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BIFURCATIONS NEAR A DOUBLE EIGENVALUE OF THE RECTANGULAR PLATE PROBLEM WITH A DOMAIN PARAMETER

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Let us consider the bifurcation problem of the Föppl-Kármán equations of a thin elastic rectangular plate having the length a and width b when the aspect ratio $\alpha = a/b$ varies near a value $\alpha = \alpha_c$ yielding a double buckling load of the plate. In a suitable non-dimensional formulation, the governing equations will refer to a common domain. On the other hand, there appears a small perturbation parameter say Θ in the equations which we introduce as (Matkowsky et al. 1980 [1])

$$\Theta = \frac{1}{\alpha_c^2} - \frac{1}{\alpha^2} . \tag{1}$$

Starting from a sample boundary conditions we define a variational solution to the boundary value problem using energy spaces H, V given as certain subspaces of the Sobolev space $W_2^2(\Omega)$. An introduction of suitable equivalent norms in H,V leads to the operator equations

$$w - \Theta M_1 w + \Theta^2 M_2 w - \lambda\left(\frac{1}{\alpha_c^2} - \Theta\right) Lw - \left(\frac{1}{\alpha_c^2} - \Theta\right) C(w,\emptyset) = 0 \tag{2a}$$

$$- \emptyset + \Theta A_1 \emptyset - \Theta^2 A_2 \emptyset - \frac{1}{2}\left(\frac{1}{\alpha_c^2} - \Theta\right) B(w,w) = 0 , \tag{2b}$$

where λ is the load parameter and $w \in H$, $\emptyset \in V$ refer to the plate deflection and Airy stress function, respectively. The operators L,B,C are essentially those introduced by Berger 1967 [2] for a plate with a definite domain. In addition, we have obtained linear bounded and self-adjoint operators M_1, M_2 and A_1, A_2 acting from H into itself and from V into itself, respectively. Eq.(2b) can be uniquely solved for $\emptyset = \emptyset^*(\Theta,w)$. If Θ is sufficiently small, $\emptyset^*$ may be easily found in the form of power series in Θ. Substituting this solution into (2a) for $\emptyset$ and introducing the small load parameter χ,

$$\lambda = \frac{\lambda - \lambda_c}{\lambda_c} , \tag{3}$$

where λ_c corresponds to the double buckling load of the plate, we arrive at the resulting equation

$$F(w,\chi,\Theta) = w - \Theta M_1 - \lambda_c(1+\chi)(\frac{1}{\frac{2}{c}} - \Theta)\, Lw + \frac{1}{2\alpha_c^4} C(w,B(w,w)) + \text{h.o.t.} = 0 \qquad (4)$$

to be solved for $\{w,\chi,\Theta\}$ near the origin of the space $H\times R\times R$. Obviously, $w=w^*(\chi,\Theta) \equiv 0$ is always a solution to Eq. (4).

Let H_c be the eigensubspace of the double eigenvalue λ_c and P_c the operator of the orthogonal projection of H onto H_c. We assume that the following hypotheses hold:

(H1) For $u\in H_c$, $B\ u,u = 0$ only if $u = 0$.

(H2) The operator $P_c M_1$ restricted to $u\in H_c$ has only simple eigenvalues.

(H3) $F(w,\chi,\Theta)$ commutes with respect to the group

$_S = \{I,S,-I,-S\}$

of operators on H, where S possesses the action of one of the operators S_x, S_y, S_{xy}

S_x: $H\to H$, $u(x,y) \to (S_x u)(x,y) = u(1-x,y)$

S_y: $H\to H$, $u(x,y) \to (S_y u)(x,y) = u(x,1-y)$

$S_{xy} = S_x S_y = S_y S_x$

and H_c is spanned by $\varphi_1\in H^+$, $\varphi_2\in H^-$; $\|\varphi_i\| = 1$, $i = 1,2$ where

$H^+ \equiv \{u\in H:\ Su=u\}$, $\quad H^- \equiv \{u\in H:\ Su=-u\}$.

The hypothesis (H1) is commonly used. (H2) implies a transversal splitting of the double eigenvalue λ_c into simple ones appearing as a result of perturbation. Actually we can show that near $\lambda = \lambda_c$ the eigenvalues λ_i^Θ of the linearized equation are of the form $\lambda_i^\Theta = \lambda_c + \lambda_i'\Theta +$ + h.o.t., $i=1,2$ with $\lambda_1' \neq \lambda_2'$. By the hypothesis (H3) restrictions upon the boundary and load conditions are imposed allowing for the occurence of the assumed symmetry of Eq. (4).

The study of Eq. (4) is constructive - via the Liapunov-Schmidt reduction and the implicit function theorem. Assuming

$$w = \zeta_1\varphi_1 + \zeta_2\varphi_2 + \omega \qquad (5)$$

with $\zeta\in R^2$, $\omega\in H_c$ and following Vanderbauwhede 1982 [3] we obtain that the bifurcation equations admit the form

$$G_i(\zeta,\chi,\Theta) = \zeta_i H_i(\zeta,\chi,\Theta) = 0, \qquad i=1,2 \qquad (6)$$

with

$$H_i(\pm\zeta_1, \pm\zeta_2, \lambda, \Theta) = H_i(\zeta_1, \zeta_2, \lambda, \Theta), \qquad i=1,2. \tag{7}$$

Thus, a $Z_2 \oplus Z_2$ symmetry of Eqs.(6) is present. Moreover, we may distinguish one-mode and coupled-mode solutions to Eq.(4). The one-mode solutions $w \in H^+$ or $w \in H^-$ correspond to solutions of

$$\zeta_2 = 0, \qquad H_1(\zeta_1, 0, \lambda, \Theta) = 0 \tag{8}$$

and

$$\zeta_1 = 0, \qquad H_2(0, \zeta_2, \lambda, \Theta) = 0, \tag{9}$$

respectively, while coupled-mode solutions correspond to solutions of

$$H_1(\zeta, \lambda, \Theta) = 0, \qquad H_2(\zeta, \lambda, \Theta) = 0. \tag{10}$$

Departing from Chow-Hale 1982 [4] , we estimate the small solutions to Eqs. 6 by an a priori bound based on (H1) and then scale Eqs.(6) ((8-10))by

$$\zeta = \beta\mu, \qquad \lambda = \mu^2 \sin\nu, \qquad \Theta = \mu^2 \cos\nu, \tag{11}$$

where $\beta = \beta(\nu,\mu)$, the angle determines a direction in the (λ, Θ) parameter plane and μ is a new small parameter. The scaled equations are

$$g_i(\beta, \nu, \mu) = \beta_i h_i(\beta, \nu, \mu) = 0, \qquad i=1,2 \tag{12}$$

where

$$h_1(\beta, \nu, \mu) = -\sin\nu + \frac{\lambda_1'}{\lambda_c}\cos\nu + a\beta_1^2 + b\beta_2^2 + \text{h.o.t.},$$

$$h_2(\beta, \nu, \mu) = -\sin\nu + \frac{\lambda_2'}{\lambda_c}\cos\nu + b\beta_1^2 + c\beta_2^2 + \text{h.o.t.}.$$

Due to (H1) it is $a > 0$, $c > 0$ and we assume that $b > 0$, too.

Definition 1. (Golubitsky-Schaeffer 1979 [5]) The bifurcation problem (6) is non-degenerate if $b/a \neq 1$, $b/c \neq 1$ and $b^2 \neq ac$.

According to Golubitsky-Schaeffer [5] , the ratios b/a, b/c represent the modal parameters of bifurcation problem (6). The lines of deneracy divide the studied positive quadrant of b/a, b/c plane into six regions within each of which the local features of bifurcating solutions to Eqs.(6) are topologically equivalent. The nondegenerate cases of Eqs.(6) were analysed in [5] by means of the singularity theory. Our study comprises the degenerate cases while the employed tools are simpler.

Letting $\mu \to 0$, the reduced equations $g(\beta, \nu, 0) = 0$ the scaled equivalents of (8-10))can be easily solved for $\beta = \beta^o$. If $b^2 \neq ac$, the non-trivial solutions appear at ν values forming an open subinterval within the considered interval $(-\pi/2, 3\pi/2\rangle$ of ν values, with end points differing from $-\pi/2$, $3\pi/2$. If $b^2 = ac$, the reduced system is solvable only at a certain $\nu \in (-\pi/2, 3\pi/2)$, say $\nu = \nu^s$, having then a continuum of solutions given by the equation

$$a\,(\beta_1^o)^2 + b(\beta_2^o)^2 = \sin\nu^s - \frac{\lambda_1'}{\lambda_c}\cos\nu^s . \qquad (13)$$

Successive continuations of the solutions to $\mu \neq 0$ by the implicit function theorem succed only within the open set of regular points in the parameter plane. Thus, the description of the solution set needs to be completed in a beighbourhood of the singular points (potential bifurcation curves). The first step in the analysis which appears to be crucial is the choice of (ζ_1,Θ) or (ζ_2,Θ) or $(\zeta_1,\ \zeta_2)$ as a new parameter plane if (8),(9) or (10), respectively, is solved. In this way we obtain that the small solutions to Eq. (4) form a connected set consisting of the trivial, one-mode and coupled-mode solution subsets.

The description of the solution set to Eq.(4) yields immediately primary and secondary bifurcation curves at cross-sections of trivial and one-mode or one-mode and coupled-mode solution subsets, respectively. Now the question on the possible additional bifurcation curves and the conditions of their appearing or non-existence is to be answered.

Theorem 2. Suppose (H1-3) hold and the coefficients a, b, c of bifurcation equations (6) of Eq.(4) satisfy:

(i) $b^2 \neq ac,\ b>0$

(ii) $b^2 = ac,\ b>0$

In the case (ii) let further be

$$\lim_{\beta_1^o\to 0} T(\beta^o)_{,\,11} \neq 0 \ \cap\ \lim_{\beta_2^o\to 0} T(\beta^o)_{,\,22} \neq 0,$$

where $T(\beta^o)$ is defined by

$$T(\beta^c) = bh_{1,\mu\mu}(\beta^o,\nu^s,0) - ah_{2,\mu\mu}(\beta^o,\nu^s,0)$$

over the ellipse (13)(the circles in the subscript positions denote a total differentiation of $T(\beta^o)$ with respect to β_1 or β_2) and either

(a) it holds

$$T(\beta^o)_{,\,1} = -\frac{a}{b}\,\frac{\beta_1^o}{\beta_2^o}\,T(\beta^o)_{,\,2} = 0, \quad \forall\ \beta^o:\ \beta_1^o>0,\ \beta_2^o>0$$

or

(b) for certain $\beta^o = \beta^{o*}$: $\beta_1^{o*}>0 \cap \beta_2^{o*}>0$ it is

$$T(\beta^{o*})_{,\,1} = T(\beta^{o*})_{,\,2} = 0,$$

$$T(\beta^{o*})_{,\,11} = \left(\frac{a}{b}\,\frac{\beta_1^*}{\beta_2^{o*}}\right)^2 T(\beta^{o*})_{,\,22} \neq 0$$

and

$$\lim_{\beta_1^o \to 0} T(\beta^o) \neq \lim_{\beta_2^o \to 0} T(\beta^o).$$

Then near the origin of the (χ,Θ) parameter plane, the bifurcation diagram of Eq.(4) consists of four distinct bifurcation curves: two primary and two secondary bifurcation curves, and in addition in the case (ii)(b) of a unique curve of limit points. A crossing of the primary and secondary bifurcation curves changes the number of solutions to Eq.(4) by two and four, respectively. A crossing of the limit-point curve changes the number of solutions by eight.

Proof. In order to study the set of bifurcation curves, we solve the system consisting of the scaled bifurcation equations (12) together with the condition of vanishing of the corresponding Jacobian $J_g = J_g(\beta,\nu,\mu)$. At solutions to the reduced system ($\mu = 0$ the value of the corresponding Jacobian is always zero but one of Eqs.(12), say $g_1 = 0$, and the equation $J_g = 0$ may be uniquely solved for $\beta_1 = \tilde{\beta}_1(\beta_2,\mu)$, $\nu = \tilde{\nu}(\beta_2,\mu)$near such solution. Substituting $\tilde{\beta}_1, \tilde{\nu}$ for β_1, ν in the remaining equation $g_2 = 0$ we get an equation the small solutions $\beta_2 = \beta_2^*(\mu)$ of which can be studied by Newton´s polygon method. The primary and secondatry bifurcation curves correspond to the tripple roots while the limit-point curve to the simple root of the remaining equation ($g_1 = 0$ or $g_2 = 0$). We note that $T(\beta^o)$is a polynomial of second degree in $(\beta_1^o)^2$ and $(\beta_2^o)^2$.

<u>Corollary 3.</u> There exist two pairs of one-mode (one from H^+ and the other from H^-) and in the cases (i),(ii)(a) no or two pairs while in the case ii b no, two or four pairs of coupled-mode solutions to Eq.(4), $\Theta=0$ near $w=0$ and any $\lambda > \lambda_c$ sufficiently close to λ_c.

Equation $F(w,\chi,0) = 0$ describes an important problem of plate having a double buckling load. A direct calculation of the buckled states of the plate bifurcating at the buckling load may be performed eliminating one of the unknowns and then applying the Newton polygon method to the remaining equation.

A necessary part of the analysis of the studied bifurcation problem is the investigation of stability of bifurcating solutions. Following the concept of linearized stability, a solution $w = w^*(\chi,\Theta)$ to Eq.(4) is stable if the eigenvalues ρ of the eigenvalue problem

$$F'(w^*,\chi,\Theta)\psi - \rho\psi = 0 \tag{14}$$

are positive. It is well known that the trivial solution is stable at any Θ and $\lambda > 0$ less than the first positive eigenvalue λ_+^Θ of the linearized equation. For $\lambda > \lambda_+^\Theta$ the trivial solution is always unstable.

McLeod and Sattinger 1973 [6] showed that the Liapunov-Schmidt reduction contains informations required for the stability analysis of bifurcating solutions. Later Sattinger 1979 [7] has shown that the stability of a one-parameter family of bifurcating solutions is determined, to the lowest order, by the eigenvalues of the Jacobian matrix of reduced bifurcation equations. Sattinger's theorem fails e.g. if $b^2 = ac$, since then the Jacobian matrix of the reduced equations has always one zero eigenvalue. The non-conforming degenerate cases can be treated by the following theorem:

Theorem 4. Suppose (H1-3) hold and the coefficient b of bifurcation equations (6) of Eq. (4) satisfies $b > 0$. Then, the linearized stability of any one-parameter family of isolated solutions to Eq. (4) bifurcating at the double eigenvalue λ_c is sufficiently close to the bifurcation point determined by the Jacobian of bifurcation equations (6)(positive value implies stability).

Proof. $F'(w^*, \chi, \theta)$ is an analytic and symmetric perturbation of the operator $F'(0,0,0)$. The spectrum of $F'(0,0,0)$ is discrete and non-negative with zero as a double eigenvalue having a positive isolation distance. Applying the Liapunov-Schmidt reduction to Eq. (14) we arrive at an eigenvalue problem in R^2 yielding the perturbation of zero eigenvalue. Now if we consider the eigenvalue problem for the Jacobian matrix of bifurcation equations (6') evaluated at $w = w^*$ we see that this equation differs from the former one only in the higher-order terms. Justifying the perturbation technique in both cases and comparing the perturbation equations we conclude the assertion.

Let us note that the stability analysis sometimes fails to indicate the energetically preferred equilibrium path of the plate and direct comparison of energy levels of buckled states is necessary. Such situation is encountered if $\theta = 0$, $b > a$, $b > c$, since then at $\lambda = \lambda_c$ there bifurcate two different pairs of stable solutions to Eq. (4).

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