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NONLINEAR SYSTEMS OF PARABOLIC PDE'S FOR PHASE CHANGE PROBLEMS

NOBUYUKI KENMOCHI

ABSTRACT. This paper is concerned with non-isothermal models for phase transitions. The models are described as a system of nonlinear parabolic PDEs with constraints. We discuss them from the viewpoint of abstract theory on time-dependent subdifferential operators in Hilbert spaces.

Introduction

We study the following two models for diffusive phase transitions, which are coupled systems of nonlinear parabolic PDEs.

Phase–Field System with Constraint (**PFC**):

$$\begin{aligned}
&\big(\rho(u)+w\big)_t - \Delta u = f(t,x) \qquad \text{in } Q := (0,T)\times\Omega\,,\\
&\nu w_t - \kappa\Delta w + \beta(w) + g(w) - u \ni 0 \qquad \text{in } Q\,,\\
&\frac{\partial u}{\partial n} + \alpha u = h_0(t,x)\,, \quad \frac{\partial w}{\partial n} = 0 \qquad \text{on } \Sigma := (0,T)\times\Gamma\,,\\
&u(0,\cdot) = u_0, \quad w(0,\cdot) = w_0 \qquad \text{in } \Omega\,.
\end{aligned}$$

Phase–Separation System with Constraint (**PSC**):

$$\begin{aligned}
&\big(\rho(u)+w\big)_t - \Delta u = f(t,x) \qquad \text{in } Q\,,\\
&\nu w_t - \Delta\big\{-\kappa\Delta w + \beta(w) + g(w) - u\big\} \ni 0 \qquad \text{in } Q\,,\\
&\frac{\partial u}{\partial n} + \alpha u = h_0(t,x)\,, \quad \frac{\partial w}{\partial n} = 0 \qquad \text{on } \Sigma\,,\\
&\frac{\partial}{\partial n}\{-\kappa\Delta w + \beta(w) + g(w) - u\} \ni 0 \qquad \text{on } \Sigma\,,\\
&u(0,\cdot) = u_0, \quad w(0,\cdot) = w_0 \qquad \text{in } \Omega\,.
\end{aligned}$$

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Here, Ω is a bounded domain in $\mathbb{R}^N$ $(1 \le N \le 3)$ with smooth boundary $\Gamma := \partial\Omega$, $0 < T < +\infty$, and we suppose the following conditions (ρ), (β) and (g):

(ρ) $\rho\colon \mathbb{R} \to \mathbb{R}$ is an increasing bi-Lipschitz continuous function; we denote by $C(\rho)$ a common Lipschitz constant of ρ and ρ^{-1} and by $\widehat{\rho^{-1}}$ a non-negative primitive of ρ^{-1}.

(β) β is a maximal monotone graph in $\mathbb{R} \times \mathbb{R}$ such that for some numbers σ_*, σ^* with $-\infty < \sigma_* < \sigma^* < +\infty$

$$\overline{D(\beta)} = [\sigma_*, \sigma^*];$$

under this condition, there is a non-negative proper l.s.c. convex function $\hat{\beta}$ on $\mathbb{R}$ such that $\partial\hat{\beta} = \beta$ in $\mathbb{R}$.

(g) $g\colon \mathbb{R} \to \mathbb{R}$ is a Lipschitz continuous function with compact support supp(g) in $\mathbb{R}$; under this condition, there is a primitive $\hat{g}$ of g such that $\hat{g}$ is non-negative on $[\sigma_*, \sigma^*]$.

Moreover, we suppose that $\kappa > 0$, $\nu > 0$ and $\alpha > 0$ are constants and

(f) $f \in L^2_{\mathrm{loc}}(\mathbb{R}_+; L^2(\Omega))$,

(h_0) $h_0 \in W^{1,2}_{\mathrm{loc}}(\mathbb{R}_+; L^2(\Gamma))$.

This work is concerned with the abstract treatment of systems (PFC) and (PSC); in fact, we show that in an adequate Hilbert space H the above systems can be reformulated in an evolution equation of the form

$$(AU)'(t) + \partial\varphi^t(U(t)) + p(U(t)) \ni \ell(t) \quad \text{in } H, \quad 0 < t < T,$$

$$U(0) = U_0,$$

where $\partial\varphi^t$ is the subdifferential of a time-dependent convex function φ^t on H and p is a Lipschitz continuous operator, with bounded range, in H and ℓ and U_0 are given data; further A is a linear, monotone, positive, selfadjoint and continuous operator in H. The basic idea for the reformulation as above is found in [5, 6, 12, 24]. For papers treating the related topics, see the references.

We shall mainly discuss system (PSC), since the abstract treatment of (PFC) is quite similar.

1. Evolution operators associated with (PSC)

Let $V = H^1(\Omega)$ with norm

$$|z|_V := |\nabla z|^2_{L^2(\Omega)} + \alpha |z|^2_{L^2(\Gamma)}{}^{\frac{1}{2}},$$

V^* be the dual space of V, F be the duality mapping from V onto V^* and $\langle\cdot,\cdot\rangle\ : V^* \times V \to \mathbb{R}$ be the duality pairing. Further let

$$V_0 := \left\{z \in H^1(\Omega)\,;\ \int_\Omega z\,dx = 0\right\}$$

be the Hilbert space with norm

$$|z|_{V_0} := |\nabla z|_{L^2(\Omega)}\,;$$

denote by $\langle\cdot\,,\cdot\rangle_0$ the duality pairing between V_0^* and V_0, and by F_0 the duality mapping from V_0 onto V_0^*. Note that

$$V_0 \subset L^2(\Omega)_0 := \left\{z \in L^2(\Omega)\,;\ \int_\Omega z\,dx = 0\right\} \subset V_0^*$$

with compact injections. We denote by π_0 the projection from $L^2(\Omega)$ onto $L^2\Omega_0$.

For the initial data u_0, w_0 we suppose that

$$u_0 \in L^2(\Omega)\,,\ w_0 \in L^\infty(\Omega) \quad \text{with } \sigma_* \le w_0 \le \sigma^* \text{ a.e. in } \Omega,\ \int_\Omega w_0 dx = c\,, \tag{1}$$

where c is a constant with

$$\sigma_* < \frac{c}{|\Omega|} < \sigma^*\,, \quad \text{i.e.,}\ \frac{c}{|\Omega|} \in \operatorname{int}\cdot D(\beta)\,. \tag{2}$$

For the boundary function h_0 we consider a function $h : \mathbb{R}_+ \to V$ such that for each $t \ge 0$

$$a\big(h(t), z\big) + \big(\alpha h(t) - h_0(t), z\big)_\Gamma = 0 \qquad \text{for all} \quad z \in V\,;$$

by assumption (h_0) we see that $h \in W^{1,2}_{\mathrm{loc}}(\mathbb{R}_+; V)$, where

$$a(z_1, z_2) := \int_\Omega \nabla z_1 \cdot \nabla z_2\,dx$$

and

$(\cdot\,,\cdot)_\Gamma$ denotes the inner product in $L^2(\Gamma)$,

$(\cdot\,,\cdot)$ denotes the inner product in $L^2(\Omega)$.

DEFINITION 1. (Weak formulation for (PSC)). Let $0 < T < +\infty$. Then a couple of functions u and w is called a *weak solution* of (PSC) on $[0,T]$, if the following conditions $(w1)-(w3)$ are fulfilled:

(w1) $\rho(u) \in C([0,T];V^*) \cap W^{1,2}_{\rm loc}((0,T];V^*) \cap L^2(0,T;L^2(\Omega))$,
$u \in L^2_{\rm loc}((0,T];V)$,
$w - \frac{c}{|\Omega|} \in C([0,T];\, L^2\Omega_0)) \cap L^2(0,T;V_0)$, $\quad w \in L^2_{\rm loc}((0,T];H^2(\Omega))$,
$w' \in L^2_{\rm loc}((0,T];\, V_0^*)$, $\quad \hat\beta(w) \in L^1(0,T;L^1(\Omega))$,

(w2) $\rho(u)(0) = \rho(u_0)$ and

$$\langle \rho(u)'(t) + w'(t), z\rangle + a(u(t) - h(t), z) + \alpha(u(t) - h(t), z)_\Gamma = (f(t), z) \quad \text{for all } z \in V \text{ and a.e. } t \in [0,T], \tag{3}$$

(w3) $w(0) = w_0$, $\frac{\partial w(t)}{\partial n} = 0$ a.e. on Γ for a.e. $t \in [0,T]$,
and there is $\xi \in L^2_{\rm loc}((0,T];L^2(\Omega))$ such that

$$\xi(t) \in \beta(w(t)) \quad \text{a.e. in} \quad \Omega \quad \text{for a.e.} \quad t \in [0,T],$$
$$\nu\langle w'(t), \eta\rangle_0 + \kappa(\Delta w(t), \Delta\eta) - (\xi(t) + g(w(t)) - u(t), \Delta\eta) = 0 \tag{4}$$

for all $\eta \in H^2(\Omega) \cap L^2\Omega_0$ with $\frac{\partial \eta}{\partial n} = 0$ a.e. on Γ and for a.e. $t \in [0,T]$.

For an abstract setting of (PSC), we consider a Hilbert space

$$X_0 := V^* \times L^2\Omega_0$$

with inner product $(\cdot\,,\cdot)_{X_0}$ given by

$$([e_1, v_1],\, [e_2, v_2])_{X_0} := \langle e_1, F^{-1}e_2\rangle + \nu(v_1, v_2) \quad \text{for any } [e_i, v_i] \in X_0\,,\ i = 1, 2.$$

Now, for each $t \ge 0$ we define a function $\varphi_0^t(\cdot)$ on X_0 by

$$\varphi_0^t([e,v]) := \begin{cases} \int_\Omega \hat{\rho^{-1}}(e - v - \frac{c}{|\Omega|})dx + \frac{\kappa}{2}|\nabla v|^2_{L^2(\Omega)} + \int_\Omega \hat\beta(v + \frac{c}{|\Omega|})dx - (h(t), e) \\ \qquad \text{if } [e,v] \in L^2(\Omega) \times V_0 \text{ and } \hat\beta(v + \frac{c}{|\Omega|}) \in L^1(\Omega), \\ +\infty \qquad \text{otherwise.} \end{cases} \tag{5}$$

THEOREM 1. (a) *For each* $t \geq 0$, φ_0^t *is proper, l.s.c. and convex on* X_0.

(b) *The subdifferential* $\partial\varphi_0^t$ *of* φ_0^t *in* X_0 *is characterized as follows:* $[e^*, v^*] \in \partial\varphi_0^t([e, v])$ *if and only if*

$$e^* = F\Big(\rho^{-1}\big(e - v - \frac{c}{|\Omega|}\big) - h(t)\Big), \tag{6}$$

and there is $\xi \in L^2(\Omega)$ *with* $\xi \in \beta(v + \frac{c}{|\Omega|})$ a.e. *in* Ω *such that*

$$\nu v^* = \kappa F_0 v + \pi_0\Big[\xi - \rho^{-1}\big(e - v - \frac{c}{|\Omega|}\big)\Big] \quad in \quad V_0^*, \tag{7}$$

i.e.,

$$\nu\langle v^*, \eta\rangle_0 = \kappa a(v, \eta) + \big(\xi - \rho^{-1}\big(e - v - \frac{c}{|\Omega|}\big), \eta\big) \quad \text{for all} \quad \eta \in V_0.$$

(c) *If* $[e_i^*, v_i^*] \in \partial\varphi_0^t([e_i, v_i])$, $i = 1, 2$, *then*

$$\begin{aligned}
&\big([e_1^*, v_1^*] - [e_2^*, v_2^*], [e_1, v_1] - [e_2, v_2]\big)_{X_0} = \\
&= \Big(\rho^{-1}\big(e_1 - v_1 - \frac{c}{|\Omega|}\big) - \rho^{-1}\big(e_2 - v_2 - \frac{c}{|\Omega|}\big), (e_1 - v_1) - (e_2 - v_2)\Big) + \\
&\qquad + \kappa\big|\nabla(v_1 - v_2)\big|^2_{L^2(\Omega)} + (\xi_1 - \xi_2, v_1 - v_2) \geq \\
&\geq \frac{1}{C(\rho)}\big|(e_1 - v_1) - (e_2 - v_2)\big|^2_{L^2(\Omega)} + \kappa\big|\nabla(v_1 - v_2)\big|^2_{L^2(\Omega)},
\end{aligned}$$

where ξ_i, $i = 1, 2$, *are the function* ξ *as in* (7).

Next, let A_0 be an operator in X_0 defined by

$$A_0([e, v]) := [e, \kappa F_0^{-1} v], \quad [e, v] \in X_0,$$

and G_0 defined by

$$G_0([e, v]) := \Big[0, \frac{1}{\nu}\pi_0\Big[g\big(v + \frac{c}{|\Omega|}\big)\Big]\Big], \quad [e, v] \in X_0.$$

Then it is easy to see that A_0 is linear, continuous, selfadjoint in X_0 and

$$\big(A_0([e, v]), [e, v]\big)_{X_0} = |e|^2_{V^*} + \kappa\nu|v|^2_{V_0^*} \quad \text{for all} \quad [e, v] \in X_0.$$

Hence A_0 is positive in X_0. Further G_0 is clearly Lipschitz continuous in X_0.

COROLLARY to Theorem 1. *Let* $\{u,w\}$ *be a weak solution of (PSC) on* $[0,T]$. *Then the function* $U(t) := \left[\rho(u(t)) + w(t), w(t) - \frac{c}{|\Omega|}\right]$ *satisfies that*

$$U \in C([0,T];X_0),\ U' \in L^2_{\mathrm{loc}}((0,T];V^* \times V_0^*),\ \varphi_0^t(U) \in L^1(0,T), \tag{8}$$

and

$$\begin{cases} \frac{d}{dt}A_0U(t) + \partial\varphi_0^t(U(t)) + G_0(U(T)) \ni \tilde{f}(t) \quad \text{in } X_0 \text{for a.e. } t \in [0,T], \\ U(0) = U_0 := \left[\rho(u_0) + w_0, w_0 - \frac{c}{|\Omega|}\right], \end{cases} \tag{9}$$

where $\tilde{f}(t) := [f(t), 0]$ *for a.e.* $t \in [0,T]$. *Conversely, if* $U(t) := [e(t), v(t)]$ *satisfies* (8) *and* (9), *then the couple* $\{u,w\}$ *with* $u = \rho^{-1}(e - v - \frac{c}{|\Omega|})$ *and* $w = v + \frac{c}{|\Omega|}$ *is a weak solution of (PSC) on* $[0,T]$.

The idea for the proof of Theorem 1 is found in V i s i n t i n [24], the theorem can be proved by applying it extensively (see [12; Part II] for a detail proof).

2. Abstract evolution equations in Hilbert spaces

Throughout this section, let H be a (real) Hilbert space with inner product $(\cdot,\cdot)_H$ and norm $|\cdot|_H$, and $\psi^t(\cdot)$ be a proper l.s.c. convex function on H for each $t \in \mathbb{R}_+$. Now, let us consider the abstract Cauchy problem

$$CP(\ell, v_0) \begin{cases} (Av)'(t) + \partial\psi^t(v(t)) + p(v(t)) \ni \ell(t) \quad \text{in } H,\ t > 0, \\ v(0) = v_0, \end{cases}$$

where A is a linear operator in H, p is a nonlinear operator in H, $\ell \in L^2_{\mathrm{loc}}(\mathbb{R}_+;H)$ and $v_0 \in \overline{D(\psi^0)}$. This problem is discussed for a family $\{\psi^t\}$ in $\Psi_H(a;K_0)$, specified below by a function $a \in W^{1,1}_{\mathrm{loc}}(\mathbb{R}_+)$ and a constant $K_0 > 0$.

We denote by $\Psi_H(a;K_0)$ the class of all families $\{\psi^t\}_{t\ge 0}$ of proper l.s.c. convex functions on H which satisfy the following conditions $(\Psi1)$–$(\Psi3)$:

$(\Psi1)$ $\psi^t(z) \ge K_0|z|_H^2$ for all $z \in H$ and $t \ge 0$.

$(\Psi2)$ $D(\psi^t) = D(\psi^0)$ for all $t > 0$, and

$$|\psi^t(z) - \psi^s(z)| \le |a(t) - a(s)|(1 + \psi^s(z)) \quad \text{for all } s,t \ge 0 \text{ and } z \in D(\psi^0).$$

$(\Psi3)$ For each $r \ge 0$, the set $\bigcup_{t\ge 0}\{z \in H; \psi^t(z) \le r\}$ is relatively compact in H.

Further we suppose that p is a Lipschitz continuous operator in H, the range $R(p)$ of p is bounded in H and there is a non-negative potential $P\colon H \to \mathbb{R}$ such that $\nabla P = p$; in this case, if $v \in W^{1,2}(0,T;H)$, then $P(v) \in W^{1,1}(0,T)$ and

$$\frac{d}{dt}P\big(v(t)\big) = \big(\nabla P(v(t)),\, v'(t)\big)_H \qquad \text{for a.e. } t \in [0,T].$$

Also, we suppose that A is a linear, continuous, positive (i.e., $(Az,z)_H > 0$ if $z \neq 0$) and selfadjoint operator in H; in this case, the fractional power $\frac{1}{2}$ of A, denoted by $A^{\frac{1}{2}}$, is defined as a linear, continuous, positive and selfadjoint operator in H again, and A is the subdifferential of the continuous convex function $j_A(z) := \frac{1}{2}|A^{\frac{1}{2}}z|_H^2$ for $z \in H$.

For uniqueness of a solution to $CP(\ell, v_0)$ we require the following condition $(*)$:

$(*)$ For each $\varepsilon > 0$ there is a number $C(\varepsilon) > 0$ such that

$$|z_1 - z_2|_H^2 \le \varepsilon(z_1^* - z_2^*,\, z_1 - z_2)_H + C(\varepsilon)\big|A^{\frac{1}{2}}(z_1 - z_2)\big|_H^2$$
$$\text{for all} \quad z_i \in D(\psi^t),\ z_i^* \in \psi^t(z_i), \quad i = 1,2, \quad \text{and} \quad t \ge 0.$$

DEFINITION 2. Let $0 < T < +\infty$, $\ell \in L^2(0,T;H)$ and $v_0 \in \overline{D(\psi^0)}$. Then a function $v\colon [0,T] \to H$ is a *solution* of $CP(\ell, v_0)$ on $[0,T]$, if $A^{\frac{1}{2}}v \in C([0,T];H) \cap W^{1,2}_{\mathrm{loc}}((0,T];H)$, $\psi^t(v) \in L^1(0,T)$, $(A^{\frac{1}{2}}v)(0) = A^{\frac{1}{2}}v_0$ and

$$\ell(t) - p\big(v(t)\big) - (Av)'(t) \in \partial\psi^t\big(v(t)\big) \quad \text{for a.e.} \quad t \in [0,T].$$

Remark 1. In Definition 2, note that $(Av)'(t) = A^{\frac{1}{2}}\big[(A^{\frac{1}{2}}v)'(t)\big] \in H$ for a.e. $t \in [0,T]$, since $(A^{\frac{1}{2}}v)'(t) \in H$ for a.e. $t \in [0,T]$.

Now the solvability of $CP(\ell, v_0)$ is mentioned in the following theorem.

THEOREM 2. *Assume that $\{\psi^t\} \in \Psi_H(a, K_0)$ and p is as above. Let $0 < T < +\infty$, $\ell \in W^{1,2}(0,T;H)$ and $v_0 \in \overline{D(\psi^0)}$. Then $CP(\ell, v_0)$ admits one and only one solution v on $[0,T]$ such that*

$$t^{\frac{1}{2}}(A^{\frac{1}{2}}v)' \in L^2(0,T;H), \quad t\psi^t(v) \in L^\infty(0,T).$$

In particular, if $v_0 \in D(\psi^0)$, then

$$A^{\frac{1}{2}}v \in W^{1,2}(0,T;H), \quad \psi^t(v) \in L^\infty(0,T).$$

A solution of $CP(\ell, v_0)$ is constructed as the limit of the solutions v_λ of approximate problems with parameter $\lambda > 0$ as $\lambda \to 0$:

$$CP_\lambda \begin{cases} [Av_\lambda + \lambda v_\lambda]'(t) + \partial\psi^t_\lambda(v_\lambda(t)) + p(v_\lambda(t)) = \ell(t), \quad 0 < t < T, \\ v_\lambda(0) = v_0, \end{cases}$$

where ψ^t_λ is the Yosida-approximation of ψ^t. See [12] for a detail proof of Theorem 2.

3. Asymptotic behaviour as $t \to 0$

Let p be as in the previous section and $\{\psi^t\} \in \Psi_H(a; K_0)$ and further suppose that

$$\ell' \in L^1(\mathbb{R}_+; H); \ \ell^\infty := \lim_{t \to +\infty} \ell(t) \quad \text{in } H, \tag{10}$$

$$a' \in L^1(\mathbb{R}_+), \tag{11}$$

$$\psi^t \to \psi^\infty \quad \text{(in the sense of Mosco) as} \quad t \to +\infty, \tag{12}$$

where ψ^∞ is a non-negative proper l.s.c. convex function, with $D(\psi^\infty) = D(\psi^0)$, on H. Here, by "$\psi^t \to \psi^\infty$ (in the sense of Mosco) as $t \to +\infty$" we mean that the following two conditions (M1) and (M2) are fulfilled:

(M1) if $t_n \to +\infty$ and $z_n \to z$ weakly in H, then

$$\liminf_{n \to +\infty} \psi^{t_n}(z_n) \geq \psi^\infty(z).$$

(M2) For any $z \in D(\psi^\infty)$ there is a function $w : \mathbb{R}_+ \to H$ such that

$$w(t) \to z \quad \text{in } H, \quad \psi^t(w(t)) \to \psi^\infty(z) \quad \text{as } t \to +\infty.$$

From the definition of the convergence in the sense of Mosco we immediately see that

$$\psi^\infty(z) \geq K_0 |z|^2_H \quad \text{for all } z \in H,$$

that is, ψ^∞ is coercive on H, and for each $r > 0$ the set $\{z \in H; \psi^\infty(z) \leq r\}$ is compact in H. Therefore, the stationary problem

$$\partial\psi^\infty(v_\infty) + p(v_\infty) \ni \ell^\infty \quad \text{in } H \tag{13}$$

has at least one solution v_∞ and the set of all solutions is compact in H; a solution of (13) is not unique in general, since p is not monotone in H.

THEOREM 3. *Let p be as in Section 2 and $\{\psi^t\} \in \Psi_H(a; K_0)$, and suppose that $v_0 \in \overline{D(\psi^0)}$ and (10)–(12) hold. Then, for the global solution v of $CP(\ell, v_0)$,*

(i) $(A^{\frac{1}{2}}v)' \in L^2(t_0, +\infty; H)$ *for each finite $t_0 > 0$.*
Moreover, the ω-limit set

$$\omega(v) := \{z \in H;\ v(t_n) \to z \text{ in } H \text{ for some } t_n \text{ with } t_n \to +\infty\}$$

satisfies that

(ii) *$\omega(v)$ is non-empty, connected and compact in H,*
(iii) *any point $v_\infty \in \omega(v)$ is a solution of (13),*
(iv) *for any $v_\infty \in \omega(v)$,*

$$\lim_{t\to+\infty}\{\psi^t(v(t)) + P(v(t)) - (\ell(t), v(t))_H\} = \psi^\infty(v_\infty) + P(v_\infty) - (\ell^\infty, v_\infty)_H .$$

Applying a modified technic in [19], we can prove Theorem 3. See [12] in details.

4. Application to (PSC)

Applying Theorems 2, 3 to system (PSC), we obtain not only an existence-uniqueness result, but also an asymptotic stability result for it.

THEOREM 4. *Assume that (ρ), (β), (g), (f), (h_0), (1), (2) hold and*

$$f' \in L^1(\mathbb{R}_+; L^2(\Omega)); \qquad f^\infty := \lim_{t\to+\infty} f(t) \quad \text{in} \quad L^2(\Omega);$$

and

$$h_0' \in L^1(\mathbb{R}_+; L^2(\Gamma)); \qquad h_0^\infty := \lim_{t\to+\infty} h_0(t) \quad \text{in} \quad L^2(\Gamma);$$

let h^∞ be the function in V such that

$$a(h^\infty, z) + (\alpha h^\infty - h_0^\infty, z)_\Gamma = 0 \quad \text{for all} \quad z \in V .$$

Then (PSC) admits one and only one global (in time) weak solution $\{u, w\}$. Moreover, the following statements hold.

(a) *For every finite* $T > 0$,

$$t^{\frac{1}{2}}\rho(u) \in L^\infty(0,T;L^2(\Omega)), \quad t^{\frac{1}{2}}\rho(u)_t \in L^2(0,T;V^*),$$
$$t^{\frac{1}{2}}(w - \frac{c}{|\Omega|}) \in L^\infty(0,T;V_0), \quad t^{\frac{1}{2}}w_t \in L^2(0,T;V_0^*),$$
$$t^{\frac{1}{2}}\xi \in L^2(0,T;L^2(\Omega)), \quad t^{\frac{1}{2}}\hat{\beta}(w) \in L^\infty(0,T;L^1(\Omega)),$$

where ξ *is the function as in* (w3) *of Definition 1.*

(b) *For every finite* $T > 0$,

$$\rho(u) \in L^\infty(T,+\infty;L^2(\Omega)), \quad w - \frac{c}{|\Omega|} \in L^\infty(T,\infty;V_0),$$
$$\hat{\beta}(w) \in L^\infty(T,+\infty;L^1(\Omega)),$$
$$\rho(u)_t \in L^2(T,+\infty;V^*), \quad w_t \in L^2(T,+\infty;V_0^*)$$

(c) $u(t) \to u_\infty$ *in* $L^2(\Omega)$ *as* $t \to +\infty$, *where* $u_\infty \in V$ *is the solution of*

$$a(u_\infty - h^\infty, \eta) + \alpha(u_\infty - h^\infty, \eta)_\Gamma = (f^\infty, \eta) \quad \text{for all} \quad \eta \in V.$$

(d) *The* ω*-limit set* $\omega(w)$ *of* w *as* $t \to +\infty$, *i.e.,*

$$\omega(w) := \{z \in L^2(\Omega);\ w(t_n) \to z \ \text{ in } L^2(\Omega) \text{ for some } t_n \text{ with } t_n \to +\infty\},$$

is non-empty, connected, and compact in $L^2(\Omega)$, *and furthermore any* $w_\infty \in \omega(w)$ *satisfies the system*

$$\begin{cases} \kappa a(w_\infty, \eta) + (\xi_\infty + g(w_\infty) - u_\infty, \eta) = 0 & \text{for all } \eta \in V_0, \\ w_\infty - \frac{c}{|\Omega|} \in V_0, \\ \xi_\infty \in L^2(\Omega), \quad \xi_\infty \in \beta(w_\infty) \text{ a.e. in } \Omega. \end{cases}$$

For a complete proof of Theorem 4, see [12].

In the one-dimensional case, we see further results on the ω-limit set $\omega(w)$ with the structure of the corresponding stationary problem for the isothermal phase separation model (Cahn–Hilliard model with constraints) (see [3]).

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