Svatopluk Fučík; Jindřich Nečas

Spectral theory of nonlinear operators

In: Miloš Ráb and Jaromír Vosmanský (eds.): Proceedings of Equadiff III, 3rd Czechoslovak Conference on Differential Equations and Their Applications. Brno, Czechoslovakia, August 28 - September 1, 1972. Univ. J. E. Purkyně - Přírodovědecká fakulta, Brno, 1973. Folia Facultatis Scientiarum Naturalium Universitatis Purkynianae Brunensis. Seria Monographia, Tomus I. pp. 163--174.

Persistent URL: http://dml.cz/dmlcz/700074

SPECTRAL THEORY OF NONLINEAR OPERATORS

by SVATOPLUK FUČÍK, JINDŘICH NEČAS

Introduction. It is our object to give in the present paper a survey of some works from nonlinear functional analysis and its applications to the partial differential equations. The plan is as follows:

1. Fredholm alternative for nonlinear operators.
2. Ljusternik-Schnirelmann theory, i.e. the existence at least a countable set of eigenvalues.
3. Morse–Sard theorem in infinite dimensional Banach spaces and converse of Ljusternik–Schnirelmann theory, i.e. the existence at most a countable set of eigenvalues.

Applications of the abstract results from Sections 1 and 2 to the existence of the solution of boundary value problems for general partial differential elliptic systems is possible. For the sake of simplicity, we shall restrict ourselves to the second order equations. To apply the results of Section 3 to the partial differential equations, we require the weak solution is classical. This regularity property is known for the second order equations and remains as a big open problem for the higher order equations and systems solved on the domains in three and more dimensional spaces.

The abstract results of this paper is possible apply to the existence of the solution of the nonlinear integral equations of the Lichtenstein type (see [13]) and to the existence of the solution of the boundary value problem for nonlinear ordinary differential equations.

Because the method of Sections 2 and 3 are variational we shall restrict to the gradient operators, i.e. to the equations which are Euler's equations of some functionals.

1. FREDHOLM ALTERNATIVE

One of the main goals of nonlinear functional analysis achieved late 1960's are the results about equations involving monotone operators in the infinite dimensional Banach space. The mapping $T : X \to X^*$ (X^* the adjoint space) is said be *monotone* if $(Tx - Ty, x - y) \geqq 0$ for $x, y \in X$, where (x^*, x) is the value of the functional $x^* \in X^*$ at the point $x \in X$. The type of abstract results which were obtained about such mappings is the following theorem (see e.g. F. E. BROWDER [1]).

Theorem 1. *Let X be a reflexive Banach space, $T : X \to X^*$ be monotone, demicontinuous*: (1.1)

$$x_n \to x \text{ (strong convergence)} \Rightarrow Tx_n \rightharpoonup Tx \text{ (weak convergence)}, \tag{1.2}$$

coercive:

$$\lim_{\|x\| \to \infty} \frac{(Tx, x)}{\| x \|} = \infty. \tag{1.3}$$

Then the range of T is all of X^*.

A natural modification of the condition (1.1) is for bounded mappings (they map bounded sets in X onto bounded sets in X^*) so-called *condition* (S):

$$x_n \rightharpoonup x, \quad (Tx_n - Tx, x_n - x) \to 0 \Rightarrow x_n \to x. \tag{1.4}$$

We have, see also F. E. Browder [1]:

Theorem 2. *Let X be a separable reflexive Banach space, T a continuous, coercive, bounded mapping of X into X^*, satisfying the condition* (S).

Then the range of T is all of X^*.

The condition (1.3) is some positive definitness of the operator T. The following theorem (see e.g. J. Nečas [24, 25]) deals with the *weak coerciveness* of the operator T:

$$\lim_{\|x\| \to \infty} \| Tx \| = \infty. \tag{1.5}$$

Theorem 3. *Let T be a demicontinuous, bounded mapping defined on a reflexive Banach space X with the values in X^*. Suppose T satisfies the condition* (S), *T is weakly coercive and odd*:

$$T(-x) = -Tx. \tag{1.6}$$

Then the range of T is all of X^*.

The main idea of the proof of Theorem 3 for a separable space: if X is Euclidean n-space R_n, $f \in R_n$ and $r > 0$ is such a number that $Tx - tf \neq 0$ for $\| x \| = r$, $t \in \langle 0, 1 \rangle$, then for $t = 0$ by Borsuk's theorem the Brouwer degree of the mapping T with respect to the ball $B(0, r) = \{x: \| x \| < r\}$ and the point zero is an odd integer. By homotopy property this is true for $t = 1$, too. Hence there exists $x \in B(0, r)$ such that $Tx = f$. We have $X = \overline{\bigcup_{n=1}^{\infty} X_n}$, $\dim X_n = n$, $X_1 \subset X_2 \subset \dots$. Let ψ_n be the injection of X_n into X and ψ_n^* its dual mapping. Define $T_n : X_n \to X_n^*$ as $T_n x = \psi_n^* T \psi_n x$. Let $f \in X^*$. By contradiction we easy obtain the existence of $r > 0$ and positive integer n_0 such that

$$T_n x - t\psi_n^* f \neq 0$$

for $\| x \| = r$, $x \in X_n$, $n \geqq n_0$ and $t \in \langle 0, 1 \rangle$. Hence for $n \geqq n_0$ there exists $x_n \in X_n$, $\| x_n \| < r$ such that $(Tx_n - f, y) = 0$ for each $y \in X_n$ (Galerkin's method). The condition (S) with the colaboration of other conditions gives $x_n \to x$ and $Tx = f$.

Fredholm alternative for nonlinear operators deals with the solving of nonlinear operator's equation $\lambda Tx - Sx = f$ in the dependence of the real parameter λ, where T and S are mappings from X into X^*; we can replace X^* by an other real Banach space (see e.g. S. Fučík [5, 6]). Denote $A_\lambda = \lambda T - S$. Under suitable conditions for the operators T and S we can generalized Theorem 3 to the operator A_λ. This immediate generalization is done for the purpose to obtain "Fredholm alternative". In this alternative the asymptotes in infinity of the operators T and S play the essential role. Let us define at first: an operator $F_0 : X \to X^*$ is said to be *a-homogeneous*, $a > 0$, if $F_0(tu) = t^a F_0 u$ for each $u \in X$ and all $t \geqq 0$. An operator $F : X \to X^*$ has *a-asymptote* $F_0 : X \to X^*$ in infinity, if

$$\lim_{\|x\| \to \infty} \frac{\| Fx - F_0 x \|}{\| x \|^a} = 0 \tag{1.7}$$

and F_0 is *a-homogeneous*. Fredholm alternative says: if $\lambda \neq 0$ is not an eigenvalue for the couple (T_0, S_0) (i.e. $\lambda T_0 u - S_0 u = 0 \Rightarrow u = 0$), where T_0 and S_0 are *a-asymptotes* of T and S, respectively, then $\lambda T - S$ is onto. More precisely:

Theorem 4 (see J. NEČAS [24, 25]). *Let X be a real, reflexive and separable Banach space. Let the operators $T, S : X \to X^*$ have a-asymptotes T_0 and S_0, respectively. Suppose that T, T_0* are *bounded and demicontinuous, S, S_0 completely continuous (i.e. they are continuous and map bounded sets in X into compact sets in X^*). Let T_0, S_0 be odd and the operator $tT + (1 - t)\, T_0$ satisfy the condition (S) for each $t \in \langle 0, 1 \rangle$.*

Then $(\lambda T - S)(X) = X^*$ provided $\lambda \neq 0$ is not an eigenvalue for the couple (T_0, S_0). Moreover, if $\lambda \neq 0$ is not an eigenvalue for the couple (T_0, S_0), then there exists $c > 0$ such that

$$\| u \| \leqq c(1 + \| A_\lambda u \|)^{1/a} \tag{1.8}$$

for each $u \in X$. If (1.8) *is satisfied, then λ is not an eigenvalue for the couple (T_0, S_0).*

First results about Fredholm alternative for nonlinear operators were obtained independently by S. I. POCHOŽAJEV [27] and J. NEČAS [23]. Other generalizations were given by M. KUČERA [18], W. V. PETRYSHYN [26], P. HESS [15], J. R. L. WEBB [30], S. FUČÍK [5, 6], J. NEČAS [24, 25] and others.

Let Ω be a bounded domain in $R_n (n \geqq 2)$ with lipschitz boundary and let $1 < m < \infty$. Denote by $W_m^{(1)}(\Omega)$ the well-known Sobolev space of the functions which are together with the first derivatives in the space $L_m(\Omega)$ and by $\mathring{W}_m^{(1)}(\Omega)$ the subspace of $W_m^{(1)}(\Omega)$ of all functions with zero traces. Consider the real functions $a_i(x, u, p) \in C(\bar{\Omega} \times R_{n+1})$ $(i = 0, 1, \dots, n)$ and let us suppose the following conditions to be hold (we do not write the most general case):

$$\sum_{i=0}^{n} | a_i(x, u, p) | \leqq c(1 + | u | + | p |)^{m-1}, \tag{1.9}$$

$$\sum_{i=1}^{n} a_i(x, u, p)\, p_i \geqq c_1 | p |^m - c_2 | u |^m, \tag{1.10}$$

$$\sum_{i=1}^{n} \big(a_i(x, u, p') - a_i(x, u, p)\big)(p_i' - p_i) > 0 \tag{1.11}$$

for each $x \in \Omega$, $u \in R_1$, $p', p \in R_n$, $p \neq p'$.

Let $f_i \in L_{m^*}(\Omega)$ $(1/m + 1/m^* = 1)$ and let us look for the weak solution in $\mathring{W}_m^{(1)}(\Omega)$ of the equation

$$-\lambda \sum_{i=1}^{n} \frac{\partial}{\partial x_i} a_i(x, u, \operatorname{grad} u) + a_0(x, u, \operatorname{grad} u) = \sum_{i=1}^{n} \frac{\partial f_i}{\partial x_i} + f_0 . \tag{1.12}$$

Further suppose the existence of the functions $A_i(x, u, p) \in C(\bar{\Omega} \times R_{n+1})$ $(i = 0, 1, \dots, n)$ satisfying

$$A_i(x, tu, tp) = t^{m-1} A_i(x, u, p), \tag{1.13}$$

$$A_i(x, -u, -p) = -A_i(x, u, p), \tag{1.14}$$

$$\left| \frac{a_i(x, tu, tp)}{t^{m-1}} - A_i(x, u, p) \right| \leqq c(t)(1 + |u| + |p|)^{m-1} \tag{1.15}$$

for each $x \in \Omega$, $u \in R_1$, $p \in R_n$, $t > 0$, where $c(t) \to 0$ for $t \to \infty$ and suppose the validity of the relation (1.10) with A_i.

Theorem 5. *Under previous assumptions the equation* (1.12) *has a weak solution for each right hand side, provided the equation*

$$-\lambda \sum_{i=1}^{n} \frac{\partial}{\partial x_i} A_i(x, u, \operatorname{grad} u) + A_0(x, u, \operatorname{grad} u) = 0$$

has in $\mathring{W}_m^{(1)}(\Omega)$ the trivial solution only.

For the proof of Theorem 5 set

$$(Tu, v) = \sum_{i=1}^{n} \int_\Omega a_i(x, u, \operatorname{grad} u) \frac{\partial v}{\partial x_i} \, dx,$$

$$(Su, v) = \int_\Omega a_0(x, u, \operatorname{grad} u) \, v(x) \, dx,$$

$$(T_0 u, v) = \sum_{i=1}^{n} \int_\Omega A_i(x, u, \operatorname{grad} u) \frac{\partial v}{\partial x_i} \, dx,$$

$$(S_0 u, v) = \int_\Omega A_0(x, u, \operatorname{grad} u) \, v(x) \, dx$$

for each $u, v \in \mathring{W}_m^{(1)}(\Omega) = X$. We can see that the operators T, T_0, S, S_0 satisfy the conditions of Theorem 4. (For details see J. NEČAS [23].)

2. LJUSTERNIK–SCHNIRELMANN THEORY

Let $T, S : X \to X^*$. Denote by Λ the set of all eigenvalues for the couple (T, S), i.e. $\lambda \in \Lambda$ iff the equation $\lambda Tu - Su = 0$ has a nontrivial solution $u \in X$. Previous Section shows the importance of the investigation of the structure of the set Λ. The lower bound for the number of points in the set Λ is included in the following theorem which was proved in the paper [8].

Theorem 6. *Let f and g be two even functionals defined on a reflexive infinite-dimensional Banach space X with a Schauder basis. Suppose that there exist Fréchet derivative f' and g' on X of the functionals f and g. Let $r > 0$ and denote $M_r(f) = \{x \in X : f(x) = r\}$. Assume that the following assumptions are valid:*

$$f(x) = 0 \Leftrightarrow x = 0, \quad \lim_{\|x\| \to \infty} f(x) = \infty, \tag{2.1}$$

$$\inf_{x \in M_r(f)} (f'(x), x) \geqq c(r) > 0, \tag{2.2}$$

$$f' : X \to X^* \quad \textit{is a bounded operator}, \tag{2.3}$$

$$f' \textit{ satisfies the condition } (S) \textit{ (see (1.4))}, \tag{2.4}$$

$$g(x) \geqq 0 \quad \textit{for each} \quad x \in X, \quad g(x) = 0 \Leftrightarrow x = 0, \tag{2.5}$$

$$g'(x) = 0 \Leftrightarrow x = 0, \tag{2.6}$$

$$g' : X \to X^* \quad \textit{is a completely continuous mapping}, \tag{2.7}$$

$$x_n \rightharpoonup x, \quad g'(x_n) \to 0, \quad 0 < t_n < 1 \Rightarrow g'(t_n x_n) \to 0, \tag{2.8}$$

$$f' \textit{ and } g' \textit{ are uniformly continuous on each bounded set.} \tag{2.9}$$

Then for each positive integer k there exists $x_k \in M_r(f)$ such that

$$\frac{(g'(x_k), x_k)}{(f'(x_k), x_k)} \cdot f'(x_k) = g'(x_k) \tag{2.10}$$

and

$$g(x_k) = \gamma_k(r), \quad x_k \rightharpoonup 0, \tag{2.11}$$

where $\{\gamma_k(r)\}$ is a monotone sequence of positive numbers converging to zero.

The idea for the proof of Theorem 6 goes back to the well-known Courant–Weinstein minimax principle for the calculation of all eigenvalues of the positive, selfadjoint, linear, completely continuous operators in Hilbert spaces. In the nonlinear case we define the function ord which domain is a family of all closed subsets of a Banach space X and its range is the set of all nonnegative integers, as follows: If $K \subset X$ is a closed subset, $x \in K \Rightarrow -x \notin K$, then ord $K = 1$ and, inductively, ord $K = n$ iff $K = \bigcup_{i=1}^{n} K_i$, ord $K_i = 1$ and n is minimal positive integer with the previous property. For arbitrary integer k let $V_k(r)$ be a system of all compact sets $K \subset M_r(f)$ such that ord $K \geqq k$. Such system is invariant to the continuous and odd transformations of $M_r(f)$ into $M_r(f)$. The number $\gamma_k(r)$ is defined by the relation

$$\gamma_k(r) = \sup_{K \in V_K(r)} \inf_{x \in K} g(x). \tag{2.12}$$

If we consider the eigenvalue problem

$$\left.\begin{aligned} \lambda f'(u) &= g'(u) \\ u &\in M_r(f) \end{aligned}\right\} \tag{2.13}$$

then the solution $u \in M_r(f)$ is the *critical point* of the functional g with respect to the manifold $M_r(f)$. The value of the functional g at the critical point is called the *critical level.* Denote $\Gamma(r)$ the set of all critical levels. The assertion of Theorem 6 is that $\Gamma_0(r) = \{\gamma_k(r)\} \subset \Gamma(r)$ and thus the set $\Gamma(r)$ is at least countable. Denote $\Lambda(r)$ the set of all eigenvalues $\lambda \in R_1$ for which the eigenvalue problem (2.13) has a solution. Let

$$\Lambda_0(r) = \{\lambda_k\},$$

where

$$\lambda_k = \frac{(g'(x_k), x_k)}{(f'(x_k), x_k)}, \qquad k = 1, 2, \ldots,$$

(from the relation (2.10)). Note that if f is a-homogeneous and g is b-homogeneous, then

$$\begin{aligned} \Gamma(r) &= r^{b/a}\Gamma(1) = r^{b/a}(a/b)\,\Lambda(1), \\ \Gamma_0(r) &= r^{b/a}\Gamma_0(1) = r^{b/a}(a/b)\,\Lambda_0(1), \end{aligned}$$

for $u \in X \Rightarrow (f'(u), u) = af(u)$, $(g'(u), u) = bg(u)$. In this case from Theorem 6 it follows that the set of all "normalized eigenvalies" $\Lambda_0(1)$ is at least countable.

Analogous assertion to Theorem 6 was firstly proved by L. A. Ljusternik and L. G. Schnirelmann [21, 22] for X is a Hilbert space. For Hilbert spaces such a theorem is contained in the book of M. A. Krasnoselskij [17]. For Banach spaces such a theorem was proved by E. S. Citlanadze [3] and F. E. Browder [2]. They used the notion "*the category of the sets*" in the sense of L. Schnirelmann which is similar to "ord" or "*genus*" (see [17]).

The eigenvalue problem for Dirichlet boundary data and second order equation given in Section 1 is to find $u \in \mathring{W}_m^{(1)}(\Omega)$ such that

$$-\lambda \sum_{i=1}^{n} \frac{\partial}{\partial x_i} a_i(x, u, \operatorname{grad} u) + a_0(x, u, \operatorname{grad} u) = 0, \tag{2.14}$$

holds in the weak sense.

For (2.14) would be an Euler's equation we shall suppose there exists a continuous function $\mathscr{F}(x, u, p) \in C(\bar{\Omega} \times R_{n+1})$ with continuous partial derivatives up to the second order with respect to p_i, u and a function $\mathscr{G}(x, u) \in C(\bar{\Omega} \times R_1)$ with continuous first and second derivatives with respect to the variable u, such that

$$\mathscr{F}(x, 0, 0) = \partial\mathscr{F}/\partial p_i(x, 0, 0) = \partial\mathscr{F}/\partial u(x, 0, 0) = 0, \tag{2.15}$$

$$\mathscr{F}(x, -u, -p) = \mathscr{F}(x, u, p), \quad \mathscr{G}(x, -u) = \mathscr{G}(x, u), \tag{2.16}$$

$$c_2(1 + |p|)^{m-2} \sum_{i=0}^{n} \eta_i^2 \geqq \sum_{i,j=1}^{n} \frac{\partial^2 \mathscr{F}}{\partial p_i \partial p_j} \eta_i \eta_j +$$

$$+ 2 \sum_{i=1}^{n} \frac{\partial^2 \mathscr{F}}{\partial p_i \partial u} \eta_i \eta_0 + \frac{\partial^2 \mathscr{F}}{\partial u^2} \eta_0^2 \geqq c_1(1 + |p|)^{m-2} \sum_{i=1}^{n} \eta_i^2, \tag{2.17}$$

$$\sum_{i=1}^{n} \frac{\partial \mathscr{F}}{\partial p_i}(x, u, p)\, p_i + \frac{\partial \mathscr{F}}{\partial u}(x, u, p)\, u \geqq c_3 |p|^m, \tag{2.18}$$

$$\mathscr{G}(x, 0) = \frac{\partial \mathscr{G}}{\partial u}(x, 0) = 0, \tag{2.19}$$

$$c_4(1 + |u|)^{m-2} \geqq \frac{\partial^2 \mathscr{G}}{\partial u^2}(x, u) > 0 \qquad (\text{for } u \neq 0), \tag{2.20}$$

for each $x \in \Omega$, $u \in R_1$, $p \in R_n$ and $\eta \in R_{n+1}$.

Theorem 7. *Under assumption* (2.15) – (2.20) *Theorem* 6 *is applicable to the equation*

$$-\lambda \sum_{i=1}^{n} \frac{\partial}{\partial x_i}\left(\frac{\partial \mathscr{F}}{\partial p_i}(x, u, \operatorname{grad} u)\right) + \lambda \frac{\partial \mathscr{F}}{\partial u}(x, u, \operatorname{grad} u) = \frac{\partial \mathscr{G}}{\partial u}(x, u). \tag{2.21}$$

For the proof of Theorem 7 set

$$f(u) = \int_\Omega \mathscr{F}(x, u, \operatorname{grad} u)\, \mathrm{d}x,$$

$$g(u) = \int_\Omega \mathscr{G}(x, u)\, \mathrm{d}x$$

for each $u \in \mathring{W}_m^{(1)}(\Omega)$. The existence of a Schuder basis in the space $\mathring{W}_m^{(1)}(\Omega)$ is proved in the paper [7].

3. MORSE-SARD THEOREM IN INFINITE-DIMENSIONAL BANACH SPACES AND CONVERSE OF LJUSTERNIK–SCHNIRELMANN THEORY

Let the notations of previous Section be observed. It can be easy proved (Courant-Weinstein principle, see e.g. N. Dunford – J. T. Schwartz [4]) that in the linear case $\Gamma_0(1) = \Gamma(1)$ and thus $\Lambda_0(1) = \Lambda(1)$. In the nonlinear case (for instance in the case of homogeneous functionals) this is not generally true (see example in [9]). However, the situation is not so bad. Under certain conditions it can be proved again that the set $\Gamma(r) = \Gamma$ of all critical levels is a sequence of positive numbers converging to zero. This is the main goal of

Theorem 8 (see [10, 11, 13]). *Let X_1, X_2, X_3 be three real Banach spaces, let X_3 be reflexive and $X_1 \subset X_3$. Let the identity mapping from X_1 into X_3 be continuous.*

Suppose that $\langle .,. \rangle$ *is a bilinear form on* $X_1 \times X_2$ *continuous on* X_2 *for fixed* $h \in X_1$ *and such that the following implication holds:*

$$\langle h, x \rangle = 0 \quad \text{for each} \quad h \in X_1 \Rightarrow x = 0.$$

Let $f, g : X_3 \to R_1$ *be two functionals with Fréchet derivatives* f' *and* g' *on* X_3 *such that* f, g *are real-analytic on* X_1 *(in the sense of* E. HILLE – R. PHILIPS [16]). *Suppose that for each* $x \in X_1$ *there exists a couple* $F(x)$, $G(x) \in X_2$ *(if there exists at least one couple then it is unique) such that*

$$df(x, h) = \langle h, F(x) \rangle, \tag{3.1}$$

$$dg(x, h) = \langle h, G(x) \rangle \tag{3.2}$$

for each $h \in X_1$ *and let*

$$F, G : X_1 \to X_2 \text{ be the real-analytic operators on } X_1 \tag{3.3}$$

and

$$G : X_1 \to X_2 \text{ is completely continuous on } X_1. \tag{3.4}$$

Denote

$B_1 = \{x \in M_r(f) \cap X_1$: *there exists* $\lambda \in R_1$ *such that* $\lambda df(x, h) = dg(x, h)$ *for each* $h \in X_1\}$,

$B = \{x \in M_r(f)$: *there exists* $\lambda \in R_1$ *such that* $\lambda f'(x) = g'(x)\}$,

$B(\delta) = \{x \in M_r(f)$: *there exists* $\lambda \in R_1, |\lambda| \geqq \delta$, *such that* $\lambda f'(x) = g'(x)\}$.

If $x_0 \in B_1$ *and corresponding* $\lambda_0 \neq 0$ *(from the definition of the set* B_1) *let*

$$F'(x_0) = J + K, \tag{3.5}$$

where J *is an isomorphism from* X_1 *onto* X_2 *and* $K : X_1 \to X_2$ *is a completely continuous linear operator.*

Moreover, suppose

$$f' : X_3 \to X_3^* \text{ is continuous and bounded,} \tag{3.6}$$

$$f'(x) \neq 0 \quad \text{for} \quad x \in M_r(f), \tag{3.7}$$

$$M_r(f) \text{ is a bounded subset of } X_3, \tag{3.8}$$

$$\text{for each } \delta > 0 \text{ the set } B(\delta) \text{ is a compact subset of } X_1 \tag{3.9}$$

(*so-called the regularity assumption*),

$$g' : X_3 \to X_3^* \text{ is a completely continuous mapping, } g(0) = 0,\ g'(0) = 0, \tag{3.10}$$

$$u_n \rightharpoonup u, \quad g'(u_n) \to 0, \quad 0 < t_n < 1 \Rightarrow g'(t_n u_n) \to 0. \tag{3.11}$$

Then the set $g(B) - \{0\}$ *is isolated and the set* $\Gamma = g(B)$ *has only one possible limit point, namely zero.*

Theorem 8 is not lucid at the first sight for his much assumptions. This is the reason for the formulation of the following theorem for the case of Hilbert spaces.

Theorem 9 (see [9, 11]). *Let* X *be a real Hilbert space with the inner product* $(.,.)$. *Let us suppose*

$$f \textit{ is a real-analytic functional on } X, f(0) = 0, f(u) > 0 \textit{ for } u \neq 0, \tag{3.12}$$

$$\textit{there exists a continuous and nondecreasing function } c(t) > 0 \tag{3.13}$$

for $t > 0$ *such that for all* $u, h \in X$

$$d^2 f(u, h, h) \geqq c(f(u)) \| h \|^2,$$

$$f' \textit{ is a bounded mapping and } M_r(f) \textit{ is the bounded set, } \inf_{x \in M_r(f)} (f'(x), x) > 0, \tag{3.14}$$

$$g' \textit{ is real-analytic and completely continuous, } g(0) = 0, g'(0) = 0, \tag{3.15}$$

$$u_n \to u, \quad g'(u_n) \to 0, \quad 0 < t_n < 1 \Rightarrow g'(t_n u_n) \to 0. \tag{3.16}$$

Then the set $g(B) - \{0\}$ *is isolated and thus* $\Gamma = g(B)$ *has only one possible limit point, namely zero.*

While the determination of the lower bound for the number of points of the set Γ is based on the topological methods, the upper bound (Theorems 8 and 9) is found on the basis of properties of real-analytic functions, namely, of the real-analytic version of the Morse–Sard theorem for functionals in infinite-dimensional spaces. Define

$$Y_1 = X_1 \times R_1 \quad \text{and} \quad \psi : Y_1 \to R_1 \quad \text{by} \quad \psi : [x, \lambda] \to \lambda(f(x) - r) - g(x).$$

Then $x \in B_1$ iff there exists $\lambda \in R_1$ such that $[x, \lambda]$ is the critical point of the functional ψ, i.e. $\psi'(x, \lambda) = 0$, and $\lambda df(x, .) = dg(x, .)$ and $\psi(x, \lambda) = -g(x)$. Thus for the investigation of the set $\Gamma = g(B)$ it is sufficient to investigate the set $\psi(\{y \in Y_1 : \psi'(y) = 0\})$. The well-known theorem about real-valued functions, so-called Morse or Morse–Sard theorem, says that if G is an open subset of R_N and $f \in C^N(G)$ is a real function, then the Lebesgue measure of the set $f(B)$ is zero, where $B = \{x \in G : \operatorname{grad} f(x) = 0\}$. It is proved in the paper [29] that in the case of real-analytic function f for each $x \in B$ there exists a neighborhood $U \subset G$ of the point x such that $f(B \cap U)$ is a one-point set and thus $f(B \cap K)$ is finite for every compact set $K \subset G$. Hence $f(B)$ is at most countable. There exists an interesting example of functional $f \in C^\infty$ on the separable Hilbert space such that the set $f(B)$ has nonzero Lebesgue measure (see [20]). S. I. Pochožajev [28] introduced the notion of the *Fredholm functional* and he proved under some additional assumptions the set $f(B)$ has a zero measure. One from his assumptions is that

$$\sup_{x \in G} \dim \operatorname{Ker} f''(x) < \infty$$

and this assumption is very difficult to verify.

Recall that a linear operator A defined on the Banach space X with values in Banach space Y is said to be *Fredholm operator* if the following conditions are fulfilled:

(i) $A(X)$ is a closed subspace of Y,

(ii) $Y/A(X)$ has a finite dimension,

(iii) $A^{-1}(0)$ is a finite-dimensional subspace of X.

The proof of Theorems 8 and 9 is based on

Theorem 10 (see [10, 12]). *Let f be a real-analytic functional defined on an open subset G of Banach space X and let Y be another Banach space. Suppose that there exists a bilinear form $\langle .,. \rangle$ on $X \times Y$ such that for fixed $x \in X$, $\langle x, . \rangle$ is continuous on Y and $\langle x, y \rangle = 0$ for all $x \in X \Rightarrow y = 0$. Let there exists a real-analytic mapping $F : X \to Y$ such that*

$$\mathrm{d}f(x, h) = \langle h, F(x) \rangle \text{ for } x \in G,\ h \in X, \tag{3.17}$$

$$F'(x_0) \text{ is a Fredholm operator for each } x_0 \text{ with } f'(x_0) = 0. \tag{3.18}$$

Then there exists a neighborhood $U(x_0)$ in X of the point $x_0 \in B$ such that $f(B \cap \cap U(x_0))$ is a one-point set and, in the case of a separable Banach space X, the set $f(\{x \in G : f'(x) = 0\})$ is at most countable.

In the proof of Theorem 10 we rewrite using Implicite Function Theorem our problem to the finite-dimensional space $[F'(x_0)]^{-1}(0)$ and we use the finite-dimensional Morse-Sard theorem due to J. Souček – V. Souček [29].

For the function $f \in C^{k,\alpha}(G)$, $G \subset R_N$ the problem about Hausdorff measure of the set $f(B)$ is solved in the paper [19]. In infinite-dimensional spaces the analog of Morse-Sard theorem and the result about Hausdorff measure of the set Γ for the functionals only with certain Hölderian derivatives are proved in [14].

Let us consider the egienvalue problem (2.21) and some sufficient conditions to be the assumptions of Theorem 8 valid. First let us suppose that the boundary $\partial\Omega$ of the considered domain Ω is infinitely differentiable. Suppose $\mathscr{F}(x, u, p) \in C^3(\bar{\Omega} \times \times R_{n+1})$. Let $\mathscr{O}$ be an open set in the complex plane $\mathscr{C}$ containing R_1 and $\mathscr{F}$ is a restriction of the function $\tilde{\mathscr{F}}$ defined on $\bar{\Omega} \times \mathscr{O}^{n+1}$. Suppose that $\tilde{\mathscr{F}}(x, u, p)$ and $\dfrac{\partial\tilde{\mathscr{F}}}{\partial x_1}(x, u, p), \ldots, \dfrac{\partial\tilde{\mathscr{F}}}{\partial x_n}(x, u, p)$ are continuous on $\bar{\Omega} \times \mathscr{O}^{n+1}$ and complex analytic on $\mathscr{O}^{n+1}$ for each $x \in \bar{\Omega}$. Analogously let $\mathscr{G}(x, u) \in C^3(\bar{\Omega} \times R_1)$ be a restriction of a function $\tilde{\mathscr{G}}(x, u) \in C(\bar{\Omega} \times \mathscr{O})$ which is a complex analytic function on $\mathscr{O}$ for each $x \in \bar{\Omega}$. We have

Theorem 11 (see [10]). *Suppose that $\mathscr{F}$ and $\mathscr{G}$ satisfy the assumptions above, conditions* (2.15) – (2.20) *and*

$$\sum_{i,j=1}^{n} \left| \frac{\partial^2 \mathscr{F}}{\partial p_i \partial x_j}(x, u, p) \right| \leqq c(1 + |p|)^{m-1}. \tag{3.19}$$

Then the set Γ for the eigenvalue problem

$$\left.\begin{array}{l} \int_\Omega \mathscr{F}(x, u, \operatorname{grad} u)\, \mathrm{d}x = r, \quad u \in \mathring{W}_m^{(1)}(\Omega) = X_3 \\ (2.21) \end{array}\right\} \tag{3.20}$$

is a sequence of positive numbers which is convergent to zero.

This Theorem is the consequence of Theorems 7 and 8 setting $X_1 = W_p^{(2)}(\Omega) \cap \cap \mathring{W}_m^{(1)}(\Omega)$, $p > n$, $1 < m < \infty$, $X_2 = L_p(\Omega)$ and $\langle .,. \rangle$ the L_2-duality between X_1 and X_2. The crucial regilarity assumption (3.9) is obtained from the $C^{2,\mu}$-regularity of the solution of considered eigenvalue problem (see [10, Section 5]).

The examples of the higher order equations and application of Theorem 9 are studied in the paper [9].

REFERENCES

[1] F. E. BROWDER: *Nonlinear elliptic boundary value problems, II.* Trans. Amer. Math. Soc. 117 (1965), 530–550.

[2] F. E. BROWDER: *Infinite dimensional manifolds and nonlinear eigenvalue problems.* Annals of Math. 82 (1965), 459–477.

[3] E. S. CITLANADZE: *Existence theorems for minimax points in Banach spaces.* (Russian) Trudy Mosk. Mat. Obšč. 2 (1953), 235–274.

[4] N. DUNFORD and J. T. SCHWARTZ, *Linear operators II.* Interscience Publ. N. Y. 1963.

[5] S. FUČÍK: *Note on Fredholm alternative for nonlinear operators.* Comment. Math. Univ. Carolinae 12 (1971), 213–226.

[6] S. FUČÍK: *Fredholm alternative for nonlinear operators in Banach spaces and its applications to the differential and integral equations.* Čas. Pěst. Mat. 96 (1971), 371–390.

[7] S. FUČÍK, O. JOHN and J. NEČAS: *On the existence of Schauder bases in Sobolev spaces.* Comment. Math. Univ. Carolinae 13 (1972), 163–175.

[8] S. FUČÍK and J. NEČAS: *Ljusternik-Schnirelmann theorem and nonlinear eigenvalue problems.* Math. Nachr. 53 (1972), 163–175.

[9] S. FUČÍK, J. NEČAS, J. SOUČEK and V. SOUČEK: *Upper bound for the number of eigenvalues for nonlinear operators.* Ann. Scuola Norm. Sup. Pisa (to appear).

[10] S. FUČÍK, J. NEČAS, J. SOUČEK and V. SOUČEK: *Upper bound for the number of critical levels for nonlinear operators in Banach spaces of the type of second order nonlinear partial differential operators.* Journ. Funct. Anal. 11 (1972), 314—333.

[11] S. FUČÍK, J. NEČAS, J. SOUČEK and V. SOUČEK: *Strengthening upper bound for the number of critical levels of nonlinear functionals.* Comment. Math. Univ. Carolinae 13 (1972), 297–310.

[12] S. FUČÍK, J. NEČAS, J. SOUČEK and V. SOUČEK: *New infinite dimensional versions of Morse-Sard theorem* (to appear).

[13] S. FUČÍK, J. NEČAS, J. SOUČEK and V. SOUČEK: *Note to nonlinear spectral theory: Application to the nonlinear integral equations of the Lichtenstein type* (to appear).

[14] S. FUČÍK, M. KUČERA, J. NEČAS, J. SOUČEK and V. SOUČEK: *Morse-Sard theorem in infinite-dimensional Banach spaces and investigation of the set of all critical levels* (to appear).

[15] P. HESS: *On nonlinear mappings of monotone type homotopic to odd operators.* Journ. Funct. Anal. (to appear).

[16] E. HILLE and R. PHILIPS: *Functional analysis and semigroups.* Providence, RI, 1957.

[17] M. A. KRASNOSELSKIJ: *Topological methods in the theory of nonlinear integral equations*, Pergamon Press, N. Y. 1964.

[18] M. KUČERA: *Fredholm alternative for nonlinear operators.* Comment. Math. Univ. Carolinae 11 (1970), 337–363.

[19] M. KUČERA: *Hausdorff measures of the set of critical values of functions of the class $C^{k,\lambda}$.* Comment. Math. Univ. Carolinae 13 (1972), 333–350.

[20] I. Kupka: *Counterexample to the Morse-Sard theorem in the case of infinite-dimensional manifolds.* Proc. Amer. Math. Soc. 16 (1965), 954–957.

[21] L. A. Ljusternik: *On a class of nonlinear operators in Hilbert space.* (Russian) Izv. Akad. Nauk SSSR No 5, 1939, 257–264.

[22] L. A. Ljusternik and L. G. Schnirelmann: *Topological methods in variational problems and their application to the differential geometry of a surface.* (Russian) Uspechi Mat. Nauk II (1947), 166–217.

[23] J. Nečas: *Sur l'alternative de Fredholm pour les opérateurs non linéaires avec applications aux problémes aux limites.* Ann. Scuola Norm. Sup. Pisa 23 (1969), 331–345.

[24] J. Nečas: *Remark on the Fredholm alternative for nonlinear operators with applications to integral equations of generalized Hammerstein type.* Comment. Math. Univ. Caroline 13 (1972), 109–120.

[25] J. Nečas: *Fredholm alternative for nonlinear operators and applications to partial differential equations and integral equations.* Čas. Pěst. Mat. 97 (1972), 65–71.

[26] W. V. Petryshyn: *Nonlinear equations involving noncompact operators.* Proc. Symp. Pure Math., Nonlinear Functional Analysis, Vol. 18, Part 1, 1970, Providence, RI, pp. 206–233.

[27] S. I. Pochožajev: *On the solvability of nonlinear equations with odd operators.* (Russian) Funkc. Analiz i Prilož. 1 (1967), 66–73.

[28] S. I. Pochožajev: *On the set of critical levels of the functionals.* (Russian) Mat. Sbornik 75 (117) (1968), 106–111.

[29] J. Souček and V. Souček: *The Morse-Sard theorem for real-analytic functions.* Comment. Math. Univ. Carolinae 13 (1972), 45–51.

[30] J. R. L. Webb: *A fixed point theorem and applications to functional equations in Banach spaces.* Boll. Unione Mat. Italiana 4 (1971), 755–788.

Authors' address:
Svatopluk Fučík, Jindřich Nečas
Department of Mathematical Analysis
Faculty of Mathematics and Physics
Sokolovská 83, Praha 8 Karlín
Czechoslovakia