HERMITE INTERPOLATION ON SIMPLEXES IN THE FINITE ELEMENT METHOD

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1. GENERAL CONSIDERATIONS

The theory of triangular elements for solving elliptic boundary value problems of an arbitrary order is well-known [1, 3, 4]. The following theorem completes this theory.

Theorem 1. *Let the parameters uniquely determining a polynomial $p(x, y)$ on an arbitrary triangle be given in such a way that at the vertices of the triangle there are prescribed all derivatives up to the order μ inclusive. If the polynomial $p(x, y)$ generates piecewise polynomial and m-times continuously differentiable functions in an arbitrarily triangulated domain then*

$$\mu \geqq 2m. \tag{1}$$

Theorem 1 is proved in [5]. We mention here the idea of the proof only: If (1) does not hold, i.e. if

$$m \leqq \mu \leqq 2m - 1,$$

then for preserving the $C^{(m)}$-continuity we waste on the sides of the triangle so many conditions that we are short of conditions in the interior of the triangle which are necessary for the unique determination of the polynomial.

From Theorem 1 and from the results of [1] and [4] we get immediately:

Theorem 2. *The simplest polynomial $p(x, y)$, which generates piecewise polynomial and m-times continuously differentiable functions in an arbitrarily triangulated domain, is of the degree $n = 4m + 1$ and is on the triangle uniquely determined by the values*)*

$$D^{\alpha}p(P_i), \ |\alpha| \leqq 2m, \quad i = 1, 2, 3 \tag{2}$$

$$D^{\alpha}p(P_0), \ |\alpha| \leqq m - 2 \tag{3}$$

$$\partial^k p(Q_r^{(k)})/\partial v^k, \quad r = 1, \ldots, 3k; \ k = 1, \ldots, m \tag{4}$$

where P_1, P_2, P_3 are the vertices of the triangle, P_0 its centre of gravity, $Q_1^{(k)}, \ldots, Q_{3k}^{(k)}$ the points dividing the sides of the triangle into $k + 1$ equal parts and $\partial p/\partial v$ the normal derivative.

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*) The following notation is used:

$$\alpha = (\alpha_1, \alpha_2), \qquad |\alpha| = \alpha_1 + \alpha_2, \qquad D^{\alpha}p = \partial^{|\alpha|}p/\partial x^{\alpha_1}\partial y^{\alpha_2}.$$

The conditions (2)–(4) uniquely determining the polynomial $p(x, y)$ are of such a form that considering $p(x, y)$ along the side P_iP_j of the triangle, i.e. setting

$$x = x_i + (x_j - x_i)\, s,\ y = y_i + (y_j - y_i)\, s,\ 0 \leqq s \leqq 1,$$

we obtain a polynomial $p(s)$ of the degree $4m + 1$ determined in such a way that it generates, as a polynomial in one variable s, piecewise polynomial and $2m$-times continuously differentiable functions.

Extrapolating this fact to the case of three variables it may be expected that the simplest polynomial $p(x, y, z)$ on the tetrahedron generating piecewise polynomial functions which are m-times continuously differentiable should be of such a degree and determined by such conditions that considering it on the triangular face $P_iP_jP_k$ of the tetrahedron we obtain a polynomial $p(s, t)$ which generates, as a polynomial in two variables s, t, $2m$-times continuously differentiable and piecewise polynomial functions. Thus the degree of such a polynomial should be $8m + 1$.

Extrapolating this result by induction to the case of the d-dimensional simplex one gets the following conjecture:

The simplest polynomial on the d-dimensional simplex which generates piecewise polynomial and m-times continuously differentiable functions is of the degree

$$n = 2^d m + 1. \tag{5}$$

The case $m = 0$ is trivial for every d. As to $d \geqq 3$, $m \geqq 1$ the conjecture was confirmed to be true in the cases $d = 3$, $m = 1$ and $d = 3$, $m = 2$ (see [5]).

In the following text we present the results concerning the case $d = 3$, $m = 1$ and mention briefly the case $d = 3$, $m = 2$.

2. NOTATION

A given closed tetrahedron will be denoted by $\bar{U}$, its interior by U. The vertices and the centre of gravity of $\bar{U}$ will be denoted by P_i $(i = 1, \ldots, 4)$ and P_0, respectively. The centres of gravity of the triangular faces $P_2P_3P_4$, $P_1P_3P_4$, $P_1P_2P_4$ and $P_1P_2P_3$ are denoted by Q_1, Q_2, Q_3 and Q_4, respectively. The symbols $Q_{jk}^{(1,s)}, \ldots, Q_{jk}^{(s,s)}$ denote the points dividing the segment P_jP_k into $s + 1$ equal parts.

The symbols s_{jk}, t_{jk} mean two arbitrary but fixed directions such that the directions P_jP_k, s_{jk}, t_{jk} are perpendicular to one another.

The symbol n_i denotes the normal to the triangular face the centre of gravity of which is the point Q_i. The symbols s_i and t_i mean two arbitrary but fixed directions such that n_i, s_i, t_i are perpendicular to one another.

Let P_j, P_k be two vertices of the triangular face the centre of gravity of which is the point Q_i. The symbol v_{ijk} denotes the direction perpendicular to the directions n_i and P_jP_k.

Let f be a function of the variables x, y, z and $\alpha_1 \geqq 0, \alpha_2 \geqq 0, \alpha_3 \geqq 0$ three arbitrary integers. Setting

$$\alpha = (\alpha_1, \alpha_2, \alpha_3), \ |\alpha| = \alpha_1 + \alpha_2 + \alpha_3$$

the operator D^α is defined by

$$D^\alpha f = \partial^{|\alpha|} f / \partial x^{\alpha_1} \partial y^{\alpha_2} \partial z^{\alpha_3}.$$

Let $\beta_1 \geqq 0, \beta_2 \geqq 0$ be two arbitrary integers. Setting

$$\beta = (\beta_1, \beta_2), \quad |\beta| = \beta_1 + \beta_2$$

the operators D_i^β and D_{jk}^β are defined by

$$D_{jk}^\beta f = \partial^{|\beta|} f / \partial s_{jk}^{\beta_1} \partial t_{jk}^{\beta_2}, \qquad D_i^\beta f = \partial^{|\beta|} f / \partial s_i^{\beta_1} \partial t_i^{\beta_2}.$$

3. INTERPOLATION THEOREM IN THE CASE $d = 3, m = 1$

Theorem 3. *Let the function $w(x, y, z)$ be continuous on a closed tetrahedron $\bar{U}$ and have bounded derivatives of the tenth order in the interior U of $\bar{U}$:*

$$|D^\alpha w(x, y, z)| \leqq M_{10}, \ |\alpha| = 10, \ (x, y, z) \in U. \tag{6}$$

Let

$$D^\alpha w(P_i) = 0, \quad |\alpha| \leqq 4 \tag{7}$$

$$D_{jk}^\beta w(Q_{jk}^{(r,s)}) = 0, \quad |\beta| = s, \qquad r = 1, \ldots, s; s = 1, 2 \tag{8}$$

$$w(Q_i) = 0 \tag{9}$$

$$D_i^\beta \frac{\partial w(Q_i)}{\partial n_i} = 0, \quad |\beta| \leqq 2 \tag{10}$$

$$D^\alpha w(P_0) = 0, \quad |\alpha| \leqq 1 \tag{11}$$

where $i = 1, \ldots, 4, j = 1, 2, 3, k = 2, 3, 4$ $(j < k)$. Then it holds on $\bar{U}$

$$|D^\alpha w(x, y, z)| \leqq \frac{K}{q^{|\alpha|} V^{|\alpha|+1}} M_{10} h^{10-|\alpha|}, \quad |\alpha| \leqq 8 \tag{12}$$

where h is the length of the largest edge of the tetrahedron $\bar{U}$ and K is a constant independent on $\bar{U}$ and on $w(x, y, z)$. The constant V is defined by

$$V = \min(V_1, \ldots, V_4), \tag{13}$$

V_i being the volume of the unit parallelepiped having edges parallel to the edges of $\bar{U}$ which intersect at the vertex P_i. The quotient q is defined by

$$q = \max_{i=1,\ldots,4} \min(a_i/h, b_i/h, c_i/h), \tag{14}$$

a_i, b_i and c_i being the lengths of the edges having the vertex P_i as a common point.

Theorem 3 is proved in [6]. The method of the proof of Theorem 3 is a modification of the method which was developed in the case of two variables in [3] and then generalized in [4].

It should be noted that the quantity V is a three-dimensional analogy of $\sin \omega$, ω being the smallest angle of a given triangle, because $\sin \omega$ is the measure of the unit rhombus having sides parallel to the sides making the angle ω.

In the two-dimensional case the estimates for derivatives depend just on h and $\sin \omega$ (see [1, 3, 4]) because $1/q < 2$. In the thre-dimensional case the quantity $1/q$ is unbounded. (To prove it it suffices to consider the tetrahedron with vertices $P_1(0, 0, 0)$, $P_2(1, 0, 0)$, $P_3(0, \varepsilon, 0)$ and $P_4(1, 0, \varepsilon)$.)

The interpolation character of Theorem 3 follows from the following

Corollary 1. *A polynomial of the ninth degree*

$$p(x, y, z) = a_1 + a_2 x + a_3 y + a_4 z + \ldots + a_{220} z^9 \tag{15}$$

is uniquely determined by the conditions (7)–(11) *where*

$$w(x, y, z) = p(x, y, z) - f(x, y, z), \tag{16}$$

$f(x, y, z)$ being a function four-times continuously differentiable on the tetrahedron $\bar{U}$. Further, if the function $f(x, y, z)$ has bounded derivatives of the tenth order in the interior U of $\bar{U}$,

$$| D^\alpha f(x, y, z) | \leqq M_{10}, \quad | \alpha | = 10, \quad (x, y, z) \in U,$$

then the difference (16) *satisfies the inequality* (12).

Proof. The number of the conditions (7)–(11) is equal to 220. If $w(x, y, z)$ is of the form (16) then the conditions (7)–(11) form a system of 220 linear equations for the 220 unknown coefficients $a_1, \ldots, a_{220}$. It is sufficient to prove that the determinant of this system is different from zero.

Let us assume that the function $w(x, y, z) = p(x, y, z)$ satisfies the conditions (7)–(11). As, according to Eq. (15),

$$D^\alpha p(x, y, z) \equiv 0, \quad | \alpha | = 10$$

it follows from Theorem 3 that $p(x, y, z) \equiv 0$. The inverse implication is trivial. Corollary 1 is proved.

4. APPLICATIONS

Let Ω be a bounded domain in E_3 with the boundary Γ consisting of a finite number of polyhedrons Γ_i $(i = 0, \ldots, s)$; $\Gamma_1, \ldots, \Gamma_s$ lie inside of Γ_0 and do not intersect. Let $\mathfrak{M}$ be a set of a finite number of closed tetrahedrons having the following properties: 1. The union of all tetrahedrons is $\bar{\Omega}$; 2. Two arbitrary tetrahedrons are either disjoint or have a common vertex or a common edge or a common face. The tetrahedrons of $\mathfrak{M}$ will be denoted by $\bar{U}_i$ ($i = 1, \ldots, N$).

Let us define by means of Corollary 1 on each tetrahedron $\bar{U}_i$ a polynomial of the ninth degree $p_i(x, y, z)$. Let the parameters prescribed at a common vertex (or on a common edge or on a common face) are the same for all tetrahedrons having this vertex (or edge or face) common. Then the following theorem, which is proved in [5], holds:

Theorem 4. *The function*

$$g(x, y, z) = p_i(x, y, z), \quad (x, y, z) \in \bar{U}_i \quad (i = 1, \ldots, N) \tag{17}$$

is once continuously differentiable on the closed domain $\bar{\Omega}$.

Let us denote the set of all functions of the type (17) by $G(\mathfrak{M})$. The set $G(\mathfrak{M})$ is a finite dimensional space and it is clear that $G(\mathfrak{M}) \subset W_2^{(2)}(\Omega)$. Thus we can use the functions of the type (17) as trial functions in the finite element procedure for solving three-dimensional boundary value problems of elliptic equations of the fourth order. We restrict ourselves to the variational formulation of the problem.

Let $H \subset W_2^{(2)}(\Omega)$ be a real Hilbert space with the norm induced by $W_2^{(2)}(\Omega)$. Let $a(v, w)$ be a real bilinear form continuous on $H \times H$, i.e. a mapping $(v, w) \rightarrow a(v, w)$ from $H \times H$ into the field of real numbers which is linear in both v and w and bounded:

$$| a(v, w) | \leqq M \| v \|_{W_2^{(2)}(\Omega)} \| w \|_{W_2^{(2)}(\Omega)}, \quad \forall v, w \in H \tag{18}$$

where M is a constant independent on v, w. Further, let the form $a(v, w)$ be symmetric,

$$a(v, w) = a(w, v), \quad \forall v, w \in H, \tag{19}$$

and H-elliptic, i.e.

$$a(v, v) \geqq \varkappa \| v \|^2_{W_2^{(2)}(\Omega)}, \quad \forall v \in H \tag{20}$$

where $\varkappa > 0$ is a constant independent on v. Finally, let $L(v)$ be a linear functional continuous on H. It is well-known that under these conditions there exists just one function $u \in H$ minimizing sharply on H the functional

$$F(v) = \frac{1}{2} a(v, v) - L(v). \tag{21}$$

The space H is determined by the stable homogeneous boundary conditions of the boundary value problem to which the given variational problem corresponds. In our case of tetrahedral elements we must restrict our considerations to such cases when the part Γ' of Γ on which the stable boundary conditions are prescribed can be covered by a finite number of triangles. In this case we can choose the division $\mathfrak{M}$ in such a way that Γ' is a union of some triangular faces.

The approximate solution of the given variational problem is then defined as the function which minimizes the functional (21) on the space $G(\mathfrak{M}) \cap H$. It follows immediately from (20) that there exists just one function of this property.

Now, let $\{\mathfrak{M}_h\}$ be a set of divisions of $\bar{\Omega}$ into closed tetrahedrons with the following properties:

$$h \to 0,\ q_h \geqq q_0 > 0,\ V_h \geqq V_0 > 0, \tag{22}$$

h being the length of the largest edge in $\mathfrak{M}_h$, q_h the smallest quantity (14) in $\mathfrak{M}_h$ and V_h the smallest quantity (13) in $\mathfrak{M}_h$. Let $H_h = G(\mathfrak{M}_h) \cap H$ and u_h be the approximate solution of the given variational problem on H_h. The following two convergence theorems hold.

Theorem 5. *Under the assumptions* (18)–(20) *and* (22) *it holds*

$$\lim_{h \to 0} \| u_h - u \|_{W_2^{(2)}(\Omega)} = 0, \tag{23}$$

u being the exact solution of the given variational problem.

Using Theorem 3 the proof of Theorem 5 goes in the same lines as the proof of the convergence theorem introduced in [7]. Further, Theorem 3 allows to state a sufficient condition for the maximum rate of convergence:

Theorem 6. *Let the conditions* (18)–(20) *and* (22) *be satisfied and the exact solution $u(x, y, z)$ have bounded derivatives of the tenth order in Ω,*

$$| D^\alpha u(x, y, z) | \leqq M_{10},\ | \alpha | = 10,\ (x, y, z) \in \Omega. \tag{24}$$

Then

$$\| u_h - u \|_{W_2^{(2)}(\Omega)} \leqq C M_{10} h^8 \tag{25}$$

where the constant C does not depend on the division $\mathfrak{M}$ and on the exact solution $u(x, y, z)$.

Proof. According to [2], p. 365, it holds

$$\| u_h - u \|_{W_2^{(2)}(\Omega)} \leqq M^{\frac{1}{2}} \varkappa^{-\frac{1}{2}} \| u - v \|_{W_2^{(2)}(\Omega)},\quad \forall v \in H_h .$$

Let φ be the function from H_h having the same values at the nodal points of the division $\mathfrak{M}_h$ as the exact solution u. Making use of Corollary 1 we can state

$$\| u - \varphi \|_{W_2^{(2)}(\Omega)} \leqq C' M_{10} h^8$$

where the constant C' depends on q_0, V_0 and mes Ω only. As $\varphi \in H_h$ the last two inequalities imply the estimate (25). Theorem 6 is proved.

5. THE CASE $d = 3$, $m = 2$

Theorem 7. *A polynomial of the seventeenth degree*

$$p(x, y, z) = a_1 + a_2 x + a_3 y + a_4 z + \ldots + a_{1140} z^{17} \tag{26}$$

is uniquely determined by the following 1140 *parameters*:

$$D^\alpha p(P_i), \ |\alpha| \leqq 8, \qquad i = 1, \ldots, 4 \tag{27}$$

$$D^\beta_{jk} p(Q^{(r,s)}_{jk}), \ \ |\beta| = s, \ \ r = 1, \ldots, s; \ s = 1, \ldots, 4; \quad j = 1, 2, 3, \ k = 2, 3, 4 \ (j < k) \tag{28}$$

$$D^\beta_{iP} p(Q_i), \ |\beta| \leqq 2, \qquad i = 1, \ldots, 4 \tag{29}$$

$$D^\beta_i \frac{\partial p(Q_i)}{\partial n_i}, \ \ |\beta| \leqq 4, \qquad i = 1, \ldots, 4 \tag{30}$$

$$D^\beta_i \frac{\partial^2 p(Q_i)}{\partial n_i^2}, \ \ |\beta| \leqq 3, \qquad i = 1, \ldots, 4 \tag{31}$$

$$\frac{\partial^5 p(Q^{(r,5)}_{jk})}{\partial v^3_{ijk}\, \partial n_i^2}, \quad \begin{array}{l} r = 1, \ldots, 5; \ i = 1, \ldots, 4; \ j = 1, 2, 3, \\ k = 2, 3, 4 \ \ (j \neq i, k \neq i, j < k) \end{array} \tag{32}$$

$$D^\alpha p(P_0), \ \ |\alpha| \leqq 5. \tag{33}$$

The polynomial of the seventeenth degree determined in such a way generates piecewise polynomial and twice continuously differentiable functions.

Theorem 7 is proved in [5]. The great number of parameters is the reason that both the polynomial of the ninth degree and the polynomial of the seventeenth degree are not applicable in numerical computations. So we may speak about a good luck that we meet three-dimensional boundary value problems the order of which is greater than two very rarely in practical applications.

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