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SOME DIFFERENTIAL EQUATIONS WITH DELAY

by C. CORDUNEANU

During the last few years, a good deal of research activity has been concentrated on the investigation of a class of difference-integral operators formally given by

$$(Ax)(t) = \sum_{j=0}^{\infty} A_j x(t - t_j) + \int_0^t B(t - s)\, x(s)\, \mathrm{d}s, \qquad t \in R_+ . \tag{1}$$

The following two hypotheses are usually assumed with respect to A:

$$\sum_{j=0}^{\infty} \| A_j \| < +\infty, \tag{2}$$

and

$$\| B(t) \| \in L(R_+, R), \tag{3}$$

where $\| . \|$ denotes the euclidean norm of a (square) matrix and L stands for the space of Lebesgue integrable functions. It is also assumed that the operator A acts on certain vector-function spaces whose elements are defined on the positive half-axis R_+ and take the values in R^n (the euclidean space of dimension n with the usual norm). The meaning of the symbol $(AX)(t)$, where X denotes a square matrix of order n, is obvious.

Two recent monographs contain a considerable amount of results related to the operators of the form (1) acting on various function spaces. The first one is a "pure mathematics" product (see I. C. GOCHBERG and I. A. FELDMAN [4]) while the second is dedicated to some applied topics and emphasizes the significance of these operators in the theory of feedback systems (see J. C. WILLEMS [6]). These monographs display consistent lists of references, though, there is no attempt to give a complete covering of the mathematical and engineering literature related to this subject.

The aim of this paper is to establish some stability results concerning the differential systems of the form

$$\dot{x}(t) = (Ax)(t), \tag{S_0}$$

or

$$\dot{x}(t) = (Ax)(t) + f(t), \tag{S}$$

or

$$\dot{x}(t) = (Ax)(t) + f(t; x), \tag{S_1}$$

all of them considered on the positive half-axis R_+ and under suitable initial conditions. In the system (S_1), $f(t; x)$ stands for an operator acting on convenient function spaces: $f(t; x) = (fx)(t)$, $t \in R_+$. A particular case of (S_1), namely

$$\dot{x}(t) = (Ax)(t) + b\varphi(\sigma), \sigma = \langle c, x \rangle, \tag{S_2}$$

where b, $c \in R^n$ and $\langle .,. \rangle$ denotes the scalar product, will be also investigated in view to obtain a criterion of absolute stability. As usual, we shall assume that φ is a map of R into itself.

A condition we shall assume throughout this paper concerns the sequence $\{t_j; j = 0, 1, 2, \ldots\}$. In order that A be a Volterra operator, or—using applied terminology—a causal operator, it is necessary to assume

$$t_j \geqq 0, j = 0, 1, 2, \ldots \tag{4}$$

The considerations we shall develop below are valid for both bounded or unbounded sequences. In other words, the infinite delays are allowed.

The first task we undertake is to construct a fundamental matrix-solution for the system (S_0). More precisely, we shall find a matrix-function $X(t)$ verifying

$$\dot{X}(t) = (AX)(t), \ t > 0, \tag{5}$$

and

$$X(t) = 0 \quad \text{for} \quad t < 0, \quad X(0+) = I, \tag{6}$$

where I denotes the unit matrix of order n and $X(0+)$ stands for the limit of $X(t)$ at the right of $t = 0$.

We shall construct the matrix $X(t)$ as being the unique solution of the integral equation

$$X(t) = I + \int_0^t (AX)(s) \, ds, \quad t \in R_+ \tag{7}$$

with $X(t) = 0$ for $t < 0$.

The proof of the existence of $X(t)$ can be obtained by the method of successive approximations applied to the equation (7), starting with $X_0(t)$ defined as follows: $X_0(t) = 0$ for $t < 0$ and $X_0(t) = I$ for $t \geqq 0$. We define then

$$X_k(t) = I + \int_0^t (AX_{k-1})(s) \, ds, \quad t \in R_+, \quad k \geqq 1, \tag{8}$$

with $X_k(t) = 0$ for $t < 0$. In order to be sure that (8) makes sense for all $k \geqq 1$, it suffices to remark that for any continuous matrix-function $X(t)$ and $t > 0$, we have

$$\| (AX)(t) \| \leqq M \sup_{u \leqq t} \| X(u) \|, \tag{9}$$

with

$$M = \sum_{j=0}^{\infty} \| A_j \| + \int_0^{\infty} \| B(s) \| \, ds. \tag{10}$$

Since $(AX)(t)$ is measurable for any $X(t)$ contiouous on R_+ and such that $X(t) = 0$ for $t < 0$, the inequality (9) implies $(AX)(t) \in L^\infty$, on any bounded interval of R_+. Therefore, each $X_k(t)$ is defined on R_+ and is absolutely continuous on any bounded interval of R_+. We obtain easily from (8) and (9)

$$\| X_{k+1}(t) - X_k(t) \| \leqq M \int_0^t (\sup_{u \leqq s} \| X_k(u) - X_{k-1}(u) \|) \, ds,$$

which gives for $t > 0$ and $k \geqq 1$

$$\sup_{u \leqq t} \| X_{k+1}(u) - X_k(u) \| \leqq M \int_0^t (\sup_{u \leqq s} \| X_k(u) - X_{k-1}(u) \|) \, ds. \tag{11}$$

By standard arguments, from the inequality (11) we obtain

$$\sup_{u \leqq t} \| X_{k+1}(u) - X_k(u) \| \leqq C(t) \frac{(Mt)^k}{k!}, \tag{12}$$

where $C(t) = \sup \| X_1(u) - X_0(u) \|$ for $u \leqq t$. From the inequality (12) we see that $\{X_k(t)\}$ converges uniformly on any bounded interval of R_+ to a continuous matrix-function $X(t)$. It is now a simple matter to show that $X(t)$ satisfies (7) and is the unique continuous solution of this equation. Since $X(t)$ is absolutely continuous on any bounded interval of R_+, there results that it satisfies a.e. the equation (5).

We shall prove now a result that we need in the sequel. It regards the integrability of $X(t)$ on R_+, but it can be also viewed as a result of asymptotic stability for the system (S_0).

First, let us associate with the operator A (see I. C. GOCHBERG and I. A. FELDMAN [4]) the matrix-function $\mathscr{A}(s)$ defined by

$$\mathscr{A}(s) = \sum_{j=0}^{\infty} A_j \exp(-t_j s) + \int_0^{\infty} B(t) \exp(-ts) \, ds, \qquad \operatorname{Re} s \geqq 0. \tag{13}$$

It appears in a natural way in connection with the Laplace transform of the function $(Ax)(t)$. If we assume, for instance, that $x \in L(R_+, R^n)$, then Ax is also integrable on R_+ and simple calculations show that

$$(\widetilde{Ax})(s) = \mathscr{A}(s) \tilde{x}(s), \quad Re\, s \geqq 0. \tag{14}$$

We agree to denote by $\tilde{x}(s)$ the Laplace transform of the function x.

Lemma 1. *Consider the matrix function $X(t)$ as constructed above and assume that conditions* (2), (3) *and* (4) *hold true. Moreover, if*

$$\det [sI - \mathscr{A}(s)] \neq 0 \text{ for } Re\, s \geqq 0, \tag{15}$$

then

$$\| X(t) \| \in L(R_+, R). \tag{16}$$

Proof. We shall prove first that $X(t)$ satisfies a convenient integral equation. Let $\Phi(t)$, $t \in R_+$, be a matrix-function of type n by n, satisfying the following conditions: 1) $\Phi(t)$ is absolutely continuous on R_+ and $\| \Phi(t) \|$, $\| \dot{\Phi}(t) \| \in L(R_+, R)$; 2) $\Phi(t) = 0$ for $t < 0$ and $\Phi(0+) = I$; 3) $\tilde{\Phi}(s)$ is nonsingular for $Re\, s \geqq 0$; 4) $\Phi(t)$ commutes with any square matrix of order n. An example of such a matrix-function is given by $\Phi(t) = I \exp(-t)$ on R_+ and $\Phi(t) = 0$ for $t < 0$. If we multiply both sides of the equation

$$\dot{X}(u) = \sum_{j=0}^{\infty} A_j X(u - t_j) + \int_0^u B(u - v) X(v) \, dv, \qquad u > 0,$$

by $\Phi(t-u)$ and integrate with respect to u from 0 to t, we get after performing an integration by parts in the first member and some elementary transformatios in the second one

$$X(t) + \int_0^t K(t-u)\, X(u)\, \mathrm{d}u = \Phi(t), \quad t \in R_+, \tag{17}$$

where

$$K(t) = \dot{\Phi}(t) - (A\Phi)(t), t \in R_+. \tag{18}$$

According to our conditions, we have $\| K(t) \| \in L(R_+, R)$. We get further $\tilde{K}(s) = s\tilde{\Phi}(s) - I - \mathscr{A}(s)\tilde{\Phi}(s) = [sI - \mathscr{A}(s)]\tilde{\Phi}(s) - I$. Therefore, $I + \tilde{K}(s) = [sI - \mathscr{A}(s)]\tilde{\Phi}(s)$ is a nonsingular matrix for Re $s \geqq 0$. This implies (see R. K. MILLER [5], p. 207) the relation $[I + K(s)]^{-1} = I + K_1(s)$, where $K_1(s)$ is the Laplace transform of a certain matrix-function $K_1(t)$, with $\| K_1(t) \| \in L(R_+, R)$. Consequently, the solution of (17) can be expressed by means of the resolvent kernel $K_1(t)$ in the form

$$X(t) = \Phi(t) + \int_0^t K_1(t-u)\, \Phi(u)\, \mathrm{d}u, \quad t \in R_+. \tag{19}$$

From (19) we obtain (16) if we take into account that both $\| \Phi(t) \|$ and $\| K_1(t) \|$ belong to $L(R_+, R)$.

Lemma 1 is thus proved.

Remark 1. From (16) and (5) there results $\| \dot{X}(t) \| \in L(R_+, R)$, if we consider that $\| (AX)(t) \| \in L(R_+, R)$. Under the additional condition that $B(t)$ is absolutely continuous and $\| \dot{B}(t) \| \in L(R_+, R)$, we get also $\| \ddot{X}(t) \| \in L(R_+, R)$.

Remark 2. We have already mentioned that Lemma 1 can be regarded as a stability result. Indeed, from (16) and $\| \dot{X}(t) \| \in L(R_+, R)$ one obtains $\| X(t) \| \to 0$ as $t \to \infty$. This shows that the zero solution of (S_0) is asymptotically stable. The precise meaning of this statement will become clear after investigating the initial value problems for the systems (S) and (S_1).

Let us consider now the system (S), with the functional-initial conditions

$$x(t) = h(t) \text{ for } t < 0 \text{ and } x(0+) = x^0 \in R^n, \tag{20}$$

where $h(t)$ is a given function with values in R^n. In the case when the sequence $\{t_j\}$ is bounded, it is enough to prescribe the values of $x(t)$ only for $t \in (-T, 0\rangle$, for a convenient $T > 0$. In order to unify the discussion of the problem, we agree to extend $h(t)$ at the whole negative half-axis, setting $h(t) = 0$ for $t < -T$.

The main result we have in view is to prove the variation of constant formula for the system (S), with the initial conditions (20).

Lemma 2. *Assume that (S) and h satisfy the following conditions*: 1) *the operator A is such that* (2), (3) *and* (15) *hold true*; 2) *$f(t)$ is a continuous map from R_+ into R^n*; 3) *$h(t)$ is a map from the negative half-axis R_- into R^n, such that*

$$h(t) \in L(R_-, R^n). \tag{21}$$

Then the unique solution of the system (S), defined on R_+ and satisfying the initial conditions (20), *is given by the formula*

$$x(t) = X(t)\, x^0 + Y(t; h) + \int_0^t X(t-u) f(u)\, du, \quad t \in R_+, \tag{22}$$

where the operator Y is defined by

$$Y(t; h) = (Yh)(t) = \sum_{j=0}^{\infty} \int_{-t_j}^{0} X(t - t_j - u)\, A_j h(u)\, du, \quad t \in R_+. \tag{23}$$

Proof. The existence and uniqueness follow easily by successive approximations. We have to find the solution in the form (22). From

$$\dot{x}(t) = \sum_{j=0}^{\infty} A_j x(t - t_j) + \int_0^t B(t-u)\, x(u)\, du + f(t),$$

we obtain by formal application of the Laplace transform

$$s\tilde{x}(s) - x^0 = \sum_{j=0}^{\infty} A_j \int_0^{\infty} x(t - t_j) \exp(-st)\, dt + \tilde{B}(s)\, \tilde{x}(s) + \tilde{f}(s).$$

But taking into account (20) we can write

$$\int_{-t_j}^{\infty} x(t - t_j) \exp(-st)\, dt = \exp(-st_j) \int_0^{\infty} x(t) \exp[-s(t - t_j)]\, dt =$$
$$= \exp(-st_j) \int_{-t_j}^{\infty} x(t) \exp(-st)\, dt = \exp(-st_j) \int_0^{\infty} x(t) \exp(-st)\, dt +$$
$$\exp(-st) \int_{-t_j}^{0} h(u) \exp(-su)\, du = \exp(-st_j)\, \tilde{x}(s) + \exp(-st_j) \int_{-t_j}^{0} h(u) \exp(-su)\, du.$$

We can write then

$$s\tilde{x}(s) - x^0 = \sum_{j=0}^{\infty} A_j \exp(-st_j)\, \tilde{x}(s) + \sum_{j=0}^{\infty} A_j \exp(-st_j) \int_{-t_j}^{0} h(u) \exp(-su)\, du +$$
$$+ \tilde{B}(s)\, \tilde{x}(s) + \tilde{f}(s),$$

from which we get

$$[sI - \mathscr{A}(s)]\, \tilde{x}(s) = x^0 + \sum_{j=0}^{\infty} A_j \exp(-st_j) \int_{-t_j}^{0} h(u) \exp(-su)\, du + \tilde{f}(s).$$

On the other hand, (5) and (15) yield

$$\tilde{X}(s) = [sI - \mathscr{A}(s)]^{-1}, \quad \operatorname{Re} s \geqq 0. \tag{24}$$

Therefore,

$$\tilde{x}(s) = \tilde{X}(s)\, x^0 + \sum_{j=0}^{\infty} \tilde{X}(s)\, A_j \exp(-st_j) \int_{-t_j}^{0} h(u) \exp(-su)\, du + \tilde{X}(s) \tilde{f}(s). \tag{25}$$

In order to obtain (22) from (25), it suffices to remark that

$$\tilde{X}(s)\, A_j \exp(-st_j) \int_{-t_j}^{0} h(u) \exp(-su)\, \mathrm{d}u = \int_{-t_j}^{0} \tilde{X}(s)\, A_j \exp[-s(t_j + u)]\, h(u)\, \mathrm{d}u =$$
$$= \int_{-t_j}^{0} \Big(\int_{0}^{\infty} X(t - t_j - u) \exp(-st)\, \mathrm{d}t \Big) A_j h(u)\, \mathrm{d}u =$$
$$= \int_{0}^{\infty} \Big(\int_{-t_j}^{0} X(t - t_j - u)\, A_j h(u)\, \mathrm{d}u \Big) \exp(-st)\, \mathrm{d}t.$$

It remains now to prove that (22) defines indeed the solution of (S), with the initial conditions (20).

First, it is obvious that $x_0(t) = X(t)\, x^0$, $x^0 \in R^n$, represents a solution of the homogeneous system (S_0), such that $x_0(t) \to x^0$ as $t \to 0_+$. According to the construction of $X(t)$, $x_0(t)$ corresponds to the initial function $h(t) = 0$ for $t < 0$.

Second, the vector-function

$$y(t) = (Yh)(t) = \sum_{j=0}^{\infty} \int_{-t_j}^{0} X(t - t_j - u)\, A_j h(u)\| \, \mathrm{d}u, \tag{26}$$

is also a solution of the homogeneous system (S_0), corresponding to the initial conditions

$$y(t) = h(t) \quad \text{for} \quad t < 0, \quad y(0_+) = 0. \tag{27}$$

Indeed, the boundedness of $X(t)$ and condition (21) allow us to write

$$\Big\| \int_{-t_j}^{0} X(t - t_j - u)\, A_j h(u)\, \mathrm{d}u \Big\| \leqq \| A_j \| (\sup \| X(t) \|) \int_{-\infty}^{0} \| h(u) \|\, \mathrm{d}u.$$

Consequently, the series occuring in the definition of $y(t)$ is uniformly convergent on R_+. One can see by a similar argument that the series obtained by formal differentiation from (26) is also uniform convergent. We have further

$$\lim \int_{-t_j}^{0} X(t - t_j - u)\, A_j h(u)\, \mathrm{d}u = 0, \quad t \to 0_+,$$

due to the fact that we can interchange the order of the limit sign and of the integral (the dominated convergence theorem applies). We can therefore state that $y(0_+) = 0$. The most relevant feature in this case consists in the fact that $y(t)$ is integrable on R_+. One obtains easily

$$\int_{0}^{\infty} \| y(t) \|\, \mathrm{d}t \leqq \Big(\sum_{j=0}^{\infty} \| A_j \| \Big) \Big(\int_{0}^{\infty} \| X(t) \|\, \mathrm{d}t \Big) \int_{-\infty}^{0} \| h(u) \|\, \mathrm{d}u.$$

Consequently, the Laplace transform considerations concerning the way of obtaining $y(t)$ are justified.

Finally, the last term in the right member of (22)

$$z(t) = \int_{0}^{t} X(t - u) f(u)\, \mathrm{d}u, \quad t \in R_+$$

gives a solution of (S), with $z(0_+) = 0$. We agree to consider $z(t)$ identically zero on the negative half-axis.

For any $t \geqq u \geqq 0$, we have

$$\dot{X}(t-u) = \sum_{j=0}^{\infty} A_j X(t-t_j-u) + \int_0^{t-u} B(v)\, X(t-u-v)\, \mathrm{d}v.$$

Multiplying both sides by $f(u)$ and integrating from 0 to t, we obtain

$$\int_0^t \dot{X}(t-u) f(u)\, \mathrm{d}u = \sum_{j=0}^{\infty} A_j \int_0^t X(t-t_j-u) f(u)\, \mathrm{d}u + \\ + \int_0^t \left(\int_0^{t-u} B(v)\, X(t-u-v)\, \mathrm{d}v \right) f(u)\, \mathrm{d}u.$$

The term by term integration of the series is allowed because we deal with a uniform convergence series on any bounded interval of R_+. Moreover, if we take into account that

$$\int_{t-t_j}^t X(t-t_j-u) f(u)\, \mathrm{d}u = 0, \quad j = 0, 1, 2, \dots$$

and change the order of integration in the double integral, we obtain

$$\int_0^t \dot{X}(t-u) f(u)\, \mathrm{d}u = \sum_{j=0}^{\infty} A_j z(t-t_j) + \int_0^t B(u)\, z(t-u)\, \mathrm{d}u.$$

On the other hand, $z(t)$ is absolutely continuous on any bounded interval of R_+. This easily follows from the absolute continuity of $X(t)$. Furthermore, elementary considerations show that

$$\lim_{u \to 0+} \frac{z(t+u) - z(t)}{u} = f(t) + \int_0^t \dot{X}(t-v) f(v)\, \mathrm{d}v,$$

at any $t \in R_+$. Consequently, we can write for almost all t in R_+

$$\dot{z}(t) - f(t) = \int_0^t \dot{X}(t-u) f(u)\, \mathrm{d}u.$$

Comparing the two expressions we have obtained for $\int_0^t \dot{X}(t-u) f(u)\, \mathrm{d}u$, there results

$$\dot{z}(t) = \sum_{j=0}^{\infty} A_j z(t-t_j) + \int_0^t B(u)\, z(t-u)\, \mathrm{d}u + f(t),$$

for almost all $t \in R_+$.

Summing up the above considerations, one obtains that $x(t) = x_0(t) + y(t) + z(t)$, as given by (22), represents the solution of the system (S) with the initial conditions (20).

Lemma 2 is thereby proved.

We can pass now to the investigation of the nonlinear system (S_1). If we are interested in finding the solution of (S_1) under initial conditions (20), then the problem can be reduced to the following nonlinear integral system:

$$x(t) = X(t)\,x^0 + (Yh)\,(t) + \int_0^t X(t-u)f(u;x)\,\mathrm{d}u, \quad t \in R_+, \tag{28}$$

The space $C_0 = C_0(R_+, R^n)$ will be chosen as underlying space. It consists of all continuous maps from R_+ into R^n, such that $\| x(t) \|$ approaches zero as $t \to \infty$. The norm is that induced by the space C of all continuous and bounded maps from R_+ into R^n: $| x |_C = \sup \| x(t) \|$ for $t \in R_+$. This choice is motivated by the fact that it appears naturally in connection with the asymptotic stability.

The following Poincaré–Liapunov type stability theorem can be easily obtained by means of the contraction mapping principle:

Theorem 1. *Consider the system* (S_1) *under the following conditions*: 1) A *satisfies* (2), (3) *and* (15); 2) h *satisfies condition* (21); 3) *the map* $x \to fx$, *from the ball* $\sum = \{x; x \in C_0(R_+, R^n), | x |_C \leqq r\}$ *into* C_0, *is such that* $f(t; 0) = 0$ *on* R_+ *and*

$$|fx - fy|_C \leqq m\,| x - y |_C. \tag{29}$$

Then there exists in Σ *a unique solution of* (S_1), *corresponding to the initial conditions* (20), *as soon as* $\| x^0 \|$, $| h |_L$ *and* m *are sufficiently small.*

Proof. We consider the following operator from Σ into $C_0(R_+, R^n)$:

$$(Tx)\,(t) = X(t)\,x^0 + (Yh)\,(t) + \int_0^t X(t-u)f(u;x)\,\mathrm{d}u, \quad t \in R_+. \tag{30}$$

As pointed out in the Remark 1 to Lemma 1, we have $\lim \| X(t) \| = 0$ as $t \to \infty$. It has been proved in Lemma 2 that $Yh \in L(R_+, R^n)$. Hence, $(Yh)^{\cdot} \in L(R_+, R^n)$ inasmuch as Yh is also a solution of (S_0). It follows then that $\lim \| (Yh)\,(t) \| = 0$ as $t \to \infty$, for any initial function satisfying (21). Consequently, $X(t)\,x^0 + (Yh)\,(t) \in C_0(R_+, R^n)$. Since C_0 is invariant with respect to the convolution operator with integrable kernel (see, for instance, [2]), there results that the last term in the right member of (30) belongs also to C_0. Therefore, $Tx \in C_0$, for any $x \in \Sigma$. If $\| x^0 \|$, $| h |_L$ and m are sufficiently small, then $T\Sigma \subset \Sigma$. Indeed, the following inequalities hold true:

$$\| X(t)\,x^0 \| \leqq (\sup \| X(t) \|)\,\| x^0 \|, \quad t \in R_+,$$

$$\| (Yh)\,(t) \| \leqq \left(\sum_{j=0}^{\infty} \| A_j \|\right)(\sup \| X(t) \|)\,| h |_L, \quad t \in R_+,$$

$$\left\| \int_0^t X(t-u)f(u;x)\,\mathrm{d}u \right\| \leqq mr \int_0^{\infty} \| X(t) \|\,\mathrm{d}t, \quad t \in R_+.$$

Moreover, T is a contraction mapping on Σ as soon as $m(\int_0^\infty \| X(t) \| \, \mathrm{d}t) < 1$. We have for $x, y \in \Sigma$

$$\| (Tx)(t) - (Ty)(t) \| \leqq \int_0^t \| X(t-u) \| \, \| f(u; x) - f(u; y) \| \, \mathrm{d}u \leqq$$

$$\leqq m \, | x - y |_C \int_0^t \| X(t-u) \| \, \mathrm{d}u \leqq m(\int_0^\infty \| X(t) \| \, \mathrm{d}t) \, | x - y |_C,$$

and taking the supremum in the first member we obtain

$$| Tx - Ty |_C \leqq m(\int_0^\infty \| X(t) \| \, \mathrm{d}t) \, | x - y |_C .$$

Theorem 1 is thus proved.

Remark. If condition 3) of Theorem 1 is replaced by the following one: the map $x \to fx$ from $\Sigma_1 = \{x; x \in C(R_+, R^n), \ | x |_C \leqq r\}$ into $C(R_+, R^n)$ is such that $| fx - fy |_C \leqq m | x - y |_C$ for any $x, y \in \Sigma_1$, then a boundedness result can be obtained using the same kind of arguments. It is also necessary to assume that $| f(t; 0) |_C$ is sufficiently small.

Another stability result we want to establish is concerned with systems of the form (S_2). They arise in the study of feedback systems and the absolute stability is the concept we shall deal with.

The system (S_2), with the initial conditions (20), can be reduced by means of variation of constants formula to the nonlinear scalar equation

$$\sigma(t) = \langle c, X(t) \, x^0 \rangle + \langle c, (Yh)(t) \rangle + \int_0^t \langle c, X(t-u) \, b \rangle \, \varphi(\sigma(u)) \, \mathrm{d}u, \tag{31}$$

whose kernel $k(t) = \langle c, X(t) \, b \rangle$ is integrable on R_+ (see Lemma 1). This feature is particularly adequate in view of application of frequency stability criteria (see, for instance, [2]).

Theorem 2. *Assume that the following conditions hold with respect to the system* (S_2) *and the initial conditions* (21): 1) A *satisfies* (2), (3) *and* (15); 2) h *satisfies* (21) *and* $x^0 \in R^n$; 3) b *and* c *are constant vectors from* R^n; 4) *the mapping* $\sigma \to \varphi(\sigma)$ *of* R *into itself is continuous, bounded and such that* $\sigma\varphi(\sigma) > 0$ *for* $\sigma \neq 0$; 5) *there exists* $q \geqq 0$, *such that*

$$\operatorname{Re} \{(1 + i\omega q) \langle c, [i\omega I - \mathscr{A}(i\omega)]^{-1} b \rangle\} \leqq 0, \tag{32}$$

for any real ω. *Then, there exists at least one solution* $x(t) \in C_0$ *of our problem. Moreover, any continuous (on* R_+*) solution belongs necessarily to* C_0.

Proof. We shall apply Theorem 3.2.2 in [2]. If we denote

$$f(t) = \langle c, X(t) \, x^0 \rangle + \langle c, (Yh)(t) \rangle,$$

then (31) can be written as

$$\sigma(t) = f(t) + \int_0^t k(t-u)\,\varphi(\sigma(u))\,\mathrm{d}u, \quad t \in R_+ . \tag{33}$$

The following conditions are obviously satisfied: $\| f(t) \|, \| \dot{f}(t) \|, \| k(t) \|, \| \dot{k}(t) \| \in L(R_+, R)$. Conditions 4) and 5) of the statement are nothing else but restatements of the corresponding conditions of Theorem 3.2.2 in [2]. Hence, equation (33) has at least one solution in $C_0(R_+, R^n)$, no matter how we choose $h \in L(R_-, R^n)$, $x^0 \in R^n$. We have from the variation of constants formula

$$x(t) = X(t)\,x^0 + (Yh)(t) + \int_0^t X(t-u)\,b\varphi(\sigma(u))\,\mathrm{d}u, \quad t \in R_+ ,$$

from which we get $x(t) \in C_0$ because $\varphi(\sigma(t))$ belongs to C_0.

The proof of Theorem 2 is now complete.

The system (S_2) constitutes only an example when the method used above is applicable. Related systems could be also investigated in the same manner. Let us remark that the particular case

$$(Ax)(t) = A_0 x(t) + \int_0^t B(t-u)\,x(u)\,\mathrm{d}u \tag{34}$$

has been widely discussed in [3].

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REFERENCES

[1] R. Bellman and K. L. Cooke. *Differential-Difference Equations.* New York and London, Academic Press, 1963.

[2] C. Corduneanu. *Integral Equations and Stability of Feedback Systems.* (Academic Press — In print)

[3] C. Corduneanu: *Absolute stability of some integro-differential systems. Ordinary Differential Equations,* 1971 NRL — MRC Conference. New York and London, Academic Press, 1972.

[4] I. C. Gochberg and I. A. Feldman: *Convolution Equations and Projection Methods for their Solution* (Russian). Moscow, Nauka, 1971.

[5] R. K. Miller. *Nonlinear Volterra Integral Equations.* Menlo Park, W. A. Benjamin, 1971.

[6] J. C. Willems: *The Analysis of Feedback Systems. Cambridge, Mass.,* M. I. T. Press, 1971.

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