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ON A NONLINEAR BOUNDARY VALUE PROBLEM OF HIGHER ORDER

by V. ŠEDA

1. The theory of boundary value problems (in short BVP) for ordinary differential (d.) equations has been largely developed as one can see from a comprehensive and carefully prepared report [1]. It is characterized by a great number of problems and a variety of methods attacking them. In the last years two methods, the subfunction method and the method based on the funnel theorem of Kneser, became successful in guaranteeing the existence of solutions to BVP. Both methods have been applied to solving second-order BVP of different types in more than 15 papers. Here only [2], [3] and [4] will be mentioned. As to the BVP of higher order, the subfunction method has been used only in [5] and some BVP of the third order were solved by means of the funnel technique in [6]. Another application of the subfunction theory to a nonlinear BVP of a higher order will be presented now. The problem arose in the theory of semiconductors and it has been solved by reducing it to a set of second-order ones. The details can be found in [7].

2. Suppose $a, b, l, \alpha, \beta, \gamma, \delta, A, B, C, D, F$ are positive numbers. The problem in question is

$$y^{(4)} - a[y'' + b(y'y''' + y''^2)] = 0, \qquad (0 \leqq x \leqq l), \qquad y' = \frac{\mathrm{d}y}{\mathrm{d}x}, \qquad \text{etc.}, \tag{1}$$

$$\begin{aligned} y(0) &= \alpha\{Fy'''(0) - By'(0)\,[C + Dy''(0)]\} \\ y'(0) &= \beta\{Fy'''(0) - By'(0)\,[C + Dy''(0)]\} \\ y(l) &= A - \gamma\{Fy'''(l) - By'(l)\,[C + Dy''(l)]\} \\ y'(l) &= \delta\{Fy'''(l) - By'(l)\,[C + Dy''(l)]\} \end{aligned} \tag{2}$$

A solution y of that problem represents potential and the function z defined through y by

$$z = Fy''' - By'(C + Dy'') \tag{3}$$

means current in a semiconductor. We shall be interested to prove the existence, if possible the uniqueness, of the solution y to (1), (2) as well as to find some properties of the associated function z given by (3).

By the double integration of (1) and assuming that

$$Fab - BD = 0, \tag{4}$$

the BVP (1), (2) can be reduced to four different types of BVP.

When 1. $\beta(Fa - BC) - 1 = \delta(Fa - BC) - 1 = 0$, then (1), (2) is equivalent to

$$y'' = a\left(y + \frac{b}{2}y'^2\right) + c_2, \tag{5'}$$

$$\frac{\alpha}{\beta}y'(0) - y(0) = 0, \tag{6}$$

$$\frac{\gamma}{\delta}y'(l) + y(l) - A = 0$$

and the associated function $z = \frac{1}{\beta}y'$.

If 2. $\beta(Fa - BC) - 1 = 0 \neq \delta(Fa - BC) - 1$, then (1), (2) reduces to the BVP (5'),

$$\frac{\alpha}{\beta}y'(0) - y(0) = 0, \tag{7}$$

$$y(l) = A, y'(l) = 0.$$

The associated function $z = \frac{1}{\beta}y'$.

In the case 3. $\beta(Fa - BC) - 1 \neq 0 = \delta(Fa - BC) - 1$ instead of (1), (2) we have (5'),

$$y(0) = y'(0) = 0 \tag{8}$$

$$\frac{\gamma}{\delta}y'(l) + y(l) - A = 0$$

and $z = \frac{1}{\delta}y'$.

When 4. $\beta(Fa - BC) - 1 \neq 0 \neq \delta(Fa - BC) - 1$ is assumed, the BVP (1), (2) means the BVP

$$y'' = a\left(y + \frac{b}{2}y'^2\right) + c_1x + c_2, \tag{5}$$

$$\begin{aligned} y(0) &= \frac{\alpha Fc_1}{1 - \beta(Fa - BC)}, & y'(0) &= \frac{\beta Fc_1}{1 - \beta(Fa - BC)}, \\ y(l) &= A - \frac{\gamma Fc_1}{1 - \delta(Fa - BC)}, & y'(l) &= \frac{\delta Fc_1}{1 - \delta(Fa - BC)} \end{aligned} \tag{9}$$

and the function $z = (Fa - BC)y' + Fc_1$. In all mentioned cases $c_1, c_2 \in R$ are arbitrary. Since the BVP (5), (6) contains all the others as special cases, first this problem will be studied. Nevertheless, the problem can be further generalized and the existence, uniqueness and a comparison property of the solution to the more general BVP will be established in the next section.

3. Consider the d. equation

$$y'' = f(x, y, y') \tag{10}$$

with f being defined and continuous on the set $S = \{(x, y, y') : 0 \leqq x \leqq l, |y| + |y'| < +\infty\}$. The notions of *a lower (an upper) solution of* (10) as well as of *a Nagumo condition for* f have the usual meaning and they are defined in [3], p. 460. Further, if φ is a lower and ψ an upper solution of (10) and $\varphi(0) < \psi(0)$ ($\varphi(l) < \psi(l)$), then by $H_1(H_2)$ will be denoted the class of all continuous functions $g = g(y, y')$ ($h = h(y, y')$) defined on $\langle\varphi(0), \psi(0)\rangle \times R$ ($\langle\varphi(l), \psi(l)\rangle \times R$) which are nondecreasing in y' and satisfy $g[\varphi(0), \varphi'(0)] \geqq 0$, $g[\psi(0), \psi'(0)] \leqq 0$ ($h[\varphi(l), \varphi'(l)] \leqq 0$, $h[\psi(l), \psi'(l)] \geqq 0$). $H_3(H_4)$ will mean the class of all continuous functions $g = g(y, y')$ ($h = h(y, y')$) defined on R^2 which are nonincreasing in y (nondecreasing in y) and nondecreasing in y'. The following existence theorem has been proved as Theorem 3.6 in [3], p. 464.

Theorem 1. *Let φ (ψ) be a lower (an upper) solution of the d. equation* (10) *with $\varphi \leqq \psi$ on $\langle 0, l\rangle$ and let $\varphi(0) < \psi(0)$, $\varphi(l) < \psi(l)$. Let f satisfy a Nagumo condition with respect to the pair φ, ψ and let $g \in H_1$, $h \in H_2$. Then there is a solution y of the BVP* (10),

$$g[y(0), y'(0)] = 0 = h[y(l), y'(l)] \tag{11}$$

which satisfies $\varphi \leqq y \leqq \psi$ on $\langle 0, 1\rangle$.

With a help of this theorem another existence as well as a comparison theorem will be proved.

Theorem 2. *Suppose $f_1 = f_1(x, y, y')$, $f_2 = f_2(x, y, y')$ are continuous and $f_1 \leqq f_2$ ($f_1 \geqq f_2$) on S. Suppose $\lim_{y \to -\infty} f_2(x, y, y') = -\infty$ ($\lim_{y \to +\infty} f_2(x, y, y') = +\infty$) uniformly in x, y' on any compact subset of $\langle 0, l\rangle \times R$. Assume further f_2 satisfies a Nagumo condition with respect to any pair $\chi, \omega \in C_0(\langle 0, l\rangle)$, $\chi \leqq \omega$. Suppose $g = g(y, y') \in H_3$ and $h = h(y, y') \in H_4$. Then, if there is a solution y_1 of the problem* (11),

$$y'' = f_1(x, y, y'), \tag{12}$$

there also exists a solution y_2 of the BVP (11),

$$y'' = f_2(x, y, y') \tag{13}$$

and a number $c > 0$ such that $y_1 - c \leqq y_2 \leqq y_1$ ($y_1 \leqq y_2 \leqq y_1 + c$) on $\langle 0, l\rangle$.

Proof. Only the case $f_1 \leqq f_2$ will be considered. Since $f_1 \leqq f_2$, y_1 is an upper solution of (13). By the assumption of f_2, there is a y_0 such that for all $x \in \langle 0, l\rangle$ and all $y \leqq y_0$ $f_2(x, y, y_1'(x)) \leqq m = \min_{x \in \langle 0, l\rangle} f_1(x, y_1(x), y_1'(x))$. Then for any $c > 0$ with $y_1(x) - c \leqq y_0$ ($x \in \langle 0, l\rangle$) the function $y_1 - c$ is a lower solution of (13). g and

h being from H_3 and H_4 respectively, $g \in H_1$ and $h \in H_2$. The result follows on the basis of Theorem 1.

The following theorem generalizes a uniqueness result in [8], p. 108.

Theorem 3. *Let $f = f(x, y, y')$ be defined and increasing in y on S. Let $g = g(y, y')$ $(h = h(y, y'))$ be decreasing in y (increasing in y) and nondecreasing in y' on R^2. Then there exists at most one solution of the BVP* (10), (11).

Proof. Suppose there are two solutions y_1, y_2 of the considered BVP. By the properties of g and h, $y_1(0) > y_2(0)$ $(y_1(0) < y_2(0))$ implies that $y_1'(0) > y_2'(0)$ $(y_1'(0) < y_2'(0))$. Similarly, $y_1(l) > y_2(l)$ $(y_1(l) < y_2(l))$ can stand only with $y_1'(l) < y_2'(l)$ $(y_1'(l) > y_2'(l))$. All four cases lead to the existence of either a positive local maximum or a negative minimum of $y_1 - y_2$. This cannot happen, with respect to the assumption on f. Thus, $y_1(0) = y_2(0)$, $y_1(l) = y_2(l)$ and $y_1 = y_2$ on $\langle 0, l \rangle$.

Remark. Theorems 1 up to 3 are valid also in the case when one or both of the functions g, h show inverse monotonic properties.

4. The above stated theorems can be applied to the BVP (5), (6). By Theorem 3, there exists at most one solution of that problem. On the basis of Theorem 1, the existence of a solution to that BVP can be assured by finding suitable lower and upper solutions to (5). Taking linear functions for lower and upper solutions we get

Lemma 1. *There exists a unique solution y to the problem* (5), (6). *This solution satisfies the inequalities*

$$-\frac{c_1}{a}x + q \leqq y(x) \leqq -\frac{c_1}{a}x + q_1 \quad (0 \leqq x \leqq l), \tag{14}$$

where

$$q = \min\left[\frac{\alpha}{\beta}\left(-\frac{c_1}{a}\right),\ A + \frac{c_1}{a}\left(\frac{\gamma}{\delta} + l\right),\ -\frac{c_2}{a} - \frac{bc_1^2}{2a^2}\right],$$

$$q_1 = \max\left[\frac{\alpha}{\beta}\left(-\frac{c_1}{a}\right),\ A + \frac{c_1}{a}\left(\frac{\gamma}{\delta} + l\right),\ -\frac{c_2}{a} - \frac{bc_1^2}{2a^2}\right].$$

The mentioned solution will be denoted as $y(c_1, c_2)$. Some of its properties are given in the following lemmas.

Lemma 2. *If $c_1'x + c_2' > c_1x + c_2$ on $\langle 0, l \rangle$ and*

$$c = \max_{x \in \langle 0, l \rangle} \frac{1}{a}[c_1'x + c_2' - (c_1x + c_2)], \tag{15}$$

then

$$y(c_1, c_2) - c \leqq y(c_1', c_2') < y(c_1, c_2) \text{ on } \langle 0, l \rangle. \tag{16}$$

Proof. By Theorem 2 and by the uniqueness of $y(c_1', c_2')$, there is a $c > 0$ such that $y(c_1, c_2) - c \leqq y(c_1', c_2') \leqq y(c_1, c_2)$ on $\langle 0, l \rangle$. A direct calculation gives (15). The strict inequality in (16) can be easily shown.

Lemma 3. *For any two numbers c_1, d there exist uniquely determined numbers c_2', c_2'' such that $y(c_1, c_2')(0) = d = y(c_1, c_2'')(l)$.*

Proof. (16) implies that the functions $y(c_1, c_2)(0)$, $y(c_1, c_2)(l)$ of the variable c_2 are continuous and decreasing in c_2. With respect to this and to the comparison properties of $y(c_1, c_2)$ only the unboudedness of $y(0, c_2)(0)$, $y(0, c_2)(l)$ has to be proved. Using some properties of $y(0, c_2)$ we get the following statements. First, (14) implies the inequalities

$$y(0, c_2) > 0 \quad \text{for all} \quad c_2 < 0, \text{ and } y(0, c_2) < A \quad \text{for all} \quad c_2 > 0, \\ \text{on } \langle 0, l \rangle. \tag{17}$$

Secondly, if one of the functions $y(0, c_2)(0)$, $y(0, c_2)(l)$ is bounded, there exists a linear function $M + Nx$ such that

$$|y(0, c_2)(x)| < M + Nx \qquad (x \in \langle 0, l \rangle) \tag{18}$$

for all c_2 sufficiently great in absolute value. At the same time

$$y'^2(0, c_2) < M_1 \quad \text{on} \quad \langle 0, l \rangle \tag{19}$$

is true since on both intervals of monotonicity of $y(0, c_2)$ the function $v(0, c_2) = y'^2(0, c_2)/2$ satisfies the d. equation $\dfrac{dv}{dy} = abv + (ay + c_2)$. When e.g. $c_2 < 0$ is sufficiently great in absolute value, then $\dfrac{dv(0, c_2)}{dy} < 0$ and hence $v(0, c_2)$ is smaller than $v[y(0, c_2)(0)]$ or $v[y(0, c_2)(l)]$ which are bounded. From (18) and (19) it follows that for any $K < 0$ there is a $c_2 < 0$ such that $y''(0, c_2) < K$ on $\langle 0, l \rangle$. This gives a contradiction with (17).

Lemma 4. *When $y''(c_1, c_2) \geqq 0$ in $\langle 0, l \rangle$, then*

$$y(c_1, c_2)(0) \leqq \frac{\alpha}{\beta} y(c_1, c_2)(l) \Big/ \left(l + \frac{\alpha}{\beta}\right),$$

$$y(c_1, c_2)(l) \leqq \frac{\gamma}{\delta} y(c_1, c_2)(0) \Big/ \left(l + \frac{\gamma}{\delta}\right) + A\left[\left(l + \frac{\alpha}{\beta}\right) \times \right. \\ \left. \times \left(l + \frac{\gamma}{\delta}\right) - \frac{\alpha}{\beta}\frac{\gamma}{\delta}\right] \Big/ \left[\left(\frac{\alpha}{\beta} + l + \frac{\gamma}{\delta}\right)\left(l + \frac{\gamma}{\delta}\right)\right].$$

If $y''(c_1, c_2) \leqq 0$ *in $\langle 0, l \rangle$, then*

$$y(c_1, c_2)(0) \geqq \frac{\alpha}{\beta} y(c_1, c_2)(l) \Big/ \left(l + \frac{\alpha}{\beta}\right)$$

and

$$y(c_1, c_2)(l) \geqq \frac{\gamma}{\delta} y(c_1, c_2)(0) \Big/ \left(l + \frac{\gamma}{\delta}\right).$$

Proof. The corresponding homogeneous BVP to (5), (6) has only the trivial solution, therefore there exists its Green's function $G = G(x, t)$. A direct calculation yields that $G(0, t) \leqq \frac{\alpha}{\beta} G(l, t) \Big/ \left(l + \frac{\alpha}{\beta}\right)$ and $G(l, t) \leqq \frac{\gamma}{\delta} G(0, t) \Big/ \left(l + \frac{\gamma}{\delta}\right)$ for $0 \leqq t \leqq l$. From these two inequalities the lemma follows.

5. By means of the foregoing lemmas the BVP (1), (2) in the first case is solved in

Theorem 4. *Let $Fab - BD = 0$, $\beta = \delta$, $Fa - BC = \frac{1}{\beta}$. Then for each number y_0 there exists a unique solution y_{y_0} of the BVP (1), (2) such that $y_{y_0}(0) = y_0$. The solutions y_{y_0} possess the following properties: When $y_0 < y_1$, then $y_{y_0} < y_{y_1}$ on $\langle 0, l\rangle$ and y_{y_0} continuously depend on y_0. Further there is a unique c_0, $0 < c_0 < A$, such that for $y_0 > c_0$ or for $y_0 < 0$ there exists a number $d_0 = d_0(y_0)$, $0 < d_0 < l$, for which the following is true:*

When $y_0 > c_0$, then $y'_{y_0}(x) > 0$ for $0 \leqq x < d_0$ and $y'_{y_0}(x) < 0$ for $d_0 < x \leqq l$. If $y_0 < 0$, then $y'_{y_0}(x) < 0$ for $0 \leqq x < d_0$ as well as $y'_{y_0}(x) > 0$ for $d_0 < x \leqq l$.

When $0 \leqq y_0 \leqq c_0$, then $y'_{y_0}(x) \geqq 0$ on $\langle 0, l\rangle$ and there is at most one zero point of y'_{y_0}.

Proof. The first part of the theorem, concerning the existence of y_{y_0}, its monotony and continuity property follows from Lemmas 1, 2 and 3. By (14), $0 < y(0, 0)(0) \leqq$ $\leqq y(0, 0)(x) \leqq y(0, 0)(l) < A$, $0 < y'(0, 0)(x)$, $0 < y''(0, 0)(x)$ on $\langle 0, l\rangle$. Hence y_{y_0}, $y_0 < 0$, behaves according to the statement of the theorem. Further there is a $c_{20} > 0$ such that $y_0 = y(0, c_{20})$ and $y''(0, c_{20}) > 0$, $y'(0\ c_{20}) \geqq 0$ on $\langle 0, l\rangle$ with the only zero-point at 0. By Lemma 3, there is a $c_{21} < 0$ such that $y(0, c_{21})(l) =$ $= A$, and by the properties of $y(0, c_2)$ it follows that $y''(0, c_{21}) < 0$, $y'(0, c_{21}) \geqq 0$ on $\langle 0, l\rangle$, the only zero-point being at l. When $c_0 = y(0, c_{21})$ is put, y_{y_0}, $y_0 > c_0$, shows the properties mentioned in the theorem. In the case $c_{21} < c_2 < c_{20}$ $y(0, c_2)$ has no local extrema and thus, it is increasing in $\langle 0, l\rangle$.

In the case 2 (3) each solution to (5′), (7) ((5′), (8)) also satisfies the BVP (5′), (6) and the solutions $y(0, c_{21})$, $y(0, c_{20})$ are the unique solutions of the mentioned problems, so we have the following

Corollary. *Let* (4) *be satisfied, $\beta \neq \delta$, $Fa - BC = \frac{1}{\beta}$ $\left(Fa - BC = \frac{1}{\delta}\right)$. Then there exists one and only one solution y of the BVP* (1), (2). *y satisfies the inequalities $y \geqq 0$, $y' \geqq 0$, $y'' \leqq 0$ ($y \geqq 0$, $y' \geqq 0$, $y'' \leqq 0$) on $\langle 0, l\rangle$, the only zero-point of y' being at l (at 0).*

Consider now the last case. By Lemma 3, for any c_1 there is a unique $c_2 = c_2(c_1)$ such that

$$y(c_1, c_2(c_1))(0) = \frac{\alpha F c_1}{1 - \beta(Fa - BC)}. \tag{20}$$

$y(c_1, c_2(c_1))$ is continuous on $R \times \langle 0, l \rangle$ and it satisfies the *d.* equation (5) with $c_2 = c_2(c_1)$, and the first half of the conditions (9). The second half will be satisfied (for a c_1) iff c_1 is a root of the equation $S(c_1) = T(c_1)$, where

$$S(c_1) = y(c_1, c_2(c_1))(l) \quad (-\infty < c_1 < +\infty) \tag{21}$$

and

$$T(c_1) = A - \frac{\gamma F c_1}{1 - \delta(Fa - BC)} \qquad (-\infty < c_1 < +\infty). \tag{22}$$

The function S is continuous. Its further properties are given by

Lemma 5. *When* $1 - \beta(Fa - BC) < 0$, *S is decreasing. If* $1 - \beta(Fa - BC) > 0$ *and*

$$l \leqq \frac{a\alpha F}{1 - \beta(Fa - BC)}, \tag{23}$$

then S is increasing.

Proof. Suppose $c_1 > c_1'$. The difference $u = y(c_1, c_2(c_1)) - y(c_1', c_2(c_1'))$ satisfies

$$u'' - p(x)\, u' - au = H(x) \tag{24}$$

$$u(0) = \frac{\alpha F}{1 - \beta(Fa - BC)}(c_1 - c_1'), \quad u'(0) = \frac{\beta F}{1 - \beta(Fa - BC)}(c_1 - c_1'), \tag{25}$$

$$\frac{\gamma}{\delta} u'(l) + u(l) = 0$$

where $p(x) = \dfrac{ab}{2}\,[y'(c_1, c_2(c_1))(x) + y'(c_1', c_2(c_1'))(x)]$, $H(x) = (c_1 - c_1')\,x + c_2(c_1) - c_2(c_1')$, $(x \in \langle 0, l \rangle)$.

When $1 - \beta(Fa - BC) < 0$, by (25), u must attain a negative local minimum at ξ, $0 < \xi < l$, and by (24) it has no local maximum in $(\xi, l\rangle$. Thus $u(l) < 0$.

If $1 - \beta(Fa - BC) > 0$, then there is a positive local maximum of u at a point $\lambda \in (0, l)$. Since $au(\lambda) + H(\lambda) \leqq 0$, by (23), we get $H(l) \leqq 0$ and thus $H \leqq 0$ in $\langle 0, l \rangle$. If $u(\mu) = 0$, $\lambda < \mu < l$, then there exists a non-positive local minimum of u at ν, $\mu < \nu < l$. But $au(\nu) + H(\nu) < 0$, what gives a contradiction. When $u(l) = 0$, and hence $u'(l) = 0$, the comparing u with the solution v of the BVP

$$v'' - p(x)\, v' - av = 0 \quad (x \in \langle 0, l \rangle) \tag{26}$$

$$v(0) = u(0), \quad v(l) = u(l)$$

gives that $u \geqq v$ in $\langle 0, l \rangle$ and hence $v'(l) > 0$, from where the existence of two zeros of v follows. This contradicts the disconjugacy of (26). The proof is complete.

From the last lemma the next theorem follows immediately.

Theorem 5. *Let $Fab - BD = 0$ and let either a) $1 - \beta(Fa - BC) > 0$, $1 - \delta(Fa - BC) > 0$, $l \leqq \alpha\alpha F/[1 - \beta(Fa - BC)]$ or b) $1 - \beta(Fa - BC) < 0$, $1 - \delta(Fa - BC) < 0$. Then there exists a unique solution y of the BVP (1), (2). In the case a) the corresponding associated function z to that solution is positive, while in the case b) z is either positive or its only local extremum is a local minimum. When 1 is sufficiently small, this minimum is positive.*

Proof. In the proof of Theorem 4 the inequalities $0 < S(0) < A$ were shown. Since S and T are continuous, $T(0) = A$, Lemma 5 implies the first part of the theorem. The associated function $z = (Fa - BC)\,y' + Fc_1$ of the established solutions satisfies the BVP

$$z'' = az - \frac{abFc_1}{Fa - BC} z' + \frac{ab}{Fa - BC} zz' - BCc_1, \tag{27}$$

$$z(0) = \frac{Fc_1}{1 - \beta(Fa - BC)}, \qquad z(l) = \frac{Fc_1}{1 - \delta(Fa - BC)}. \tag{28}$$

(28) imply that $z(0) > 0$, $z(l) > 0$. In the case a) from (27) it follows that z cannot *have a non-positive local minimum and hence, $z > 0$ in $\langle 0, l \rangle$. In the case b) z may have a local minimum. Carrying on some considerations we get that this minimum is positive, when l is sufficiently small.*

Theorem 6. *Let $Fab - BD = 0$, $1 - \beta(Fa - BC) < 0$, $1 - \delta(Fa - BC) > 0$. Let, further*

$$\frac{\gamma}{1 - \delta(Fa - BC)} > -\left(1 + \frac{l\beta}{\alpha}\right)\frac{1}{1 - \beta(Fa - BC)} \tag{29}$$

or

$$\frac{\gamma}{1 - \delta(Fa - BC)} < -\frac{\gamma}{\delta}\,\frac{\alpha}{1 - \beta(Fa - BC)}\,\frac{\alpha}{l + \dfrac{\gamma}{\delta}}. \tag{30}$$

Then there exists at least one solution y of the BVP (1), (2). In the former case (the latter case) the associated function z shows the properties: $z(0) < 0$, $z(l) > 0$, $z'(x) \geqq 0$ for all x in $\langle 0, l \rangle$($z(0) > 0$, $z(l) < 0$, $z'(x) \leqq 0$ in $\langle 0, l \rangle$).

Proof. When $c_1 > 0$, the investigation of $y(c_1, c_2(c_1))$ as well as of $y'(c_1, c_2(c_1))$ gives that $y''(c_1, c_2(c_1)) \geqq 0$ in $\langle 0, l \rangle$ and hence, Lemma 4 can be applied. Thus, when (29) is true, there is a $c_1 > 0$ such that $S(c_1) > T(c_1)$ what establishes the existence of a solution to the BVP (1), (2). (28) guarantee the inequalities $z(0) < 0$, $z(l) > 0$ and $z' = (Fa - BC)\,y'' \geqq 0$. If $c_1 < 0$ is sufficiently great in absolute value, one can prove that $y''(c_1, c_2(c_1)) \leqq 0$ in $\langle 0, l \rangle$ and again, with a help of Lemma 4, on the basis of (30), the result can be proved.

When $1 - \beta(Fa - BC) > 0$, $1 - \delta(Fa - BC) < 0$, then the situation is a little more complicated. Still the methods from the last proof are efficient to prove

Theorem 7. *Let* $Fab - BD = 0$, $1 - \beta(Fa - BC) > 0$, $1 - \delta(Fa - BC) < 0$. *Let, further,*

$$\frac{\gamma}{1 - \delta(Fa - BC)} < -\left(1 + \frac{l\beta}{\alpha}\right)\frac{\alpha}{1 - \beta(Fa - BC)},$$

or

$$\frac{\gamma}{1 - \delta(Fa - BC)} >$$

$$> \max\left\{-\frac{\gamma}{\delta}\alpha\Big/\left[\left(l + \frac{\gamma}{\delta}\right)[1 - \beta(Fa - BC)]\right],\ -\frac{\gamma}{\delta}\frac{\beta}{1 - \beta(Fa - BC)}\right\}.$$

Then there exists at least one solution of the BVP (1), (2). *In the former* (*the latter*) *case the associated function z to the mentioned solution shows the properties* $z(0) < 0$, $z(l) > 0$ *and either* $z' \geqq 0$ *on* $\langle 0, l \rangle$ *or there is a subinterval* $\langle 0, l_1 \rangle$ *such that* $z' \geqq 0$ *on* $\langle 0, l_1 \rangle$ *and* $z' \leqq 0$ *in* $\langle l_1, l \rangle$ ($z(0) > 0$, $z(l) < 0$, *and either* $z' \leqq 0$ *on* $\langle 0, l \rangle$ *or there is a subinterval* $\langle 0, l_1 \rangle$ *such that* $z' \leqq 0$ *in* $\langle 0, l_1 \rangle$ *and* $z' \geqq 0$ *in* $\langle l_1, l \rangle$).

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