# PERIODIC SOLUTIONS OF $\ddot{x} = f(x, \dot{x})$

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Recently several authors ([1], [2], [3], [4]) have found sufficient conditions for the existence of periodic solutions of the period $\omega$ of the differential equation

$$\ddot{x} + f(x)\,\dot{x}^{2n} + g(x) = \mu p(t)$$

where $f$, $g$ and $p$ are continuous for all $x$ and $t$, $p(t + \omega) = p(t)$ and $\mu$ is a sufficiently small parameter. Their investigations were based on a lemma due to I. BERNŠTEIN and A. HALANAY [5]:

*In the system*

$$\frac{\mathrm{d}x}{\mathrm{d}t} = X(x) + \mu Y(x, \dot{x}, t, \mu),\ x \in R_2$$

*let $X$, $Y$ be continuous vector-functions, $Y$ $\omega$-periodic and let every initial-value problem be unique in a neighbourhood $U$ of the origin. If the degenerate system $\frac{\mathrm{d}x}{\mathrm{d}t} = X(x)$ has in $U$ a periodic solution with the period $\tilde{\omega} \neq \omega$, then the former system has at least an $\omega$-periodic solution, provided $|\mu|$ is sufficiently small.*

A similar approach can be applied to a more general equation

$$\ddot{x} = f(x, \dot{x}) + \mu g(x, \dot{x}, t, \mu)$$

where $f$, $g$ are continuous functions, $f(x, -\dot{x}) = f(x, \dot{x})$ and $g$ is periodic. Using the above lemma, the problem is to show the existence of periodic solutions of

$$\ddot{x} = f(x, \dot{x}) \tag{1}$$

with different periods. The purpose of this paper is to establish some sufficient conditions for the existence of periodic solutions of (1) and to derive the estimates of their periods.

Consider the phase-plane with coordinates $x$ and $y = \dot{x}$. Then the equation (1) is equivalent to the system

$$\begin{aligned} \dot{x} &= y, \\ y &= f(x, y). \end{aligned} \tag{2}$$

In what follows, let us suppose

(*i*) $f(x, y) \in C^0(I \times R_1)$ *where $I = (a, b)$ is an open bounded or unbounded interval containing the origin and $R_1 = (-\infty, \infty)$.*

(*ii*) *The solutions of* $z' = 2f(x, \sqrt{z})$ *are uniquely determined by initial conditions in the halfplane* $z \geqq 0$[1]).

(*iii*) $$f(x, -y) = f(x, y) \text{ on } I \times R_1 .$$

(*iv*) $$xf(x, 0) < 0 \text{ for } x \neq 0.$$

First of all note that the orbits of (2) are symmetric with respect to the $x$-axis. In fact, if $(x(t), y(t))$ is a solution of (2), then $(x(-t), -y(-t))$ is a solution as well. From the first equation (2) one can also see that the point $A(x(t), y(t))$ of an orbit $\Gamma$ of (2) moves from left to right if $A$ is in the half-plane $y > 0$ and in an opposite direction in the half-plane $y < 0$. Especially, if $\Gamma$ is closed, $A$ moves in a clockwise direction.

From this consideration it follows immediately that each part of any trajectory $\Gamma : x = \varphi(t)$, $y = \psi(t)$ of (2) situated in the half-plane $y \geqq 0$ may be written in the form $y = u(x)$ where $u(x) = \psi[\varphi^{-1}(x)]$. In the half-plane $y > 0$ the function $u(x)$ is differentiable and in view of $\dfrac{d}{dt} = \dfrac{d}{dx}\dfrac{dx}{dt} = y\dfrac{d}{dx}$ it holds

$$uu' = f(x, u). \tag{3}$$

If $y = u(x)$ is an orbit of (2) for $x \in [c, d] \subset (a, b)$ such that $u(x) > 0$ for $x \in (c, d)$ and $u(c) = u(d) = 0$, then $u(x) \sim \sqrt{2f(c, 0)(x - c)}$ for $x \to c_+$ and $u(x) \sim \sqrt{2f(d, 0)} \,.\, \sqrt{x - d}$ for $x \to d_-$ by (*iv*) and (3). Every solution $x(t)$ of (1) satisfying at any time $t_0$ initial conditions $x(t_0) = x_0$, $\dot{x}(t_0) = \dot{x}_0$ where $\dot{x}_0 = u(x_0)$ is periodic and its period $\omega$ is given by the formula

$$\omega = 2 \int_c^d \frac{ds}{u(s)} . \tag{4}$$

**Theorem 1.** *Under the assumptions* (*i*) – (*iv*) *every solution of* (1) *satisfying small enough initial conditions is periodic.*

Proof. Consider a point $A(0, \eta_1)$, $\eta_1 > 0$ and the solution $x = \varphi_1(t)$, $y = \psi_1(t)$ of (2) satisfying initial conditions $\varphi_1(0) = 0$, $\psi_1(0) = \eta_1$. Then there is a closed interval $[t_1, t_2]$, $t_1 < 0 < t_2$, in which this solution exists and $\psi_1(t) > 0$. Denote $c = \varphi_1(t_1)$, $d = \varphi_1(t_2)$ so that $c < 0$, $d > 0$. Let $x = \varphi_2(t)$, $y = \psi_2(t)$ be the solution of (2) satisfying $\varphi_2(0) = c$, $\psi_2(0) = 0$. Since $\dot{\psi}_2(0) = f(c, 0) > 0$ by (*iv*), it follows that the orbit $\Gamma : x = \varphi_2(t)$, $y = \psi_2(t)$ is situated for $t > 0$ in the half-plane $y > 0$ and cannot meet the $x$-axis for $x \leqq 0$. But with respect to the uniqueness this orbit cannot meet the orbit $x = \varphi_1(t)$, $y = \psi_1(t)$ so that it is forced to cross the line $x = 0$ in a point $B(0, \eta_2)$, $0 < \eta_2 < \eta_1$. If this orbit meets the real axis at a point of $(0, d]$,

[1]) I am very grateful to Prof. J. Butler for his advice referring to this hypothesis in the preprint of this paper.

the proof is complete. Otherwise consider the solution $x = \varphi_3(t)$, $y = \psi_3(t)$, $\varphi_3(0) = d$, $\psi_3(0) = 0$ for $t \leqq 0$. The orbit representing this solution neither meets the $x$-axis for $x \geqq 0$ nor the orbit $x = \varphi_1(t)$, $y = \psi_1(t)$ so that it is forced to cross the $x$-axis at any point belonging to the interval $(c, 0)$ to prove the theorem.

**Theorem 2.** *Let the assumptions (i) – (iv) be satisfied. If there is a positive function $\gamma(x) \in C^0(a, b)$ such that either*

$$\left.\begin{aligned} \gamma(x)\, D_R\gamma(x) - f(x, \gamma(x)) \geqq 0 \quad &\text{for} \quad a < x < 0, \\ \gamma(x)\, D_L\gamma(x) - f(x, \gamma(x)) \leqq 0 \quad &\text{for} \quad 0 < x < b, \end{aligned}\right\} \tag{5}$$

or

$$\gamma(x)\, D_R\gamma(x) - f(x, \gamma(x)) \geqq 0 \quad \text{for} \quad a < x < b, \tag{6}$$

or

$$\gamma(x)\, D_L\gamma(x) - f(x, \gamma(x)) \leqq 0 \quad \text{for} \quad a < x < b,^1) \tag{7}$$

*then there exists at least one periodic solution of* (1) *with the period*

$$\omega > 2 \min \left\{ \int_a^0 \frac{\mathrm{d}s}{\gamma(s)}, \int_0^b \frac{\mathrm{d}s}{\gamma(s)} \right\}. \tag{8}$$

**Remark.** If both integrals in (8) are divergent, the last inequality is to be read "$\omega$ is arbitrarily large".

Proof. Let us consider the vector field determined by (2) on the curves $y = 0$ and $y = \gamma(x)$ in the phase plane. For the sake of brevity denote $I_- = (a, 0]$, $I_+ = [0, b)$.

Suppose (5). We shall prove that there exists an orbit $\Gamma : y = u_L(x)$ of (2) defined on an interval containing $I_-$, $u_L(x) > 0$ on $I_-$ and such that every trajectory $x = \varphi(t)$, $y = \psi(t)$ of (2) starting at a point $A(0, u_0)$, $0 < u_0 < u_L(0)$ crosses the $x$-axis at a point $c$, $a < c < 0$.

In order to prove this let $\xi \in I_-$ and let $(\varphi(t), \psi(t))$ be the solution of (2) determined by the initial conditions $\varphi(0) = \xi$, $\psi(0) = 0$. In view of (*iv*) the trajectory $\Gamma_\xi : x = \varphi(t)$, $y = \psi(t)$ enters the second quadrant with the increasing time and $\varphi(t) < 0$, $0 < \psi(t) < \gamma[\varphi(t)]$ for $t > 0$ sufficiently small. For this reason we may write $\Gamma_\xi$ in a right neighbourhood of $\xi$ in the form $y = u(x)$ and $u(x)$ is a solution of (3). Moreover, in view of (*iv*) and (5) it follows by a well known comparison theorem (see, e.g. P. Hartman [7] pp. 28, (*b*)) $0 < u(x) < \gamma(x)$ for $a < x \leqq 0$. Especially $u(0) = \eta$, $0 < \eta < \gamma(0)$. Let us denote $\eta_0 = \text{l.u.b.}$ $\{\eta :$ the orbit $y = u(x)$ of (2) starting at the point $(0, \eta)$ cuts the $x$-axis at a point $c \in (a, 0)$. Evidently $\eta_0 \leqq \gamma(0)$ and the orbit $\Gamma : y = u_L(x)$, $u_L(0) = \eta$ has the desired properties.

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1) Here $D_R\gamma(x)$, $D_L\gamma(x)$ denote right and left derivatives of $\gamma(x)$.

Replacing $x$ by $-x$, the above assertion can be modified for the interval $I_+$ as follows: there exists an orbit $y = u_R(x)$ of (2) defined on an interval containing $I_+$, $u_R(x) > 0$ on $I_+$ and such that every trajectory $x = \varphi(t)$, $y = \psi(t)$ of (2) starting at a point $(0, \zeta_0)$, $0 < \zeta_0 < u_R(0)$ crosses the $x$-axis at a point $d$, $0 < d < b$.

With respect to the uniqueness it is evidently either $u_L(x) = u_R(x)$ on $I$ or $u_L(x) < u_R(x)$ or $u_L(x) > u_R(x)$ on a common interval of existence. For the sake of definiteness suppose $u_L(x) \leqq u_R(x)$ and define

$$\eta = \begin{cases} b \text{ if } u_L(x) \text{ is positive on } I_+, \\ \text{the first zero of } u_L(x) \text{ on the right of the origin.} \end{cases}$$

The solution $x(t)$ of (1), $x(0) = 0$, $\dot{x}(0) = u_L(0)$ is defined at least on an interval $(t_1, t_2)$ where

$$t_1 = \int_0^a \frac{ds}{u_L(s)}, \qquad t_2 = \int_0^\eta \frac{ds}{u_L(s)}.$$

Let $0 < \lambda_n \uparrow u_L(0)$ and let $\Gamma_n : x = \varphi_n(t)$, $y = \psi_n(t)$ be the trajectory of (2), $\varphi_n(0) = 0$, $\psi_n(0) = \lambda_n$. Then $\Gamma_n$ is a periodic orbit of (2) crossing the $x$-axis at $a_n$, $b_n$, $a < a_n < b_n < \eta$. The corresponding solution $x_n(t)$, $x_n(0) = 0$, $\dot{x}_n(0) = \lambda_n$ is periodic and its period $\omega_n$ is given by

$$\omega_n = 2 \int_{a_n}^{b_n} \frac{ds}{u_n(s)}.$$

It holds

$$\int_{a_n}^{b_n} \frac{ds}{u_n(s)} > \int_{a_n}^{b_n} \frac{ds}{\gamma(s)} \uparrow \int_a^\eta \frac{ds}{\gamma(s)} > \int_a^0 \frac{ds}{\gamma(s)} \geqq$$

$$\geqq \min \left\{ \int_a^0 \frac{ds}{\gamma(s)}, \int_0^b \frac{ds}{\gamma(s)} \right\}$$

for $n \to \infty$. Hence for $n$ sufficiently large the period $\omega_n$ of $x_n(t)$ satisfies (8).

If $u_L(x) > u_R(x)$, one proceeds in a similar way to prove the assertion under the assumption (5).

Suppose (6). In this case there exists an orbit $\Gamma : y = u_L(x)$ with the same properties on $I_-$ as above. By (6) $\Gamma$ cannot meet the curve $y = \gamma(x)$ for $x > 0$ so that it cuts either the $x$-axis at a $d \in (0, b)$ or the function $u_L(x)$ is positive on $I$. In the latter case one constructs the orbit $y = u_R(x)$ as above; then we have $u_R(x) < u_L(x)$ and the proof can be finished as in the case 1.

Suppose (7). Making the transformation $x = -X$, this case is reduced to the preceding one.

The proof is complete.

**Theorem 3.** *Suppose (i) – (iv). Suppose that there are functions $H(x)$, $K(x)$, $h(x)$, $k(x)$ with the following properties*

$$H(x) \in C^0(I_-),\ K(x) \in C^0[0, \infty),\ K(x) > 0, \tag{9}$$

$$\int_a^0 H(s)\,\mathrm{d}s \leqq \int_0^\infty \frac{s\,\mathrm{d}s}{K(s)}, \tag{10}$$

$$f(x, y) \leqq H(x)\,K(y) \quad \text{for} \quad x \in I_-,\ y \geqq 0, \tag{11}$$

$$h(x) \in C^0(I_+),\ k(x) \in C^0[0, \infty),\ k(x) > 0, \tag{12}$$

$$\int_b^0 h(s)\,\mathrm{d}s \leqq \int_0^\infty \frac{s\,\mathrm{d}s}{k(s)}, \tag{13}$$

$$h(x)\,k(y) \leqq f(x, y) \quad \textit{for} \quad x \in I_+,\ y \geqq 0. \tag{14}$$

*Then for every $\omega_0$*

$$0 < \omega_0 < 2 \min \left\{ \operatorname*{l.u.b.}_{a < \alpha < 0} \int_\alpha^0 \frac{\mathrm{d}s}{U_\alpha(s)},\ \operatorname*{l.u.b.}_{0 < \beta < b} \int_0^\beta \frac{\mathrm{d}s}{V_\beta(s)} \right\} \tag{15}$$

*where $U_\alpha(x)$, $V_\beta(x)$ are defined by*

$$\int_\alpha^x H(s)\,\mathrm{d}s = \int_0^{U_\alpha(x)} \frac{s\,\mathrm{d}s}{K(s)}, \qquad \int_\beta^x h(s)\,\mathrm{d}s = \int_0^{V_\beta(x)} \frac{s\,\mathrm{d}s}{k(s)}, \tag{16}$$

*there exists a periodic solution with the period $\omega > \omega_0$.*

Proof. First of all note that $H(x) > 0$ for $x \in (a, 0)$ as it follows in view of (*iv*) from (11) for $y = 0$. Therefore, the function $U_\alpha(x)$ defined by (16) exists on the whole interval $[\alpha, 0]$ in view of (10) is positive on $(\alpha, 0]$, $U_\alpha(x) \in C^1$ and

$$U'_\alpha(x) = \frac{H(x)\,K(U_\alpha(x))}{U_\alpha(x)}. \tag{17}$$

Let $x = \varphi(t)$, $y = \psi(t)$ be the solution of (2) satisfying initial conditions $\varphi(0) = \alpha$, $\psi(0) = 0$. From (*iv*) it follows that this solution may be written for all $t > 0$ at which $\psi(t) > 0$ in the form $y = u(x)$. It is evidently $u(\alpha) = U_\alpha(\alpha) = 0$; it will be shown

$$0 < u(x) \leqq U_\alpha(x) \quad \text{for} \quad \alpha < x \leqq 0.$$

The first part of these inequalities is clear by (*iv*). To prove $u(x) \leqq U_\alpha(x)$ compare (3) with (17) on the interval $(\alpha, 0]$. By (11) and from the *Comparison theorem* it follows that the inequality $0 < u(\xi) \leqq U_\alpha(\xi)$ at a $\xi \in (\alpha, 0)$ implies $u(x) \leqq U_\alpha(x)$ for $x \in [\xi, 0]$. Hence it is sufficient to prove this inequality in a neighbourhood of $\alpha$.

Since the functions $u(x)$, $U_\alpha(x)$ have positive derivatives in a right neighbourhood of $\alpha$, there exist here inverse functions $y = u^{-1}(x)$, $Y = U^{-1}(x)$, resp. and it holds

$$\frac{dy}{dx} = \frac{x}{f(y, x)}, \qquad \frac{dY}{dx} = \frac{x}{H(Y)K(x)}, \qquad u^{-1}(0) = U_\alpha^{-1}(0) = \alpha.$$

In view of (11) it is $u^{-1}(x) \geqq U_\alpha^{-1}(x)$ in a right neighbourhood of 0. But this implies $u(x) \leqq U_\alpha(x)$ in a certain right neighbourhood of $\alpha$ and the assertion is proved.

In the same way one deduces from (12), (13), (14) that the orbit $y = v(x)$ of (2) starting at the point $(\beta, 0)$, $0 < \beta < b$ satisfies the inequalities $0 < v(x) \leqq V_\beta(x)$ for $0 \leqq x < \beta$. At least one of the orbits $y = u(x)$, $y = v(x)$ crosses twice the $x$-axis and that is sufficient for the existence of a periodic solution of (1) satisfying (15). The proof is complete.

Note. One can often use as a "comparison equation" $\ddot{x} + g(x) = 0$ the properties of which have been described in many papers (see, e.g., references in [6]).

**Theorem 4.** *Suppose* (*i*) - (*iv*) *and*

$$f(x, y) \leqq 0 \quad \text{for} \quad 0 < x < b, y > 0. \tag{18}$$

*Suppose the existence of* $H(x)$, $K(x)$ *satisfying* (9), (10), (11). *Then for every* $\omega_0$

$$0 < \omega_0 < 2 \operatorname*{l.u.b.}_{a<\alpha<0} \left\{ \min \int_\alpha^0 \frac{ds}{U_\alpha(s)}, \frac{\beta}{U_\alpha(0)} \right\}, \tag{19}$$

*where* $U_\alpha(x)$ *is defined by* (16) *and* $0 < \beta < b$ *there exists a periodic solution with the period* $\omega > \omega_0$.

Proof. As in *Theorem* 3 the assumptions (*i*) – (*iv*) with (9), (10), (11) imply that the orbit $\Gamma_1 : x = \varphi_1(t)$, $y = \psi_1(t)$, $\varphi_1(0) = \alpha$, $\psi_1(0) = 0$ can be written for $t > 0$ for which $\psi_1(t) > 0$ in the form $y = u(x)$ and $0 < u(x) \leqq U_\alpha(x)$ for $\alpha < x \leqq 0$. Since for $x > 0$ the inequality $u(x) > 0$ implies by (18) $u(x)$ nonincreasing, two possibilities can occur: the orbit $\Gamma_1$ either cuts the $x$-axis a a point $c \in (0, b)$ or it is $u(x) > 0$ on $(a, b)$. In the former case the solution $x = \varphi_1(t)$, $y = \psi_1(t)$, $\varphi_1(0) = \alpha$, $\psi_1(0) = 0$ is $\omega$-periodic and it holds

$$\omega = 2 \int_\alpha^c \frac{ds}{u(s)} > 2 \int_\alpha^0 \frac{ds}{u(s)} \geqq 2 \int_\alpha^0 \frac{ds}{U_\alpha(s)}.$$

In the latter case, consider the trajectory $\Gamma_2 : x = \varphi_2(t)$, $y = \psi_2(t)$, $\varphi_2(0) = \beta$, $\psi_2(0) = 0$ where $\beta \in (0, b)$ for $t < 0$. In view of (*iv*) this trajectory enters the first quadrant and wih respect to (*iv*) and (*ii*) it is forced to cut the $y$-axis at a point $\eta > 0$. Let $\eta_0 = \text{l.u.b.}\ \{\eta :$ the orbit $y = u_\eta(x)$ of (2) passing through the point $(0, \eta)$, $0 < \eta < \eta_0$ cuts the $x$-axis for $x \in (0, b)\}$. Then every orbit $y = u_\eta(x)$ of (2) passing through $(0, \eta)$, $0 < \eta < \eta_0$ cuts the $x$-axis at $a_\eta$, $b_\eta$ and $a_\eta \downarrow c \geqq \alpha$, $b_\eta \uparrow b$ if $\eta \uparrow \eta_0$. Hence for any $b_\eta > \beta$ the solution $x(t)$ of (1), $x(0) = 0$, $\dot{x}(0) = \eta$ is periodic with the period

$$\omega = 2 \int_{a_\eta}^{b_\eta} \frac{\mathrm{d}s}{u_\eta(s)} \geqq 2 \int_{a_\eta}^{0} \frac{\mathrm{d}s}{u_\eta(s)} + 2 \int_{0}^{b_\eta} \frac{\mathrm{d}s}{u_\eta(0)} > 2 \frac{b_\eta}{U_{a_\eta}(0)} > 2 \frac{\beta}{U_\alpha(0)} .$$

Theorem is proved.

**Theorem 5.** *Suppose (i), (ii), (iii),*

$$f(x, 0) > 0 \quad \textit{for} \quad a < x < 0, \tag{20}$$

$$f(x, y) \textit{ nonincreasing with respect to } y \textit{ for } 0 < x < b,\ y > 0. \tag{21}$$

*Moreover, suppose that there exists a positive function* $\gamma(x) \in C^0(a, 0]$

$$\gamma(x)\, D_R \gamma(x) - f(x, \gamma(x)) \geqq 0 \quad \textit{for} \quad a < x < 0 \tag{22}$$

*and that*

$$2 \operatorname*{g.l.b.}_{0<x<b} \int_0^x f(s, 0)\, \mathrm{d}s < -\gamma^2(0). \tag{23}$$

*Then every solution $x(t)$ of* (1) *determined by initial conditions* $x(0) = \xi$, $\dot{x}(0) = 0$, $\xi \in I_-$ *is periodic and there are periodic solutions with the period* $\omega$

$$\omega > 2 \int_a^0 \frac{\mathrm{d}s}{\gamma(s)} . \tag{24}$$

Proof. To prove this theorem consider the orbit $\Gamma$ of (2) passing through the point $(0, \gamma(0))$. We will prove that $\Gamma$ is forced to cross the $x$-axis at a point $d$, $d \in I_+$. To prove this, assume that $\Gamma$ is situated over the $x$-axis. Then $\Gamma$ may be written in the form $y = u(x)$, $u(x) > 0$ for $x \in I_+$. Since the function $f(x, y)$ is nonincreasing, we conclude that $u(x)$ (which is a solution of (2)) is defined on the whole interval $[0, b)$. But with respect to (23) there is a $x_1$, $0 < x_1 < b$ such that $2 \int_0^{x_1} f(s, 0)\, \mathrm{d}s < -\gamma^2(0)$ and we have

$$-\gamma^2(0) = -u^2(0) < u^2(x_1) - u^2(0) = 2 \int_0^{x_1} u(s)\, u'(s)\, \mathrm{d}s =$$

$$= 2 \int_0^{x_1} f(s, u(s))\, \mathrm{d}s \leqq 2 \int_0^{x_1} f(s, 0)\, \mathrm{d}s < -\gamma^2(0).$$

This contradiction proves the assertion.

Now, let $0 < c_n \downarrow a$ and let $\Gamma_n : x = \varphi_n(t)$, $y = \psi_n(t)$ be the trajectory of (2), $\varphi_n(0) = c_n$, $\psi_n(0) = 0$. In view of (22) and the preceding consideration, $\Gamma_n$ is a periodic orbit of (2) crossing the $x$-axis at $c_n$, $d_n$, $c_n < d_n < d$ and the part of $\Gamma_n$ between $c_n$ and $d_n$ can be written in the form $y = u_n(x)$. Since the sequence $\{d_n\}$ is increasing and bounded, there exists $\lim d_n$, say $d_0$. The period $\omega_n$ of $x_n(t)$ is given by the formula

$$\omega_n = 2\int_{c_n}^{d_n} \frac{ds}{u_n(s)} = 2\int_{c_n}^{0} \frac{ds}{u_n(s)} + 2\int_{0}^{d_n} \frac{ds}{u_n(s)} > 2\int_{c_n}^{0} \frac{ds}{\gamma(s)} +$$

$$+ 2\int_{0}^{d_n} \frac{ds}{u(s)} \to 2\int_{a}^{0} \frac{ds}{\gamma(s)} + 2\int_{0}^{d_0} \frac{ds}{u(s)} > 2\int_{a}^{0} \frac{ds}{\gamma(s)}.$$

From this relation it follows the exitence of a periodic solution $x_N(t)$, $x_N(0) = a_N$, $\dot{x}_N(0) = 0$ with the period $\omega_N$ satisfying (24). The proof is complete.

**Theorem 6.** *Suppose* (*i*), (*ii*), (*iii*) *and* (20), (21). *Suppose further that there are functions* $H(x)$, $K(x)$ *satisfying* (9), (10), (11). *If there is an* $\alpha$, $a < \alpha < 0$ *such that the function* $U_\alpha(x)$ *defined by* (16) *satisfies*

$$2 \operatorname*{g.l.b.}_{0<x<b} \int_0^x f(s, 0)\, ds < -U_\alpha^2(0),$$

*then the solution* $x(t)$ *of* (1), $x(0) = \alpha$, $\dot{x}(0) = 0$ *is periodic with the period* $\omega$

$$\omega > 2\int_{\alpha}^{0} \frac{ds}{U_\alpha(s)}. \tag{25}$$

Proof. From the first part of the proof of Theorem 3 it follows that the trajectory $x = \varphi(t)$, $y = \psi(t)$ satisfying initial contitions $\varphi(0) = \alpha$, $\psi(0) = 0$ is situated above the $x$-axis for such $t$ for which $\psi(t) > 0$ so that it cuts the $y$-axis at a point $(0, \eta)$, $0 < \eta < U_\alpha(0)$. Moreover, an easy modification of introductory part of the proof of *Theorem* 5 assures that this trajectory crosses the $x$-axis at a point $d$, $0 < d < b$ so that the solution $x(t)$ of (1), $x(0) = \alpha$, $\dot{x}(0) = 0$ is periodic and its period $\omega$ satisfies (25).

The following theorem concerns two differential equations

$$\ddot{x} = f_i(x, \dot{x}), \quad i = 1, 2 \tag{$26_i$}$$

and states sufficient conditions under which the existence of periodic solutions of the equation ($26_2$) implies the existence of periodic solutions of ($26_1$).

**Theorem 7.** (*Comparison theorem*). *Let the functions* $f_i$, $i = 1, 2$ *satisfy* (*i*) – (*iv*). *Suppose further that* ($26_2$) *has a periodic solution and that either*

$f_1(x, y) \leqq f_2(x, y)$ *or* $f_1(x, y) \leqq f_2(x, y)$ *for* $x \in I$ *and* $y \leqq 0$. *Then* ($26_1$) *also has a periodic solution.*

Proof. In the phase-plane consider the system

$$\dot{x} = y, \; y = f_i(x, y). \tag{$27_i$}$$

Let $\Gamma_2 : x = \varphi_2(t)$, $y = \psi_2(t)$ be a periodic orbit of ($27_2$). Then there are values $t_1 < t_2$ such that $\psi_2(t_1) = \psi_2(t_2) = 0$, $\varphi_2(t_1) < 0 < \varphi_2(t_2)$ and $\psi_2(t) > 0$ for $t_1 < t < t_2$. If the first inequality is satisfied, let $\Gamma_1 : x = \varphi_1(t)$, $y = \psi_1(t)$ be a solution of ($27_1$) satisfying initial conditions $\varphi_1(0) = \varphi_2(t_1)$, $\psi_1(0) = 0$. The trajectory $\Gamma_1$ enters the second quadrant for increasing $t$ and neither can meet the orbit $\Gamma_2$ nor the negative $x$-axis as it follows from an easy modification of the proof of *Theorem* 3. Hence $\Gamma_1$ is forced to cross the positive $x$-axis and in view of (*iii*), $\Gamma_1$ is a periodic orbit of ($27_1$), too.

If the second inequality is satisfied, we investigate the orbit $\Gamma_3 : x = \varphi_3(t)$, $y = \psi_3(t)$, $\varphi_3(0) = \varphi_2(t_2)$, $\psi_3(0) = 0$, which corresponds to a periodic solution of ($27_1$). Theorem is proved. Replacing $x$ by $-x$ we obtain easily the following modifications of *Theorems* 4,5 and 6.

**Theorem 8.** *Suppose* (*i*) – (*iv*) *and* $f(x, y) \geqq 0$ *for* $a < x < 0$, $y > 0$. *Suppose the existence of* $h(x)$, $k(x)$ *satisfying* (12), (13), (14). *Then for every* $\omega_0$

$$0 < \omega_0 < 2 \operatorname*{l.u.b.}_{0<\beta<b} \min \left\{ \int_0^\beta \frac{\mathrm{d}s}{V_\beta(s)}, \frac{\alpha}{V_\beta(0)} \right\},$$

*where* $V_\beta(x)$ *is defined by* (16) *and* $0 < \alpha < -a$, *there exists a periodic solution with the period* $\omega > \omega_0$.

**Theorem 9.** *Suppose* (*i*), (*ii*), (*iii*) *and* $f(x, 0) < 0$ *for* $0 < x < b$, $f(x, y)$ *nondecreasing with respect to* $y$ *for* $a < x < 0$, $y > 0$. *Moreover, suppose that there exists a positive function* $\gamma(x) \in C^0(0, b]$, $\gamma(x)\, D_L\gamma(x) - f(x, \gamma(x)) \leqq 0$ *for* $0 < x \leqq b$ *and that* $2 \operatorname*{l.u.b.}_{a<x<0} \int_x^0 f(s, 0)\,\mathrm{d}s > \gamma^2(0)$. *Then every solution* $x(t)$ *of* (1) *determined by inital conditions* $x(0) = \xi$, $\dot{x}(0) = 0$, $\xi \in I_+$ *is periodic and there are periodic solutions with the period*

$$\omega > 2 \int_0^b \frac{\mathrm{d}s}{\gamma(s)}.$$

**Theorem 10.** *Suppose* (*i*), (*ii*), (*iii*), $f(x, 0) < 0$ *for* $0 < x < b$ *and* $f(x, y)$ *nondecreasing with respect to* $y$ *for* $a < x < b$, $y > 0$. *Suppose further that there are functions* $h(x)$, $k(x)$ *satisfying* (12), (13), (14). *If there is a* $\beta$, $0 < \beta < b$ *such that the function* $V_\beta(x)$ *defined by* (16) *satisfies*

$$2 \operatorname*{l.u.b.}_{a<x<0} \int_x^0 f(s, 0)\,\mathrm{d}s > V_\beta^2(0),$$

*then the solution $x(t)$ of* (1), $x(0) = \beta$, $\dot{x}(0) = 0$ *is periodic with the period* $\omega$

$$\omega > 2 \int_0^{\beta} \frac{\mathrm{d}s}{V_\beta(s)}\,.$$

## PERIODIC SOLUTIONS OF $\ddot{x} + f(x)\,\dot{x}^{2n} + g(x) = 0$ WITH ARBITRARILY LARGE PERIODS

In this section several applications of preceding theorems to the differential equation

$$\ddot{x} + f(x)\,\dot{x}^{2n} + g(x) = 0 \tag{28}$$

are given. Let us suppose $f(x)$, $g(x)$ to be continuous on $(-\infty, \infty)$. Moreover, let any initial problem for the equation $z' + 2f(x)\,z^n + g(x) = 0$ be unique. The assumption

$$xg(x) > 0 \quad for \quad x \neq 0 \tag{29}$$

is sufficient for the existence of periodic solutions of (28) in view of *Theorem* 1. We say the equation (28) has the property $P$ if it has solutions with arbitrarily large periods. For the sake of brevity denote $G(x) = \int_0^x g(s)\,\mathrm{d}s$.

**Corollary 1.** *Suppose* (29), $xf(x) < 0$ *for* $x \neq 0$, $g(x)/f(x)$ *bounded in a neighbourhood of the origin and that there is a constant* $A$ *such that* $x[x + f(x)\,x^{2n} + g(x)] \leqq 0$ *for* $|x| > A$. *Then* (28) *has the property* $P$.

Proof. Consider the vector-field determined by the equation $yy' + f(x)\,y^{2n} + g(x) = 0$ in the half plane $y > 0$. The points at which $y' = 0$ lie on the curves $x = 0$ and $y^{2n} = -g(x)/f(x)$. The latter curve is defined for all $x \neq 0$ and is evidently bounded for $0 < |x| \leqq A$. Let $B = \operatorname*{l.u.b.}_{0<|x|\leqq A} \sqrt[2n]{-g(x)/f(x)}$ and put $M = \max(A, B)$. Setting $\gamma(x) = M$ for $|x| \leqq M$ and $\gamma(x) = |x|$ for $|x| > M$, the assumption (5) is satisfied and *Corollary* 1 follows immediately from *Theorem* 2.

**Corollary 2.** *Suppose* (29), $f(x) < 0$ *for all* $x$ *and that there is a constant* $M > 0$ *such that* $x + f(x)\,x^{2n} + g(x) \leqq 0$ *for* $x > M$. *Then* (28) *has the property* $P$.

Proof. If we choose $\gamma(x) = M$ for $x < M$ and $\gamma(x) = x$ for $x \geqq M$, the condition (7) is satisfied and *Corollary* 2 follows from *Theorem* 2.

**Corollary 3.** *Suppose* (29), $f(x) > 0$ *for all* $x$ *and that there is a constant* $M > 0$ *such that* $x + f(x)\,x^{2n} + g(x) \geqq 0$ *for* $x < -M$. *Then* (28) *has the property* $P$.

Proof. Choosing $\gamma(x) = -x$ for $x \leqq -M$ and $\gamma(x) = M$ for $x > -M$ the assumptions of *Theorem* 2 are fulfilled with (6). This corollary was proved by G. VIL-

LARI in [2]. It generalizes a former result of S. SEDZIWY [1]. Another application of *Theorem* 2 can be obtained if we put $\gamma(x) = h(x)$ for $x \leqq 0$ and $h(x) = h(0)$ for $x > 0$.

*Suppose* (29), $f(x) > 0$ *for all* $x$ *and that there is a function* $h(x) \in C^1(-\infty, 0]$ *such that* $h(x) > 0$, $h(x)\,h'(x) + f(x)\,h^{2n}(x) + g(x) \geqq 0$,

$$\int_{-\infty}^{0} \frac{ds}{h(s)} = \infty. \tag{30}$$

This is a result of J. W. HEIDEL [4]. In the original version [3] the assumption (30) was missed.

**Corollary 4.** *Suppose* (29), $xf(x) \leqq 0$ *for all* $x$ *and*

$$\lim_{x \to -\infty} \int_{x}^{0} [G(x) - G(s)]^{-1/2}\, ds = \infty, \tag{31}$$

$$\lim_{x \to \infty} \int_{0}^{x} [G(x) - G(s)]^{-1/2}\, ds = \infty. \tag{32}$$

*Then* (28) *has the property P.*

Proof. This is a consequence of *Theorem* 3 choosing $H(x) = -g(x)$ for $x \leqq 0$, $h(x) = -g(x)$ for $x \geqq 0$, $K(x) = k(x) = 1$ for all $x$. Note that (31), (32) are, e.g., satisfied if $G(x) \sim kx^2 + m$, $k > 0$, $m \geqq 0$ for $x \to \pm\infty$.

**Corollary 5.** *Suppose* (29), (31) and $f(x) \geqq 0$ *for all* $x$. *Then* (28) *has the property P.*

Proof. This is a consequence of *Theorem* 4 choosing $H(x) = -g(x)$ for $x \leqq 0$ and $K(x) = 1$ for $x \geqq 0$.

Note that these conditions are more general than those of G. VILLARI [2], where instead of (31) it is assumed $\lim\limits_{x \to -\infty} G(x) = c < \infty$.

**Corollary 6.** *Suppose* $g(x) < 0$ *for* $x < 0$, $f(x) \geqq 0$ *for* $x > 0$ *and that there is a constant* $M > 0$ *such that* $f(x)\,M^{2n} + g(x) \geqq 0$ *for* $x < 0$ *and* $2\ \operatorname*{l.u.b.}\limits_{x>0} G(x) > M^2$. *Then* (28) *has the property P.*

Proof. This corollary is a consequence of *Theorem* 5 for $\gamma(x) \equiv M$.

**Corollary 7.** *Suppose* $g(x) < 0$ *for* $x < 0$, (31), $f(x) \geqq 0$ *for all* $x$, $\operatorname*{l.u.b.}\limits_{x>0} G(x) > C$ *if* $\lim\limits_{x \to -\infty} G(x) = C$, $\operatorname*{l.u.b.}\limits_{x>0} G(x) = \infty$ *if* $\lim\limits_{x \to -\infty} G(x) = \infty$. *Then* (28) *has the property P.*

Proof. This corollary is a consequence of *Theorem* 6 choosing $H(x) = -g(x)$, $K(x) = 1$; then we have $U_\alpha(x) = \sqrt{2[G(\alpha) - G(x)]}$ and $U_\alpha(0) = \sqrt{2G(\alpha)}$.

**Corollary 8.** *Suppose* (29), (32) and $f(x) \leqq 0$ *for all* $x$. *Then* (28) *has the property* $P$.

Proof. This corollary is a consequence of *Theorem* 8 for $h(x) = -g(x)$, $k(x) = 1$.

**Corollary 9.** *Suppose* $g(x) > 0$ *for* $x > 0$, $f(x) \leqq 0$ *for* $x < 0$ *and that there is a constant* $M > 0$ *such that* $f(x)\, M^{2n} + g(x) \leqq 0$ *for* $x > 0$ *and* $2 \operatorname*{g.l.b.}_{x<0} G(x) < -M^2$. *Then* (28) *has the property* $P$.

Proof. This corollary is a consequence of *Theorem* 9 for $\gamma(x) = M$.

**Corollary 10.** *Suppose* $g(x) > 0$ *for* $x > 0$, (32), $f(x) \geqq 0$ *for all* $x$, $\operatorname*{l.u.b.}_{x<0} G(x) > C$ *if* $\lim_{x\to\infty} G(x) = C$ *and* $\operatorname*{l.u.b.}_{x<0} G(x) = \infty$ *if* $\lim_{x\to\infty} G(x) = \infty$. *Then* (28) *has the property* $P$.

Proof. This corollary is a consequence of *Theorem* 10 for $h(x) = -g(x)$.

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