int_a^\infty t^{n-2} f(t, ct)dt = \infty \quad \text{for all } c > 0.$$

(ii) $S_b^0 \neq \phi$, $S_{+b}^1 \neq \phi, \ldots, S_{+b}^{n-1} \neq \phi$ if

$$\int_a^\infty f(t, ct^{n-1})dt < \infty \quad \text{for some} \quad c > 0 .$$

(iii) $S_{+u}^k = \phi$ for $0 \leq k \leq n-2$ with $n \equiv k \pmod 2$ if

$$\int_a^\infty t^{n-2} f(t, c)dt = \infty \quad \text{for some } c > 0 \text{ in case } n \text{ is even ,}$$

$$\int_a^\infty t^{n-3} f(t, ct)dt = \infty \quad \text{for some } c > 0 \text{ in case } n \text{ is odd ,}$$

or if

$$\int_a^\infty t f(t, ct^{n-1})dt < \infty \quad \text{for some} \quad c > 0 .$$

THEOREM 8. Let (1) be sublinear for $y > 0$.

(i) $S_{+b}^1 = \cdots = S_{+b}^{n-1} = \phi$ if

$$\int_a^\infty f(t, ct^{n-1})dt = \infty \quad \text{for all} \quad c > 0 .$$

(ii) $S_b^0 \neq \phi$, $S_{+b}^1 \neq \phi, \ldots, S_{+b}^{n-1} \neq \phi$ if

$$\int_a^\infty t^{n-1} f(t, c)dt < \infty \quad \text{for some} \quad c > 0 .$$

(iii) $S_{+u}^k = \phi$ for $0 \leq k \leq n-2$ with $n \equiv k \pmod 2$ if

$$\int_a^\infty f(t, ct^{n-2})dt = \infty \quad \text{for some} \quad c > 0 ,$$

or if

$$\int_a^\infty t^{n-1} f(t, ct)dt < \infty \quad \text{for some } c > 0 \text{ in case } n \text{ is even ,}$$

$$\int_a^\infty t^{n-2} f(t, ct^2)dt < \infty \quad \text{for some } c > 0 \text{ in case } n \text{ is odd .}$$

Stronger results can be obtained for equations of the form

$$y^{(n)} + \varphi(t)g(y) = 0 , \qquad t > a , \tag{4}$$

where $\varphi : [a,\infty) \to (0,\infty)$ and $g : \mathbb{R} \to (0,\infty)$ are continuous, $g(y)$ is nondecreasing and $\lim_{y\to-\infty} g(y) = 0$.

THEOREM 9. Suppose in addition that

$$\int_\delta^\infty \frac{dy}{g(y)} < \infty \qquad \text{for some} \quad \delta \in \mathbb{R} .$$

Then, all solutions $y(t)$ of (4) have the property $\lim_{t\to\infty} y(t) = -\infty$ if and only if

$$\int_a^\infty t^{n-1}\varphi(t)dt = \infty .$$

THEOREM 10. Suppose in addition that $g(y)/y$ is nonincreasing for $y > 0$, $h(z) = \inf_{x>0} g(xz)/g(x) > 0$ and

$$\int_0^\delta \frac{dz}{h(z)} < \infty \qquad \text{for some} \quad \delta > 0 .$$

Then, all solutions $y(t)$ of (4) have the property $\lim_{t\to\infty} y(t) = -\infty$ if and only if

$$\int_a^\infty \varphi(t)g(ct^{n-1})dt = \infty \qquad \text{for all} \quad c > 0 .$$

EXAMPLE. Consider the elliptic partial differential equation

$$\Delta^m u + \psi(|x|)e^u = 0 , \quad x \in \Omega_a , \quad m \geqq 2 , \tag{5}$$

where $\Omega_a = \{x \in \mathbb{R}^3 : |x| > a\}$, $a > 0$. A radial function $u(x) = y(|x|)$ is a solution of (5) if and only if

$$(ty)^{(2m)} + t\,\psi(t)\,e^y = 0 , \quad t > a ,$$

which is equivalent to

$$z^{(2m)} + t\,\psi(t)\,e^{z/t} = 0 , \quad t > a . \tag{6}$$

Applying any of the above theorems to (6), we have a corresponding result on the existence and asymptotic behavior of radial solutions of (5) in exterior domains. For example, we see that every radial solution $u(x)$ of the equation

$$\Delta^m u + e^u = 0 , \quad x \in \Omega_a , \tag{7}$$

has the property $\lim_{|x|\to\infty} u(x) = -\infty$, and for each k, $1 \leqq k \leqq 2m-2$, (7) has a solution $u_k(x)$ such that $\lim_{|x|\to\infty} u_k(x)/|x|^k = \text{const} < 0$; we also see that all radial solutions $u(x)$ of the equation

$$\Delta^m u + \lambda \exp(\mu|x|^\nu)e^u = 0 , \quad x \in \Omega_a ,$$

with $\lambda > 0$, $\mu > 0$ and $\nu > 2m-2$, are such that $\lim_{|x|\to\infty} u(x)/|x|^{2m-2} = -\infty$.

REMARKS. For the proofs of the above-mentioned theorems the reader is referred to the paper [1]. Generalizations of the above theory to perturbed general disconjugate equations of the form $L_n y + f(t,y) = 0$ will be published elsewhere. Closely related results are found in the papers [2 , 3].

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