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FLOQUET THEORY FOR, AND BIFURCATIONS FROM SPATIALLY PERIODIC PATTERNS

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ABSTRACT. We consider elliptic systems of PDEs on infinite long cylindrical domains allowing applications such as travelling waves in reaction-diffusion systems or fluid flow in pipes. We develop a method for studying solutions which are close to a given solution u_0 which is periodic with respect to the axial variable x. Using spatial Floquet theory we are able to construct a *spatial center manifold* and to show that all orbitally close solutions can be described by an ODE.

1. Introduction

According to Kirchgässner [Ki82] it is advantageous to study elliptic problems in cylinders by the so-called method of *spatial dynamics.* This means that the axial variable plays the role of a time-like variable. Then, the associated differential equation can be treated using tools from dynamical systems theory. For illustration we consider the following reaction-diffusion system

$$\partial_t u = D\Delta_{x,y} u + f(\lambda, u), \qquad \text{in} \quad \Omega = \mathbb{R} \times \Sigma, \quad u|_{\partial\Omega} = 0. \tag{1}$$

Here, $u \in \mathbb{R}^m$ contains the concentrations, D is the diffusion matrix, and f is the reaction term. $x \in \mathbb{R}$ is the axial variable, $y \in \Sigma$ the cross-sectional variable, and $\Delta_{x,y} = \partial_x^2 + \Delta_y$ the Laplacian. Looking for travelling waves with speed c, we can rewrite the system as a *spatial dynamical system* with respect to x. Let $\widetilde{u} = \partial_x u \in \mathbb{R}^m$ and $w = (u, \widetilde{u})$, then we are lead to

$$\frac{d}{dx} w = \mathcal{F}(w) = \begin{pmatrix} \widetilde{u} \\ -\Delta_y u - D^{-1}\big[c\widetilde{u} - f(\lambda, u)\big] \end{pmatrix}. \tag{2}$$

Now, $w(x, \cdot)$ is an element of the Hilbert space $H = \mathring{H}^1(\Sigma)^m \times L_2(\Sigma)^m$.

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We are interested in solutions which are close to a given periodic solution p with $p(x+T) = p(x)$ for all x and $T > 0$. In a neighborhood of this orbit we introduce a local coordinate system via

$$w(x(\tau)) = p(\tau) + v(\tau) \quad \text{with} \quad \langle p'(\tau), v(\tau)\rangle = 0. \tag{3}$$

We use τ as new independent time-like variable instead of x and obtain $(' = d/d\tau)$

$$\begin{aligned} x' &= 1 + a(\tau, v) = \left[\langle p'(\tau), p'(\tau)\rangle - \langle p''(\tau), v\rangle\right] \big/ \langle p'(\tau), \mathcal{F}(p(\tau)+v)\rangle, \\ v' &= \widehat{\mathcal{F}}(\tau, v) \quad = \big(1 + a(\tau, v)\big)\big(\mathcal{F}(p(\tau)+v) - \mathcal{F}(p(\tau))\big). \end{aligned} \tag{4}$$

By construction, the functions a and $\widehat{\mathcal{F}}$ are T–periodic in τ. Note that $\widehat{\mathcal{F}}(\tau, v)$ is quasilinear in v even for semilinear $\mathcal{F}(w)$.

For the study of the second equation we first treat the linear part to construct a spectral splitting corresponding to the Floquet multipliers on the unit circle (see Section 2). Writing $\widehat{\mathcal{F}}(\tau, v) = \widehat{A}(\tau)v + \mathcal{N}(\tau, v)$ with $\widehat{A}(\tau) = A + B(\tau)\colon D(A) \to H$ and $\mathcal{N} = \mathcal{O}(\|v\|^2)$ we find the following

THEOREM 1. *Let $A\colon D(A) \to H$ be a closed operator with compact resolvent such that $\|(A + i\xi)^{-1}\| \le C/(1 + |\xi|)$ for all $\xi \in \mathbb{R}$. Further assume $B \in C^{r+1}(\mathbb{R}, \mathcal{L}(D(A^\beta), H))$ for some $\beta \in [0,1)$ and that the linear operator $Lv = v' - \widehat{A}(\cdot)v$ has only a finite number of Floquet multipliers on the unit circle. Moreover, assume $\mathcal{N} \in C^{r+1}(\mathbb{R} \times D(A), H)$.*

Then, there exist projections $P(x)$ such that $P(\cdot) \in C^r(\mathbb{R}, \mathcal{L}(H,H))$, $P(x+T) = P(x)$, and $n = \dim P(x)H \le \infty$. Furthermore there exists a $(n+1)$–dimensional local center manifold $M_C \subset H$ given as

$$\{p(\tau) + v_0 + h(\tau, v_0) \in D(A)\colon P(\tau)v_0 = v_0, \quad P(\tau)h(\tau, v_0) = 0, \\ \langle p'(\tau), v_0 + h(\tau, v_0)\rangle = 0, \quad \|v_0\| \le \varepsilon\}$$

where $h \in C^r$ and $h(\tau, v_0) = h(\tau + T, v_0) = \mathcal{O}(\|v_0\|^2)$.

Note that this theorem allows the nonlinear term to have the same loss of smoothness as the linear part, hence we are able to treat certain quasilinear systems. The C^r-smoothness of M_C is just one order less than that of the nonlinearity.

This spatial center manifold contains the original periodic orbit p and all solutions $w\colon \mathbb{R} \to D(A)$ which exist on the whole infinite cylinder and stay orbitally close to p. All bifurcating solutions can now be found by studying the reduced problem

$$\frac{d}{d\tau}x = 1 + a\big(\tau, v_0 + h(\tau, v_0)\big), \quad \frac{d}{d\tau}v_0 - \widehat{A}(\tau)v_0 = P(\tau)\mathcal{N}\big(\tau, v_0 + h(\tau, v_0)\big),$$

which is an ODE with periodic coefficients. For the analysis of such systems we refer to [MS86] and to [Io88, IA92] where an associated normal form theory is developed.

The purpose of this note is to establish the proper functional analytic setup and to give the basic ideas how to be able to handle such problems. The reader should be a acquainted with the theory for elliptic systems with autonomous linear part as presented in [Mi88]. Because of the limited space we have to refer for most of the technicalities to [DFMK94]; there all details as well as applications to reaction-diffusion systems and fluid dynamics are given.

2. The spectral splitting

We have to study a linear elliptic system $Lv = \frac{d}{d\tau}v - \widehat{A}(\tau)v = f(\tau) \in L_2(\mathbb{R}, H)$ where $\widehat{A}$ satisfies the assumptions of Theorem 1. The Floquet multipliers and exponents of the problems are defined by considering the associated periodic operator

$$L_\#: D(L_\#) \to H_\# = L_2\big((0,T), H\big)\,; \quad v \mapsto v' - \widehat{A}(\cdot)v\,,$$

where $D(L_\#) = H^1_\#\big((0,T), H\big) \cap L_2\big((0,T), D(A)\big)$. Here and further on the $\#$ stands for periodic functions with period T.

We call the eigenvalues λ of this operator the Floquet exponents of the problem and $\rho = e^\lambda$ the Floquet multipliers. Note that even for constant $\widehat{A}$ we have infinitely many Floquet multipliers inside and outside the unit circle. This is due to the ellipticity of the underlying problem. Thus, it is nontrivial to show that the resolvent set of $L_\#$ is nonempty, see [Ku82, DFKM93] for positive and negative results. However, if the resolvent set is nonempty, then the standard application of Fredholm's alternative shows that the set of Floquet exponents is discrete. Since $\lambda + i\frac{2\pi}{T}$ is a Floquet exponent whenever λ is, we see that the critical part on the imaginary axis has to be infinite dimensional (if nonempty). Nevertheless, we are able to construct a spectral projection which separates this critical part.

Lemma 2. *Let $\widehat{A}$ be as above and assume that the resolvent set of $L_\#$ is nonempty. Then, there is a projection $\widehat{P}$ on $H_\#$ with the following properties: (i) $\widehat{P}L_\# = L_\#\widehat{P}$, (ii) $\widehat{P}L_\#$ has spectrum only on the imaginary axis, (iii) $L_\# + i\xi\colon (I-\widehat{P})D(L_\#) \to (I-\widehat{P})H_\#$ is invertible for all $\xi \in \mathbb{R}$, and (iv) $\widehat{P}$ has the form*

$$[\widehat{P}f](x) = P(x)f(x) \quad \textit{with} \quad P(x)g = \sum_{k=1}^{N} \langle g, \psi_k(x)\rangle \phi_k(x)\,, \tag{5}$$

where ϕ_k, $\psi_k \in D(L_\#)$.

Remark. Note that the spectral projection acts pointwise in x. The commutation relation (i) with $L_\#$ readily implies

$$P'(x) = \widehat{A}(x)P(x) - P(x)\widehat{A}(x), \quad x \in \mathbb{R}. \tag{6}$$

Proof. For simplicity assume $T = 2\pi$. We choose $\varepsilon > 0$ such that the strip $\{\lambda \in \mathbb{C}: |\mathrm{Re}\,\lambda| < \varepsilon\}$ contains exactly the spectrum of $L_\#$ which lies on the imaginary axis. For each $I \subset \mathbb{R}$ we let

$$\Sigma_I = \big\{\lambda \in \mathrm{Spectrum}(L_\#): |\mathrm{Re}\,\lambda| < \varepsilon,\ \mathrm{Im}\,\lambda \in I\big\} \subset i\mathbb{R},$$

Since the spectrum is discrete we can define the spectral projection P_I as the Dunford integral $P_I = \frac{1}{2\pi i}\int_{\Gamma_I}(L_\# - \lambda)^{-1}d\lambda$, where Γ_I is a positively oriented closed C^1-curve, such that Σ_I lies in its interior whereas $\Sigma(L_\#)\backslash\Sigma_I$ lies outside.

Our aim is to define $\widehat{P} = P_{\mathbb{R}}$ as the spectral projection of the whole strip. Since the Dunford integral does not exist for this unbounded set, we use the special structure connecting the eigenfunctions corresponding to Floquet exponents which only differ by ik, $k \in \mathbb{Z}$. Since $L_\#$ has a compact resolvent, the projection $P_{(-1/2,1/2]}$ can be written as

$$P_{(-1/2,1/2]}f(x) = \sum_{k=1}^{N}\frac{1}{2\pi}\int_0^{2\pi}\langle f(y),\, \psi_k(y)\rangle dy\, \phi_k(x)\,,$$

where N is sum of the algebraic multiplicities of the eigenvalues in $\Sigma_{(-1/2,1/2]}$. Since $e^{imx}\phi_k(x)$ $\big(e^{imx}\psi_k(x)\big)$ are eigenvectors to the shifted eigenvalue $\lambda_k + im$ $(\overline{\lambda}_k - im)$ we find

$$\begin{aligned} P_{(-n-1/2,n+1/2]}f(x) &= \sum_{m=-n}^{n}\sum_{k=1}^{N}\frac{1}{2\pi}\int_0^{2\pi}\langle f(y), e^{imy}\psi_k(y)\rangle dy\, e^{imx}\phi_k(x) \\ &= \sum_{k=1}^{N} Q_n\big(\langle f, \psi_k\rangle\big)(x)\phi_k(x)\,. \end{aligned}$$

Here Q_n is the orthogonal projection from $L_2\big((0,2\pi),\mathbb{C}\big)$ onto the span of $e^{-inx},\dots,e^{inx}$. Thus, for each $f \in H_\#$ the limit $\widehat{P}f = P_{\mathbb{R}}f$ of $P_{(-n-1/2,n+1/2]}f$ exists and satisfies (5). □

Now the linear problem $v' - \widehat{A}(\tau)v = f \in L_2(\mathbb{R}, H)$ can be solved by decoupling the critical part. Let $v_0(\tau) = P(\tau)v(\tau)$ and $v_1 = v - v_0$, then

$$\begin{aligned} v_0' - \widehat{A}(\tau)v_0 &= f_0(\tau) = P(\tau)f(\tau), \\ v_1' - \widehat{A}(\tau)v_1 &= f_1(\tau) = \big(I - P(\tau)\big)f(\tau). \end{aligned} \tag{7}$$

The v_0-equation generates a polynomially growing fundamental solution while v_1 contains the exponentially growing and decaying modes. We construct the solution operator $v_1 = K_1 f_1$ by the use of the direct integral and Fourier transform. Every function $f \in L_2(\mathbb{R}, H)$ can be written as a direct integral in the sense of [RS80]: $f(x) = \int_0^1 e^{i\omega x} F(\omega, x)\, d\omega$, where $F(\omega, \cdot) \in L_2\big((0, 2\pi), H\big)$. In fact, if $\widehat{f}(\xi)$ is the Fourier transform of f we have $F(\omega, x) = \sum_{k \in \mathbb{Z}} e^{ikx} \widehat{f}(k + \omega)$. Hence, the solution in question is given by the formula

$$v_1(x) = \int_0^1 e^{i\omega x} \big[(L_\# - i\omega)^{-1} F(\omega, \cdot)\big]\, d\omega\,.$$

As $\widehat{P}F(\omega, \cdot) = 0$ the inverse exists and is bounded over $\omega \in [0, 1]$. Standard regularity theory (cf. [Mi87]) then implies $v_1 \in H^1(\mathbb{R}, H) \cap L_2\big(\mathbb{R}, D(A)\big)$. Using the methods from [Mi87] it is then possible to generalize this result to all L_p-spaces, $p \in (1, \infty)$ as well as to exponentially weighted spaces (cf. [DFKM93]). We arrive at

LEMMA 3. *Assume that A satisfies the assumptions of Theorem 1. Then, there is a $\delta > 0$ such that for all $\alpha \in (0, \delta)$ and all $p \in (1, \infty)$ there is a constant C such that (7) has, for all (τ_0, ξ_0, f) with $\tau_0 \in \mathbb{R}$, $\xi_0 = P(\tau_0)\xi_0$, and f with $e^{-\alpha|x|} f(x) \in L_p(\mathbb{R}, H)$, a unique solution $v = K_{\tau_0}(\xi_0, f)$ with $v_0(\tau_0) = P(\tau_0)v(\tau_0) = \xi_0$ and $\big\|e^{-\alpha|x|} A v(x)\big\|_p \le C\big(|\xi_0| + \|e^{-\alpha|x|} f(x)\|_p\big)$.*

3. The center manifold

The center manifold is now constructed by a contraction mapping argument completely analogous to [Mi88]. The only difference appearing here is that the linear part is nonautonomous as well. Hence, after multiplying $\mathcal{N}(\tau, v)$ with a suitable cut-off function, we consider

$$\frac{d}{d\tau} v - \widehat{A}(\tau)v \;=\; \widetilde{\mathcal{N}}(\tau, v)\,, \tag{8}$$

where $\widetilde{\mathcal{N}}$ is globally Lipschitz continuous with small Lipschitz constant. We augment (8) with the initial condition $P(\tau_0)v(\tau_0) = \xi_0$. Then, the solution operator from Lemma 3 shows that every weakly exponentially growing solution $v: \mathbb{R} \to D(A)$ has to satisfy the integral equation

$$v = S(\tau_0, \xi_0, v) := K_{\tau_0}\Big(\xi_0, \widetilde{\mathcal{N}}(\,\cdot\,, v(\cdot))\Big).$$

Because of the small Lipschitz constant of $\widetilde{\mathcal{N}}$ the mapping $S(\tau_0, \xi_0, \cdot)$ is a contraction on the Banach space of functions v with $e^{-\alpha|x|}v(x) \in L_p(\mathbb{R}, D(A))$. Thus, we find a unique solution $v = \mathcal{V}(\tau_0, \xi_0)$. Using the fiber bundle contraction method it can be shown that $\mathcal{V}$ depends, in fact, r-times differentiable on τ_0 and ξ_0. The center manifold is now defined to be the graph of the function h defined via $h(\tau_0, \xi_0) = \big(I - P(\tau_0)\big)\mathcal{V}(\tau_0, \xi_0)(\tau_0)$. As in [Mi88] it follows that h has the desired properties, viz. it defines a locally invariant manifold containing all small bounded solutions. This completes the sketch of proof for Theorem 1.

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