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ON BICOMPACT SPACES

BY EDUARD ČECH

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The theory of bicompact spaces was extensively studied by P. Alexandroff and P. Urysohn in their paper *Mémoire sur les espaces topologiques compacts*, Verhandlingen der Kon. Akademia Amsterdam, Deel XIV, No. 1, 1929; I shall refer to this paper with the letters AU. An important result was added by A. Tychonoff in his paper *Über die topologische Erweiterung von Räumen*, Math. Annalen 102, 1930, who proved that complete regularity is the necessary and sufficient condition for a topological space to be a subset of some bicompact Hausdorff space. As a matter of fact, Tychonoff proves more, viz. that, given a completely regular space S, there exists a bicompact Hausdorff space $\beta(S)$ such that (i) S is dense in $\beta(S)$, (ii) any bounded continuous real function defined in the domain S admits of a continuous extension to the domain $\beta(S)$. It is easily seen that $\beta(S)$ is uniquely defined by the two properties (i) and (ii). The aim of the present paper is chiefly the study of $\beta(S)$.

The paper is divided into four chapters. In chapter I, I briefly resume some well known definitions adding a few simple remarks. In particular I show that an *arbitrary* topological space S determines a completely regular space $\rho(S)$ such that a good deal of topology of S reduces to the topology of $\rho(S)$, this being true in particular for the theory of real valued continuous and Baire functions. Chapter II contains the theory of the bicompact space $\beta(S)$ mentioned above. Here I shall recall only a few results of chapter II. First, if the space S is *normal*, then $\beta(S)$ may be defined without any reference to continuous real function since property (ii) may be replaced by the following: if two closed subsets of S have no common point, then their closures in $\beta(S)$ have no common point either. Second, if the space S satisfies the first countability axiom, then S is completely determined by $\beta(S)$, S being simply the set of all points of $\beta(S)$ where the first countability axiom holds true. This implies that in this case (embracing the case of metrizable spaces) the whole topology of S may be reduced to the topology of the bicompact space $\beta(S)$. Hence it is evident that it is highly desirable to carry further the study of bicompact spaces and in particular of $\beta(S)$. Of course it must be emphasized that $\beta(S)$ may be defined only formally (not constructively) since it exists only in virtue of Zermelo's theorem. If I denotes the space of integer numbers, then I think it is impossible to determine effectively (in the sense of Sierpiński) a point of $\beta(I) - I$. I was even unable to determine the cardinal number of $\beta(I)$. (The paper contains several other unsolved problems.) The space $\beta(I) - I$ furnishes incidentally a positive solution of a problem proposed by Alexandroff and Urysohn (AU, p. 54:

Existe-t-il un espace bicompact ne contenant aucun point (κ)? The authors write in this connection: La résolution affirmative de ce problème nous donnerait un exemple des espace bicompacts d'une nature toute différente de celle des espaces connus jusqu'à présent). In chapter III, I call a completely regular space S *topologically complete* if S is a G_δ in $\beta(S)$. The reason for this designation lies in the fact that, if S is metrizable, it has this property if and only if it is homeomorphic with a metric complete space. The proof is an easy adaptation of Hausdorff's well known proof of the theorem that a G_δ in a metric complete space is a homeomorph of a metric complete space. In chapter IV, I consider *locally normal* spaces and I prove that a locally normal space S is always an open subset of some normal space. This was of course to be expected but I think it would be difficult to prove without the theory of $\beta(S)$.

I

A set S is called a *topological space* (and its elements are called *points*) if there is given a class $\mathfrak{F}$ of subsets of S (called *closed* subsets of S) such that (1) the whole space S and the vacuous set 0 are closed, (2) the intersection of any family of closed sets is closed, (3) the sum of two closed sets is closed. A set $G \subset S$ is called *open*, if the complementary set $S - G$ is closed. A *neighborhood* of a set $A \subset S$ (A may consist of a single point) is an open set containing A.

The intersection of all closed sets containing a given set A is called the *closure* of A and is denoted by $\bar{A}$. The closure operation has the following properties: (1) $\bar{0} = 0$, (2) $A \subset \bar{A}$, (3) $\overline{A + B} = \bar{A} + \bar{B}$, (4) $\bar{\bar{A}} = \bar{A}$. Conversely, it is possible to define the general notion of a topological space starting with an operation $\bar{A}$ subject only to conditions (1)-(4) and defining closed sets by the condition $\bar{A} = A$.

An *open base* of a topological space S is a class $\mathfrak{B}$ of open sets such that any open set is the sum of some of the elements of $\mathfrak{B}$. The class $\mathfrak{H}$ of *all* open sets is a particular open base. Any open base $\mathfrak{B}$ has the following properties: (1) given a point $x \epsilon S$, there exists a $U \epsilon \mathfrak{B}$ such that $x \epsilon U$, (2) given a point $x \epsilon S$ and two sets U and V such that $U \epsilon \mathfrak{B}$, $V \epsilon \mathfrak{B}$, $x \epsilon UV$, there exists a set W such that $W \epsilon \mathfrak{B}$, $x \epsilon W$, $W \subset UV$. Conversely it is possible (and the possibility is utilized very frequently in practice) to define a topological space starting with a class $\mathfrak{B}$ subject only to condition (1) and (2); the closure $\bar{A}$ of a set $A \subset S$ consists then of all the points x such that

$$U \epsilon \mathfrak{B},\ x \epsilon U \text{ implies } UA \neq 0.$$

A fixed subset T of a topological space S is always considered as a topological space, defining a set $A \subset T$ to be *relatively closed* (i.e. closed in the space T) whenever A is the intersection of T with some closed subset of S. A set $A \subset T$ is *relatively open* whenever A is the intersection of T with some open subset of S. The *relative closure* of a set $A \subset T$ is the intersection $T\bar{A}$ of T with the closure of A in the space S. Any open base $\mathfrak{B}$ of S determines an open base $\mathfrak{B}_0$ of T; the elements of $\mathfrak{B}_0$ are the intersections of T with the elements of $\mathfrak{B}$.

A *mapping* f of a topological space S_1 into a topological space S_2 is an operation attaching to each point $x \in S_1$ a definite point $f(x) \in S_2$; we always suppose that, given any point $y \in S_2$, there exists at least one point $x \in S_1$ such that $f(x) = y$. The space S_1 is the *domain* of f, S_2 is its *range*. The *image* $f(A)$ of a set $A \subset S_1$ is the set of all points $f(x)$, x running over A. The *inverse image* $f^{-1}(B)$ of a set $B \subset S_2$ is the set of all points $x \in S_1$ such that $f(x) \in B$. The mapping f is *one-to-one* if

$$x_1 \in S_1, \; x_2 \in S_1, \; x_1 \neq x_2 \text{ implies } f(x_1) \neq f(x_2).$$

If f is one-to-one, then the inverse operation f^{-1} is a one-to-one mapping of S_2 into S_1. The mapping f will be called a *function* if its range consists of real numbers. The function f is *bounded* if its range is a bounded set.

The mapping f is called *continuous at a point* $x \in S_1$ if, given any neighborhood V of $f(x)$, there exists a neighborhood U of x such that $f(U) \subset V$. f is called *continuous* (simply) if it is continuous at any point $x \in S_1$. f is called *homeomorphic* if it is one-to-one and if both f and f^{-1} are continuous. f is continuous, if and only if the inverse image of any closed subset of S_2 is a closed subset of S_1.

A set $A \subset S$ is called a G_δ-set if there exists a countable sequence $\{G_n\}$ of open sets such that $A = \prod_1^\infty G_n$; A is called an F_σ-set if there exists a countable sequence $\{F_n\}$ of closed sets such that $A = \sum_1^\infty F_n$. The complement of a G_δ-set is an F_σ-set and vice-versa.

S is called a *Kolmogoroff space*[1] if the closures of any two distinct points are distinct. S is called a *Riesz space*[2] if any single point is closed. S is a Riesz space if and only if the intersection of all the neighborhoods of any point x consists of x only. S is called a *Hausdorff space* if the intersection of the closures of all the neighbhorhoods of any point x consists of x only. Any Riesz space is a Kolmogoroff space. Any Hausdorff space is a Riesz space. Any subset of a Kolmogoroff space is a Kolmogoroff space. Any subset of a Riesz space is a Riesz space. Any subset of a Hausdorff space is a Hausdorff space. Let $\mathfrak{B}$ be any open base of S. S is a Kolmogoroff space if and only if, given two distinct points x and y, there exists a set $U \in \mathfrak{B}$ containing precisely one of the points x and y. S is a Riesz space if and only if, given two distinct points x and y, there exists a set $U \in \mathfrak{B}$ containing x and not containing y. S is a Hausdorff space if and only if, given two distinct points x and y, there exist sets U and V such that $U \in \mathfrak{B}$, $V \in \mathfrak{B}$, $x \in U$, $y \in V$, $UV = 0$.

Now we proceed to prove that *the theory of general topological spaces* (in the sense precised above) *can be completely reduced to the theory of Kolmogoroff spaces.* Let S be a topological space. Two points $x \in S$ and $y \in S$ will be called equivalent (for the time being) if $\bar{x} = \bar{y}$. Let F be any closed subset of S and let x and y be two equivalent points; if $x \in F$, then $\bar{x} \subset F$, since F is closed, but $y \in \bar{y}$ and $\bar{y} = \bar{x}$, so that $y \in F$. It follows that any closed subset of S consists of complete

[1] See P. Alexandroff and H. Hopf, *Topologie* I, p. 58.

[2] See G. Birkhoff, *On the combination of topologies*, Fund. Math. 26, p. 162.

classes of mutually equivalent points. Now let us attach to each point $x \in S$ a new symbol $\tau(x)$ chosen in such manner that $\tau(x) = \tau(y)$ if and only if x and y are equivalent; let us call S_0 the set of the symbols $\tau(x)$, so that τ is a mapping of S into S_0. A set $A_0 \subset S_0$ will be considered as closed if and only if its inverse image $\tau^{-1}(A_0)$ is a closed subset of S. It is evident that S_0 is a topological space and that τ is a continuous mapping. Further it is evident that for any set $A \subset S$ we have $\tau(\bar{A}) = \overline{\tau(A)}$; in particular $\tau(\bar{x}) = \overline{\tau(x)}$ for any $x \in S$. If $\tau(x) \neq \tau(y)$, we have $\bar{x} \neq \bar{y}$; since the sets $\bar{x}$ and $\bar{y}$ are closed, it easily follows that $\tau(\bar{x}) \neq \tau(\bar{y})$, or $\overline{\tau(x)} \neq \overline{\tau(y)}$, so that S_0 is a Kolmogoroff space. Conversely, let S_0 be a Kolmogoroff space. Let τ be a mapping of a set S into S_0. Let us call closed in S the inverse image of any closed subset of S_0. Then S is the most general topological space and τ has the previous meaning. Evidently the topology of S is quite completely described by that of S_0.

S is called a *regular space* if it is a Kolmogoroff space having the following property: given a neighborhood U of a point x, there exists a neighborhood V of x such that $\bar{V} \subset U$.[3] We shall prove that any regular space S is a Hausdorff space.[4] Let x and y be two distinct points of S. If we had both $x \in \bar{y}$ and $y \in \bar{x}$, it would follow, since $\bar{x}$ and $\bar{y}$ are closed, that $\bar{x} \subset \bar{y}$ and $\bar{y} \subset \bar{x}$, i.e. $\bar{x} = \bar{y}$, which is impossible. The argument being symmetrical, we may suppose that x does not belong to $\bar{y}$, so that $S - \bar{y}$ is a neighborhood of x. Hence there exists a neighborhood U of x such that $\bar{U} \subset S - \bar{y}$. Putting $V = S - \bar{U}$, we have two open sets U and V such that $x \in U$, $y \in V$, $UV = 0$, so that S is a Hausdorff space.

Any subset of a regular space is a regular space.

S is called a *completely regular space* if it is a Kolmogoroff space having the following property: given a closed set F and a point $a \in S - F$, there exists a continuous function f (in the domain S) such that $f(a) = 0$ and $f(x) = 1$ for any $x \in F$.[5] It is easy to see that a completely regular space is regular and that any subset of a completely regular space is a completely regular space.

Now we shall start with an arbitrary topological space S and we shall attach to it a uniquely defined completely regular space $\rho(S)$ in such manner that a great deal of topology of S may be reduced to that of $\rho(S)$. Two points x and y of S will be called equivalent (for the time being) if $f(x) = f(y)$ for every continuous function f (in the domain S). To each point $x \in S$ let us attach a new symbol $\rho(x)$ chosen in such a manner that $\rho(x) = \rho(y)$ if and only if x and y are equivalent;[6] let us call S_1 the set of all the symbols $\rho(x)$, so that ρ is a mapping of S into $S_1 = \rho(S)$. We shall introduce a topology in S_1 by defining an open

[3] The neighborhoods may here be restricted to a given open base of S.

[4] This is usually done assuming *a priori* that S is a Riesz space; for this point I am indebted to Dr. K. Koutský.

[5] We may assume that $0 \leq f(x) \leq 1$ for every $x \in S$, since we could replace f with φ by defining $\varphi(x) = f(x)$ if $0 \leq f(x) \leq 1$, $\varphi(x) = 0$ if $f(x) < 0$, and $\varphi(x) = 1$ if $f(x) > 1$.

[6] It is evident that $\tau(x) = \tau(y)$ implies $\rho(x) = \rho(y)$, but of course we may restrict ourselves to Kolmogoroff spaces.

base $\mathfrak{B}$ for S_1. An element $[f, I]$ of $\mathfrak{B}$ will be defined by a continuous function f in the domain S and an open interval I, $[f, I]$ consisting of the points $\rho(x)$ of S_1 such that $f(x) \in I$. To prove that S_1 is a topological space we have to verify two things. First, for any $a \in S$, there evidently exists an $[f, I]$ containing $\rho(a)$. Second, let $\rho(a)$ belong both to $[f_1, I_1]$ and to $[f_2, I_2]$; we have to prove that there exists an $[f, I]$ such that $\rho(a) \in [f, I]$ and $[f, I] \subset [f_1, I_1]\cdot[f_2, I_2]$. There exists a number $\varepsilon > 0$ such that, for $i = 1$ and for $i = 2$, the interval $f_i(a) - \varepsilon < t < f_i(a) + \varepsilon$ is a subset of I_i. It is easy to see that we may put $f(x) = |f_1(x) - f_1(a)| + |f_2(x) - f_2(a)|$, choosing I to be the interval $-\varepsilon < t < \varepsilon$. Hence S_1 is a topological space.

Since the topology of S_1 was defined by means of *continuous* functions in the domain S, it is easy to see that ρ is a continuous mapping of S into S_1 so that, if φ is any continuous function in the domain S_1, $f(x) = \varphi[\rho(x)]$ is a continuous function in the domain S. Moreover, in our case the converse is also true: *any continuous function in the domain S has the form $f(x) = \varphi[\rho(x)]$, φ being a continuous function in the domain S_1.*

If $\rho(a)$ and $\rho(b)$ are two distinct points of S_1, then there exists a continuous function f in the domain S such that $f(a) \neq f(b)$. There exist two disjoined open intervals I_1 and I_2 such that $f(a) \in I_1$ and $f(b) \in I_2$. Then $[f, I_1]$ and $[f, I_2]$ are two disjoined open subsets of S_1 and $\rho(a) \in [f, I_1]$, $\rho(b) \in [f, I_2]$. It follows that S_1 is a Hausdorff space. As a matter of fact, S_1 is a completely regular space. Let Φ be a closed subset of S_1 not containing the point $\rho(a)$. There exists an $[f, I]$ such that $\rho(a) \in [f, I] \subset S_1 - \Phi$; we may suppose that I consists of all numbers t such that $|t - f(a)| < \varepsilon (\varepsilon > 0)$. If $|f(x) - f(a)| \geqq \varepsilon$, put $g(x) = 1$; if $|f(x) - f(a)| < \varepsilon$, put $g(x) = \varepsilon^{-1}. |f(x) - f(a)|$. Then g is a continuous function in the domain S, so that there exists a continuous function φ in the domain S_1 such that $g(x) = \varphi[\rho(x)]$. It is easy to see that $\varphi[\rho(a)] = 0$ and $\varphi(x) = 1$ for each $x \in \Phi$.

Let F be a closed subset of S. We shall prove that *a necessary and sufficient condition for the set $\rho(F)$ to be closed in S_1 is that for any point*

$$a \in S - \rho^{-1}[\rho(F)]$$

there exists a continuous function f in the domain S such that $f(a) = 0$ and $f(x) = 1$ for each $x \in F$. First suppose the condition satisfied. If $\rho(F)$ were not closed in S_1, we could choose a point a such that

$$\rho(a) \in \overline{\rho(F)} - \rho(F).$$

Since $\rho(a) \in S_1 - \rho(F)$, there would exist a continuous function f in the domain S such that $f(a) = 0$ and $f(x) = 1$ for each $x \in F$. There would exist a continuous function φ in the domain S_1 such that $f(x) = \varphi[\rho(x)]$. For $x \in \rho(F)$ we would have $\varphi(x) = 1$; since φ is continuous, it easily follows that $\varphi(x) = 1$ for $x \in \overline{\rho(F)}$, in particular $\varphi[\rho(a)] = 1$, i.e. $f(a) = 1$, which is a contradiction. Secondly, suppose $\rho(F)$ closed in S_1. Let $a \in S - \rho^{-1}[\rho(F)]$. Then $\rho(a) \in S_1 - \rho(F)$. Since S_1 is completely regular, there exists a continuous function φ in the domain

S_1 such that $\varphi[\rho(a)] = 0$ and $\varphi(x) = 1$ for each $x \in \rho(F)$. Putting $f(x) = \varphi[\rho(x)]$, we have a continuous function f in the domain S such that $f(a) = 0$ and $f(x) = 1$ for each $x \in F$.

As a corollary, we obtain that, if the space S itself is completely regular, the mapping ρ is homeomorphic.

The following property is characteristic for completely regular spaces S: *Let σ be a continuous mapping of S into a topological space R such that each continuous function f in the domain S has the form $f(x) = \varphi[\sigma(x)]$, φ being a continuous function in the domain R. Then the mapping σ is homeomorphic.* The property cannot be true if S is not completely regular, as is seen by putting $\sigma = \rho$. Hence suppose that S is completely regular. If $a \in S$, $b \in S$, $a \neq b$, there exists a continuous function f in the domain S such that $f(a) \neq f(b)$; since $f(x) = \varphi[\sigma(x)]$, we have $\sigma(a) \neq \sigma(b)$, i.e. the mapping σ is one-to-one. It remains to show that if F is a closed subset of S the set $\sigma(F)$ is closed in R. If $\sigma(F)$ is not closed, there exists a point $a \in S$ such that

$$\sigma(a) \in \overline{\sigma(F)} - \sigma(F).$$

There exists a continuous function f in the domain S such that $f(a) = 0$ and $f(x) = 1$ for each $x \in F$. We may put $f(x) = \varphi[\sigma(x)]$ and we have $\varphi[\sigma(a)] = 0$ and $\varphi(x) = 1$ for each $x \in \sigma(F)$. Since φ is continuous, we must have $\varphi(x) = 1$ for each $x \in \overline{\sigma(F)}$, hence for $x = a$, which is a contradiction.

Consider the following three properties of a topological space S: (1) If F_1 and F_2 are two closed sets such that $F_1F_2 = 0$, there exist two open sets G_1 and G_1 such that $F_1 \subset G_1$, $F_2 \subset G_2$, $G_1G_2 = 0$. (2) If F_1 and F_2 are two closed sets such that $F_1F_2 = 0$, there exists a continuous function f in the domain S such that $f(x) = 0$ for each $x \in F_1$ and $f(x) = 1$ for each $x \in F_2$.[5] (3) If F is a closed set and if φ is a bounded[7] continuous function in the domain F, there exists a continuous function f in the domain S such that $f(x) = \varphi(x)$ for each $x \in F$. It is easily seen that (2) is formally stronger than (1) and that (3) is formally stronger than (2). But Urysohn proved[8] that all three properties are equivalent to one another. A space having these properties is called *normal*. Property (2) shows that a normal Riesz space is a completely regular space (hence a regular space, therefore a Hausdorff space).

If the space S is normal, then $\rho(S)$ is normal as well. Let Φ_1 and Φ_2 be two closed subsets of $\rho(S)$ such that $\Phi_1\Phi_2 = 0$. Then $F_1 = \rho^{-1}(\Phi_1)$ and $F_2 = \rho^{-1}(\Phi_2)$ are two closed subsets of S such that $F_1F_2 = 0$. Since S is normal, there exists a continuous function f in the domain S such that $f(x) = 0$ for each $x \in F_1$ and $f(x) = 1$ for each $x \in F_2$. There exists a continuous function φ in the domain $\rho(S)$ such that $f(x) = \varphi[\rho(x)]$. Evidently $\varphi(x) = 0$ for each $x \in \Phi_1$ and $\varphi(x) = 1$ for each $x \in \Phi_2$.

If the space S is normal, then for $a \in S$, $b \in S$ we have $\rho(a) = \rho(b)$ if and only if

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[7] It is easy to prove that the word *bounded* may be omitted.

[8] P. Urysohn, *Über die Mächtigkeit zusammenhängender Mengen,* Math. Annalen 94, 1925.

$\bar{a} \cdot \bar{b} \neq 0$. Suppose first that $c \in \bar{a} \cdot \bar{b}$. If f is a continuous function in the domain S, it is easy to see that $f(a) = f(c) = f(b)$, whence $\rho(a) = \rho(b)$. Secondly, suppose that $\bar{a} \cdot \bar{b} = 0$. Since S is normal, there exists a continuous function f in the domain S such that $f(x) = 0$ for each $x \in \bar{a}$ and $f(x) = 1$ for each $x \in \bar{b}$, whence $f(a) = 0, f(b) = 1$.

If the space S is normal and if F is a closed subset of S, then $\rho(F)$ is a closed subset of $\rho(S)$. Let $a \in S - \rho^{-1}[\rho(F)]$. For $x \in F$ we have $\rho(a) \neq \rho(x)$, whence $\bar{a} \cdot \bar{x} = 0$; therefore $\bar{a} \cdot F = 0$. Hence there exists a continuous function f in the domain S such that $f(x) = 1$ for each $x \in F$ and $f(x) = 0$ for each $x \in \bar{a}$, in particular $f(a) = 0$. We know that this implies that $\rho(F)$ is closed in $\rho(S)$.

The last two theorems show that, if S is normal, the space $\rho(S)$ and its topology may be completely described without any explicit reference to continuous functions: The space $\rho(S)$ consists of symbols $\rho(x)$ attached to single points $x \in S$, $\rho(x)$ and $\rho(y)$ being identical if and only if $\bar{x} \cdot \bar{y} \neq 0$; and a set $\Phi \subset \rho(S)$ is closed in $\rho(S)$ if and only if the set $\rho^{-1}(\Phi)$ is closed in S. It is an interesting problem to give a similar description of $\rho(S)$ in the general case.

If the space S is normal, then a necessary and sufficient condition for a set $A \subset S$ to be both closed and a G_δ is the existence of a continuous function f in the domain S such that $f(x) = 0$ if and only if $x \in A$. Suppose first that such a function f exists. Then $A = \{f(x) = 0\}$ is a closed set and $G_n = \{ |f(x)| < 1/n\}$ are open sets and $A = \prod G_n$. Conversely let $A = \bar{A} = \prod G_n$, G_n being open. Since S is normal, there exist continuous functions f_n in the domain S such that $f_n(x) = 0$ for $x \in A$, $f_n(x) = 1$ for $x \in S - G_n$, $0 \leqq f_n(x) \leqq 1$ for $x \in S$. It is sufficient to put $f(x) = \sum 2^{-n} \cdot f_n(x)$.

A point x of a topological space S is called a *complete limit point* of a set $A \subset S$ if, for any neighborhood U of x, the cardinal number of the set AU is equal to the cardinal number of the set A. A family $\mathfrak{C}$ of subsets of S is called *monotonic* if for any two sets $A \in \mathfrak{C}$, $B \in \mathfrak{C}$ we have either $A \subset B$ or $B \subset A$. A family $\mathfrak{C}$ of subsets of S is called a *covering* of S if each point of S belongs to some set of $\mathfrak{C}$.

Consider the following three properties of a topological space S: (1) Every infinite subset possesses at least one complete limit point. (2) A monotonic family of non-vacuous closed subsets has a non-vacuous intersection. (3) Any covering of S consisting of open sets contains a finite covering of S. It is known that all three properties are equivalent to one another.[9] A space having these properties is called *bicompact*. It is known that *a bicompact Hausdorff space is normal*[10] (hence completely regular). *A closed subset of a bicompact space is a bicompact space.* Conversely, *a bicompact subset of a Hausdorff space is closed.*[11] It easily follows that *a one-to-one continuous mapping of a bicompact Hausdorff space is homeomorphic.*

Let $\{S_\iota\}$ be a family of sets; the subscript ι runs over an arbitrarily given set I. The cartesian product $\mathfrak{P}_\iota S_\iota$ of the family $\{S_\iota\}$ is the set of all families $x = \{x_\iota\}$,

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[9] AU, p. 8.

[10] AU, p. 26.

[11] AU, p. 47.

each x_ι belonging to S_ι. The x_ι's are called the coordinates of x. If every S_ι is a topological space, we introduce a topology into $S = \mathfrak{P}_\iota S_\iota$ by means of the following open base $\mathfrak{B}$: The elements of $\mathfrak{B}$ are sets of the form $\mathfrak{P}_\iota G_\iota$ where (1) each G_ι is an open subset of S_ι, (2) $G_\iota = S_\iota$ except for a finite number of subscripts ι. It is easy to see that S is a Kolmogoroff space, a Riesz space, a Hausdorff space, a regular space, a completely regular space, if and only if *every* factor space S_ι belongs to the corresponding category of spaces. If S is normal, every S_ι is normal as well; but the converse is false.

The cartesian product $S = \mathfrak{P}_\iota S_\iota$ of any family of bicompact spaces ia a bicompact space. Using Zermelo's theorem, we may suppose that the set I consists of all ordinal numbers less than a given ordinal number. Let there be given an infinite subset A of S. We have to construct a complete limit point $z = \{z_\iota\}$ of S. According to the way the topology of S was introduced, it is sufficient to construct the coordinates z_ι by transfinite induction, choosing each $z_\iota \epsilon S_\iota$ in such a way that it have the following property π_ι: If there is given a finite number of subscripts $\iota_n \leqq \iota$ and, for each ι_n, a neighborhood G_n of z_{ι_n} (in the space S_{ι_n}), then the cardinal number of the intersection of A with the set of those points $x = \{x_\iota\}$ for which $x_{\iota_n} \epsilon G_n$ (for each of the given subscripts ι_n) is equal to the cardinal number of A. We need only prove that the definition of the z_ι's by transfinite induction may be carried through. Hence suppose that, for a definite value $\lambda \epsilon I$, the points z_ι (with property π_ι) having already been constructed for $\iota < \lambda$, it is impossible to choose $z_\lambda \epsilon S_\lambda$ with property π_λ. Then, for every point $y_\lambda \epsilon S_\lambda$, there exist: a neighborhood $T(y_\lambda)$ of the point y_λ (in the space S_λ), a finite (perhaps vacuous) set $M(y_\lambda)$ of subscripts $\iota < \lambda$ and, for each $\iota \epsilon M(y_\lambda)$, a neighborhood $G(z_\iota, y_\lambda)$ of the point z_ι (in the space S_ι) such that the cardinal number of the set $A \cdot H(y_\lambda) \cdot K(y_\lambda)$ is less than the cardinal number of A, where $H(y_\lambda)$ is the set of all points $x = \{x_\iota\}$ for which $x_\lambda \epsilon T(y_\lambda)$ and $K(y_\lambda)$ is the set of all points $x = \{x_\iota\}$ for which $x_\iota \epsilon G(z_\iota, y_\lambda)$ for every $\iota \epsilon M(y_\lambda)$. Since the space S_λ is bicompact, there exists a finite set of points $y_\lambda^{(i)} \epsilon S_\lambda (1 \leqq i \leqq m < \infty)$ such that

$$(1) \qquad \sum_{i=1}^{m} T(y_\lambda^{(i)}) = S_\lambda.$$

The cardinal number of the set

$$(2) \qquad \sum_{i=1}^{m} A \cdot H(y_\lambda^{(i)}) \cdot K(y_\lambda^{(i)})$$

is less than the cardinal number of A. On the other hand, it follows from (1) that

$$\sum_{i=1}^{m} H(y_\lambda^{(i)}) = S$$

so that the set (2) contains the set

$$(3) \qquad A \cdot \prod_{i=1}^{m} K(y_\lambda^{(i)}).$$

It follows that the cardinal number of the set (3) is less than the cardinal number of A. But it is easy to see that this is in contradiction with property π_μ, choosing $\mu < \lambda$ and $\mu \geqq \iota$ for every $\iota \in \sum_i M(y_\lambda^{(i)})$.

II

Since a bicompact Hausdorff space is completely regular, every subset of a bicompact Hausdorff space is also completely regular. Following Tychonoff, we shall prove conversely that *every completely regular space is a subset of some bicompact Hausdorff space.*

Let S be given completely regular space. Let T denote the interval $0 \leqq t \leqq 1$. Let Φ denote the set of all continuous functions f in the domain S such that $f(S) \subset T$. Choose a set I having the same potency as the set Φ, so that there exists a one-to-one mapping of I into Φ; let f_ι be the function corresponding to $\iota \in I$. For $\iota \in I$, put $T_\iota = T$ and let R be the cartesian product $\mathfrak{P}_\iota T_\iota$. Since every T_ι is a bicompact Hausdorff space, R is also a bicompact Hausdorff space. For any $x \in S$, put $g(x) = \xi = \{\xi_\iota\} \in R$, where $\xi_\iota = f_\iota(x)$. Then g is a mapping of the space S into the space $S^* = g(S) \subset R$. It is easy to see that the mapping g is homeomorphic. For $\iota \in I$ and $\xi \in R$, put $\varphi_\iota(\xi) = \xi_\iota$. Then φ_ι is a continuous function in the domain R such that $\varphi_\iota(R) = T$. Moreover, we see that $\varphi_\iota[g(x)] = f_\iota(x)$ for $x \in S$.

If S is a completely regular space, let $\beta(S)$ designate any topological space having the following four properties: (1) *$\beta(S)$ is a bicompact Hausdorff space,* (2) $S \subset \beta(S)$, (3) *S is dense in $\beta(S)$* (i.e. the closure of S in the space $\beta(S)$ is the whole space $\beta(S)$), (4) *every bounded continuous function f in the domain S may be extended*[12] *to the domain $\beta(S)$* (i.e. there exists a continuous function φ in the domain $\beta(S)$ such that $\varphi(x) = f(x)$ for every $x \in S$).

The space $\beta(S)$ exists for every completely regular S. Using the above notation, we easily see that the closure of S^* in the space R has the properties (1)-(4) relatively to S^*, so that $\beta(S^*)$ exists. Since S and S^* are homeomorphic, $\beta(S)$ exists as well.

Given a completely regular space S, the space $\beta(S)$ is essentially unique. More precisely: If B_1 and B_2 both have properties (1)-(4) of $\beta(S)$, then there exists a homeomorphic mapping h of B_1 into B_2 such that $h(x) = x$ for each $x \in S$. This is but a particular case of the following theorem: *Let S be a completely regular space. Let B be a space having properties* (1)-(3) of $\beta(S)$ (but not necessarily property (4)). *Then there exists a continuous mapping h of $\beta(S)$ into B such that:* (i) *$h(x) = x$ for each $x \in S$,* (ii) $h[\beta(S) - S] = B - S$. *The mapping h is one-to-one* (*and consequently homeomorphic*) *if and only if B also possesses property* (4). Let I, T, R, g and S^* have the above meaning. Divide the set I into two disjoined subsets I_1 and I_2, putting $\iota \in I_1$ if and only if the continuous function f_ι may be extended to the domain B. Let R_1 denote the cartesian product $\mathfrak{P}_\iota T_\iota$ where ι runs over I_1 and $T_\iota = T$ for each ι. For any $x \in B$, put $g_1(x) = \xi = \{\xi_\iota\}_{\iota \in I_1} \in R_1$, where $\xi_\iota = \varphi_\iota(x)$, φ_ι being the extension of f_ι to the domain B.

[12] It follows easily from property (3) that the extended function is uniquely defined by f.

Then g_1 is a homeomorphic mapping of the space B into the space $B^* = g_1(B) \subset R_1$, just as g was a homeomorphic mapping of S into the space S^*. For any point $\xi = \{\xi_\iota\}_{\iota \in I} \in R$, put $k(\xi) = \{\xi_\iota\}_{\iota \in I_1} \in R_1$. Evidently k is a continuous mapping of R into R_1. For $x \in S$, it is easy to see that $k[g(x)] = g_1(x)$ so that $k(S^*) \subset B^*$. Since k is continuous, it follows that $k(\overline{S^*}) \subset \overline{B^*}$, where $\overline{S^*}$ is the closure of S^* in the space R and $\overline{B^*}$ is the closure of B^* in the space R_1. Since B^* is a homeomorph of B, B^* is a bicompact Hausdorff space, whence $\overline{B^*} = B^*$. Therefore $k(\overline{S^*}) \subset B^*$, i.e. k defines a continuous mapping k_0 of $\overline{S^*}$ into a subset of B^*. Since $\overline{S^*}$ was homeomorphic with $\beta(S)$, and B^* was homeomorphic with B, k_0 defines a continuous mapping h of $\beta(S)$ into a subspace $h[\beta(S)]$ of B; evidently $h(x) = x$ for every $x \in S$. The space $h[\beta(S)]$, as a continuous image of the bicompact space $\beta(S)$, must be bicompact. It follows that $h[\beta(S)]$ is closed in B. On the other hand, $h[\beta(S)] \supset S$ must be dense in B. Therefore, $h[\beta(S)] = B$, i.e., h is a continuous mapping of $\beta(S)$ into B. If B possesses property (4) of $\beta(S)$, we have $I_1 = I$, whence $R_1 = R$ and k is the identity. This readily implies that the mapping h is homeomorphic.

Returning to the general case, we still have to prove that $h[\beta(S) - S] = B - S$. Of course $h[\beta(S) - S] \supset B - S$. It remains to arrive at a contradiction in supposing the existence of a point $b \in \beta(S) - S$ such that $a = h(b) \in S$. Since $\beta(S)$ is a bicompact Hausdorff space, it is completely regular. Hence there exists a continuous function φ in the domain $\beta(S)$ such that $\varphi(a) = 0$, $\varphi(b) = 1$. Let Q be the set of all points $x \in S$ such that $\varphi(x) \geqq \frac{1}{2}$. Then Q is a closed subset of S, so that there exists a closed subset P of the space B such that $Q = SP$. Since B is a bicompact Hausdorff space, it is completely regular. Hence there exists a continuous function ψ in the domain B such that $\psi(a) = 0$, $\psi(x) = 1$ for each $x \in P$ and $0 \leqq \psi(x) \leqq 1$ for each $x \in B$. From property (4) of $\beta(S)$ it follows that there exists a continuous function χ in the domain $\beta(S)$ such that $\chi(x) = \psi(x)$ for each $x \in S$, whence $\chi(a) = 0$. Since h is a continuous mapping of $\beta(S)$ into B, $\psi[h(x)]$ is a continuous function in the domain $\beta(S)$. The set C of all points $x \in \beta(S)$ such that $\psi[h(x)] = \chi(x)$, is closed in $\beta(S)$ and contains the set S which is dense in $\beta(S)$; therefore $C = \beta(S)$, whence $\chi(b) = \psi[h(b)] = \psi(a) = 0$. The set D of all points $x \in \beta(S)$ such that both $\varphi(x) > \frac{1}{2}$ and $\chi(x) < \frac{1}{2}$ is open in $\beta(S)$ and is not vacuous, since $b \in D$. Since S is dense in $\beta(S)$, there exists a point $c \in S \cdot D$. Since $c \in D$, we have $\chi(c) < \frac{1}{2}$; since $c \in S$, we have $\chi(c) = \psi(c)$. Therefore $\psi(c) < \frac{1}{2}$ so that $c \in S \cdot (B - P) = S - Q$. From the definition of Q it follows that $\varphi(c) < \frac{1}{2}$; since $c \in D$, this is a contradiction.

Two subsets A_1 and A_2 of a topological space S will be called *completely separated* if there exists a continuous function f in the domain S such that $f(x) = 0$ for each $x \in A_1$ and $f(x) = 1$ for each $x \in A_2$.[5] It is easy to see that A_1 and A_2 are completely separated if and only if the closed sets $\bar{A}_1$ and $\bar{A}_2$ are completely separated. We know that S is completely regular if and only if any single point x and any closed set not containing x are always completely separated. We know that S is normal if and only if two closed sets without common points are always completely separated.

Let S be a completely regular space. We characterized the space $\beta(S)$ by the properties (1)-(4) given above. We will now show that $\beta(S)$ may be also characterized by the properties (1), (2), (3) and (4′), where (4′) means the following: *If A_1 and A_2 are two completely separated subsets of S, then the closures of A_1 and A_2 in the space $\beta(S)$ are disjoint.* Suppose first that A_1 and A_2 are two completely separated subsets of S. Then there exists a continuous function f in the domain S such that $f(x) = 0$ for each $x \epsilon A_1$ and $f(x) = 1$ for each $x \epsilon A_2$. We may suppose that $0 \leqq f(x) \leqq 1$ for each $x \epsilon S$, so that there exists a continuous extension φ of f to the domain $\beta(S)$. Letting the bar denote closures in the space $\beta(S)$, we have $\varphi(x) = 0$ for each $x \epsilon \bar{A}_1$ and $\varphi(x) = 1$ for each $x \epsilon \bar{A}_2$, so that indeed $\bar{A}_1\bar{A}_2 = 0$. Conversely, let the space B have properties (1), (2), (3), (4′). There exists a continuous mapping h of the space $\beta(S)$ into the space B such that $h(x) = x$ for each $x \epsilon S$. It is sufficient to prove that the mapping h is one-to-one. Suppose the contrary. Then there exist two points $a \epsilon \beta(S)$, $b \epsilon \beta(S)$ such that $a \neq b$, $h(a) = h(b)$. There exists a continuous function f in the domain $\beta(S)$ such that $f(a) = 0, f(b) = 1$. Let A_1 denote the set of all points $x \epsilon S$ such that $f(x) \leqq \frac{1}{3}$; let A_2 denote the set of all points $x \epsilon S$ such that $f(x) \geqq \frac{2}{3}$. It is easy to see that A_1 and A_2 are two completely separated subsets of S so that $\bar{A}_1\bar{A}_2 = 0$ where the bar designates closures in the space B. Since $h(a) = h(b)$, we shall have a contradiction if we shall prove that $h(a) \epsilon \bar{A}_1$, $h(b) \epsilon \bar{A}_2$. Let U be any neighborhood of $h(a)$ in the space B. Then $h^{-1}(U)$ is a neighborhood of a in the space $\beta(S)$. Since $f(a) = 0$ and since S is dense in $\beta(S)$, it is easy to see that $h^{-1}(U)\cdot A_1 \neq 0$, whence $U \cdot A_1 \neq 0$. Since U was an arbitrary neighborhood of $h(a)$ in the space B, we have indeed $h(a) \epsilon \bar{A}_1$ and similarly we prove that $h(b) \epsilon \bar{A}_2$.

In the particular case when *S is a normal Riesz space*, it follows from the result just proved that $\beta(S)$ may characterized by the properties (1), (2), (3) and (5) where (5) means the following: *If F_1 and F_2 are two closed subsets of S without common points, then the closures of F_1 and F_2 in the space $\beta(S)$ have no common points.* Conversely, *if there exists a space B having properties* (1), (2), (3) *and* (5), *then S is normal and $B = \beta(S)$.* Indeed, it is easy to see that property (5) is stronger than property (4′) so that $B = \beta(S)$. If F_1 and F_2 are two closed subsets of S and $F_1F_2 = 0$, then $\bar{F}_1\bar{F}_2 = 0$, the bar indicating closures in B. Since B is a bicompact Hausdorff space, it is normal, so that there exists a continuous function φ in the domain $\beta(S)$ such that $\varphi(x) = 0$ for each $x \epsilon \bar{F}_1$ and $\varphi(x) = 1$ for each $x \epsilon \bar{F}_2$. Hence it follows that S is normal.

Let S be a completely regular space. Let T be a closed subset of S; let $\bar{T}$ denote the closure of T in the space $\beta(S)$. Then *we have $\bar{T} = \beta(T)$* (i.e. $\bar{T}$ possesses the properties (1)-(4) of $\beta(T)$) *if and only if every bounded*[7] *continuous function in the domain T admits of a continuous extension to the domain S.* Suppose first that $\bar{T} = \beta(T)$ and let f be a continuous function in the domain T such that e.g. $0 \leqq f(x) \leqq 1$ for each $x \epsilon T$. Since $\bar{T} = \beta(T)$, there exists a continuous extension g of f to the domain $\bar{T}$; of course $0 \leqq g(x) \leqq 1$ for each $x \epsilon \bar{T}$. Since $\beta(S)$ is a bicompact Hausdorff space, it is normal; since $\bar{T}$ is closed in

$\beta(S)$, there exists a continuous extension φ of g to the domain $\beta(S)$. Hence f may be continuously extended to the domain $\beta(S)$ and therefore also to the domain $S \subset \beta(S)$. Conversely suppose that every bounded continuous function in the domain T may be continuously extended to the domain S. Of course $\bar{T}$ has always properties (1)-(3) (relatively to T); therefore to prove that $\bar{T} = \beta(T)$ it is sufficient to prove that $\bar{T}$ has property (4′) (again relatively to T). Hence suppose that $A_1 \subset T$ and $A_2 \subset T$ are completely separated in the space T. Then there exists a continuous function f in the domain T such that $f(x) = 0$ for each $x \epsilon A_1$, $f(x) = 1$ for each $x \epsilon A_2$ and $0 \leqq f(x) \leqq 1$ for each $x \epsilon T$. There exists a continuous extension φ of f to the domain S, whence it readily follows that A_1 and A_2 are completely separated in the space S. Since $\beta(S)$ has property (4′) (relatively to S), we have $\bar{A}_1 \bar{A}_2 = 0$, the bar indicating closures in the space $\beta(S)$. But of course $\bar{A}_1$ and $\bar{A}_2$ are closures of A_1 and A_2 in the space $\bar{T}$, so that $\bar{T}$ has indeed property (4′) relatively to T.

The theorem just proved has the following consequence: *If S is a normal Riesz space, then $\bar{T} = \beta(T)$* (the bar indicating closure in $\beta(S)$) *for every closed subset T of S. If the completely regular space S is not normal, then there exists a closed subset T of S such that $\bar{T} \neq \beta(T)$.*

If Φ is a family of neighborhoods of a point x of a topological space S, then we say that Φ is *complete* if, given an arbitrary neighborhood G of x, there exists a neighborhood U of x such that both $U \epsilon \Phi$ and $U \subset G$. The least cardinal number of a complete family of neighborhoods of x is called the *character*[13] of x (in the space S) and is denoted by $\chi(x) = \chi_S(x)$. If $T \subset S$ and $x \epsilon T$, it is easy to see that

$$\chi_T(x) \leqq \chi_S(x).$$

Let S be a completely regular space. Then for every point $a \epsilon S$ we have

$$\chi_S(a) = \chi_{\beta(S)}(a).$$

Let Φ be a complete family of neighborhoods of a in the space S whose cardinal number is equal to $\chi_S(a)$. It is sufficient to construct a complete family Ψ of neighborhoods of a in the space $\beta(S)$ such that the cardinal number of Ψ does not exceed $\chi_S(a)$. The family Ψ will be constructed as a transform of the family Φ, each $U \epsilon \Phi$ determining a $\tau(U) \epsilon \Psi$, in the following way,

$$\tau(U) = \beta(S) - \overline{S - U}$$

(the bar indicating closures in the space $\beta(S)$). Of course Ψ is a family of neighborhoods of a in the space $\beta(S)$ and the cardinal number of Ψ does not exceed $\chi_S(a)$. Hence we have only to prove that, given a neighborhood G of a in the space $\beta(S)$, there exists a $U \epsilon \Phi$ such that $\tau(U) \subset G$. There exists a continuous function f in the domain $\beta(S)$ such that $f(a) = 0$ and $f(x) = 1$ for each $x \epsilon \beta(S) - G$. Let H denote the set of all points $x \epsilon S$ such that $f(x) < \frac{1}{2}$. Then H is a neighborhood of a in the space S, so that there exists a $U \epsilon \Phi$ such that $U \subset H$.

[13] AU, p. 2.

It remains to prove that $\tau(U) \subset G$. Supposing the contrary, there exists a point $b \in \tau(U) - G$. Since $b \in \beta(S) - G$, we have $f(b) = 1$. Let V be an arbitrary neighborhood of b in the space $\beta(S)$. Since $f(b) = 1$ and since S is dense in $\beta(S)$, there exists a point $c \in SV$ such that $f(c) > \frac{1}{2}$. Since $U \subset H$, we cannot have $c \in U$. Therefore $c \in S - U$ so that $(S - U) V \neq 0$. Since V was an arbitrary neighborhood of b in the space $\beta(S)$, we have $b \in \overline{S - U} = \beta(S) - \tau(U)$, which is a contradiction.

Let S be a completely regular space. Let $A \subset \beta(S) - S (A \neq 0)$ be both closed and a G_δ in $\beta(S)$. Then the cardinal number of A is $\geqq 2^{\aleph_0}$. Since A is both closed and a G_δ in the normal space $\beta(S)$, there exists a continuous function f in the domain $\beta(S)$ such that $f(x) = 0$ for each $x \in A$ and $f(x) > 0$ for each $x \in \beta(S) - A$. The set of all points $x \in \beta(S)$ such that $f(x) < n^{-1} (n = 1, 2, 3, \cdots)$ is open and not vacuous. Since S is dense in $\beta(S)$, there exists a point $a_n \in S$ such that $f(a_n) < n^{-1}$. Since $AS = 0$, we have $f(a_n) > 0$. It is evident that the points a_n may be chosen is such a manner that $f(a_{n+1}) < f(a_n)$. Let us arrange the rational numbers of the interval $0 < t < 1$ in a simple sequence $\{r_n\}$. There exists a continuous function φ in the domain $0 < t < \infty$ such that $0 < \varphi(t) < 1$ and $\varphi[f(a_n)] = r_n (n = 1, 2, 3, \cdots)$. Since $f(x) > 0$ for each $x \in S$, we obtain a bounded continuous function g in the domain S such that $g(x) = \varphi[f(x)]$ for each $x \in S$. There exists a continuous extension h of g to the domain $\beta(S)$. Choose a real number $\alpha, 0 \leqq \alpha \leqq 1$. There exists a sequence $i_1 < i_2 < i_3 < \cdots$ such that $r_{i_n} \to \alpha$ for $n \to \infty$. Let M_n designate the set of points $a_{i_n}, a_{i_{n+1}}, a_{i_{n+2}}, \cdots$ so that $M_n \subset S$, $M_n \supset M_{n+1}$, $M_n \neq 0$. Since the space $\beta(S)$ is bicompact, there exists a point $b \in \prod \bar{M}_n$. Since the functions f and h are continuous, we have $f(\bar{M}_n) \subset \overline{f(M_n)}$, $h(\bar{M}_n) \subset \overline{h(M_n)} = \overline{g(M_n)}$, whence $f(b) \in \prod \overline{f(M_n)}$, $h(b) \in \prod \overline{g(M_n)}$. Since $f(a_{i_n}) \to 0$, $g(a_{i_n}) \to \alpha$ for $n \to \infty$, we easily see that $f(b) = 0$, $h(b) = \alpha$. Since $f(b) = 0$, we have $b \in A$. Therefore, for each α such that $0 \leqq \alpha \leqq 1$, the set A contains a point b such that $h(b) = \alpha$. Hence the cardinal number of A is at least $2^{\aleph_0}$.

Let S_1 and S_2 be two completely regular spaces satisfying the first countability axiom. Let the spaces $\beta(S_1)$ and $\beta(S_2)$ be homeomorphic. Then the spaces S_1 and S_2 are homeomorphic. We may assume that $\beta(S_1) = \beta(S_2)$. According to the preceding theorem no point $x \in \beta(S_1) - S_1$ is a G_δ in $\beta(S_1)$. But every point $x \in S_2$ satisfies the first countability axiom relatively to S_2 and, therefore, after the theorem last but one, relatively to $\beta(S_2)$ as well and hence x is a G_δ in $\beta(S_2) = \beta(S_1)$. Therefore $S_2 \subset S_1$ and similarly $S_1 \subset S_2$, so that $S_1 = S_2$.

Let I denote an infinite countable isolated space (e.g. the space of all natural numbers). It is an important problem to determine the cardinal number $\mathfrak{m}$ of $\beta(I)$. All I know about it is that

$$2^{\aleph_0} \leqq \mathfrak{m} \leqq 2^{2^{\aleph_0}}.$$

It is easily seen that each point of I is an isolated point of $\beta(I)$ so that the set I is open in $\beta(I)$. Since I is countable, it is an F_σ in $\beta(I)$. Hence $\beta(I) - I$ is both closed and a G_δ in $\beta(I)$ so that the cardinal number of $\beta(I) - I$ is $\geqq 2^{\aleph_0}$.

On the other hand, since the set I is dense in the Hausdorff space $\beta(I)$, it is easy to see that a point $x \in \beta(I)$ is uniquely determined knowing the family of all sets $A \subset I$ such that $x \in \bar{A}$, so that the cardinal number of $\beta(I)$ is at most equal to the cardinal number $2^{2^{\aleph_0}}$ of all families of subsets of I.

A topological space S is called *compact* if, given any infinite subset A of S, there exists a point $x \in S$ such that $x \in \overline{A - x}$.

Let the normal Riesz space S be not compact. Then the cardinal number of $\beta(S) - S$ is at least equal to the cardinal number of $\beta(I)$ (hence at least equal to $2^{\aleph_0}$). Since S is not compact, it is well known that S contains a closed subset F homeomorphic with I. Since S is normal, we have $\beta(I) = \bar{I} \subset \beta(S)$, so that $\beta(I) - I \subset \beta(S) - S$. But the sets $\beta(I) - I$ and $\beta(I)$ have the same cardinal number.

I do not know whether this theorem remains true if we replace normality by complete regularity. It may be shown that the assumption of normality may be replaced by the following weaker assumption[14]: If F_1 and F_2 are two closed subsets of S such that F_1 *is countable* and $F_1F_2 = 0$, there exist two open sets G_1 and G_2 such that $G_1 \supset F_1$, $G_2 \supset F_2$, $G_1G_2 = 0$.

If the space S is compact, then the set $\beta(S) - S$ may consist of a single point. Let S be the set of all ordinal numbers $<\omega_1$, ω_1 being the first uncountable ordinal number. Let S_0 be the set of all ordinal numbers $\leqq \omega_1$. The topology of S and S_0 is the usual topology of an ordered set, an open base being given by the family of all open intervals. It is well known that S is a compact normal Riesz space and that S_0 is a bicompact Hausdorff space. We shall prove that $S_0 = \beta(S)$. Since it is evident that S_0 possesses properties (1)–(3) of $\beta(S)$, it is sufficient to prove that a continuous function f in the domain S admits of a continuous extension to the domain S_0. This is an easy consequence of the following theorem. *If f is a continuous function in the domain S, then there exists a point $\xi \in S$ such that f is constant for $x \geqq \xi$.* It is sufficient to prove that, given a number $\varepsilon > 0$, there exists a point $\xi(\varepsilon) \in S$ such that $|f(x) - f(y)| < \varepsilon$ for $x \in S, y \in S, x > \xi(\varepsilon), y > \xi(\varepsilon)$. Supposing the contrary, there would exist in S two sequences $\{a_n\}$ and $\{b_n\}$ such that $a_n < b_n < a_{n+1}$ and $|f(a_n) - f(b_n)| \geqq \varepsilon$. But this is impossible, because f would then be discontinuous at α, α being the first ordinal number greater than each a_n.

We say that $x \in S$ is a *κ-point*[15], if there exists a sequence $\{x_n\} \subset S - (x)$ such that $\lim x_n = x$, i.e. that, given any neighborhood U of x, we have $x_n \in U$ except for a finite number of subscripts n. Alexandroff and Urysohn raised the question[16] whether there exists a bicompact Hausdorff space which is dense in itself and which contains no κ-point. We shall prove that the space $\beta(I) - I$ has this property. Supposing the contrary, there exists a point $c \in \beta(I) - I$ and a sequence $\{a_n\} \subset \beta(I) - I - (c)$ such that $\lim a_n = c$. We may suppose that the points a_n are all distinct from one another. Let A_n be the set of the points

[14] AU, p. 58.
[15] AU, p. 53.
[16] AU, p. 54.

$a_n, a_{n+1}, a_{n+2} \cdots$ together with the point c. It is easy to see that A_n is a closed subset of $\beta(I)$. We shall construct successively open subsets U_n of the space $\beta(I)$ as follows. U_1 contains the point a_1, but $\bar{U}_1 A_2 = 0$. If, for a certain value of n, we have already constructed the set U_n so that $\bar{U}_n \cdot A_{n+1} = 0$, let U_{n+1} be an open subset containing a_{n+1}, but such that $\bar{U}_{n+1} \cdot \bar{U}_i = 0$ for $1 \leq i \leq n$ and $\bar{U}_{n+1} \cdot A_{n+2} = 0$. It is easy to see that the successive construction of the sequence $\{U_n\}$ may be carried through. Now put $\Phi = I \cdot \sum U_{2n-1}$, $\Psi = I \cdot \sum U_{2n}$. Then $\Phi\Psi = 0$ and the sets Φ and Ψ are of course closed in I, since I is an isolated space. Since I is normal, we must have $\bar{\Phi}\bar{\Psi} = 0$, the bars indicating closures in $\beta(I)$. On the other hand, since I is dense in $\beta(I)$ and U_n is open in $\beta(I)$, it is easy to see that $\overline{IU_n} = \bar{U}_n$, so that $a_n \in \overline{IU_n}$, whence we easily get the contradiction $c \in \bar{\Phi}\bar{\Psi}$.

III

We shall say that the space S is *topologically complete* if there exists a bicompact Hausdorff space $B \supset S$ such that S is a G_δ in B. Of course S is then completely regular. *A G_δ in a topologically complete space is a topologically complete space. A closed subset of a topolologically complete space is a topologically complete space.*

A topological space S is topologically complete if and only if it is completely regular and a G_δ in $\beta(S)$. If S is a G_δ in $\beta(S)$, then it is topologically complete, since $\beta(S)$ is a bicompact Hausdorff space. Conversely suppose that S is topologically complete. Then there exists a bicompact Hausdorff space $B \supset S$ such that S is a G_δ in B. Let B_0 be the closure of S in the space B. Then B_0 is a bicompact Hausdorff space and S is dense in B_0 and a G_δ in B_0. We know that there exists a continuous mapping h of $\beta(S)$ into B_0 such that $h^{-1}(S) = S$. Since S is a G_δ in B_0, it is easy to see that $h^{-1}(S) = S$ is a G_δ in $\beta(S)$.

Let T be a completely regular[17] *space. Let $S \subset T$ be a topologically complete space. Then S is a G_δ in the closure of S in the space T.* Let S_0 be the closure of S in the space $\beta(T)$. It is sufficient to prove that S is a G_δ in S_0. Since S_0 is a bicompact Hausdorff space and since S is dense in S_0, there exists a continuous mapping h of $\beta(S)$ into S_0 such that $h[\beta(S) - S] = S_0 - S$. Since S is topologically complete, it is a G_δ in $\beta(S)$, so that $\beta(S) - S$ is an F_σ in $\beta(S)$. Hence there exist closed subsets F_n of $\beta(S)$ such that $\sum F_n = \beta(S) - S$, whence $S_0 - S = \sum h(F_n)$. Every F_n is a bicompact space, so that every $h(F_n)$ is a bicompact space. Since $h(F_n)$ is a bicompact subset of the Hausdorff space S_0, it is closed in S_0, so that $S_0 - S$ is an F_σ in S_0 and finally S is a G_δ in S_0.

Let T be a topologically complete space. Let $S \subset T$. Then S is a topologically complete space if and only if it is the intersection of a closed subset of T and a G_δ in T. If $S = FH$, where F is closed in T and H is a G_δ in T, then F is a topologically complete space and S is a G_δ in F, so that S is a topologically complete space. Conversely let S be topologically complete. Then S is a G_δ in the closure $\bar{S}$ of S in T, so that $S = \bar{S}H$, H being a G_δ in T.

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[17] I do not know whether this assumption is necessary.

Let $S \neq 0$ be a topologically complete space[18]. *Let $\{G_n\}$ be a sequence of open and dense subsets of S. Let $H = \prod G_n$. Then $H \neq 0$ and, moreover, H is dense in S.* There exists a regular compact (as a matter of fact, bicompact) space $K \supset S$ such that S is a G_δ in K. We may suppose that $\bar{S} = K$, the bar denoting closure in K. The sets G_n being open in S, there exist sets Γ_n open in K and such that $G_n = S \cdot \Gamma_n$. Since S is a G_δ in K, there exist sets Δ_n open in K and such that $S = \prod \Delta_n$. Since S is dense in K and G_n are dense in S, the sets G_n are dense in K. Choose an arbitrary point $a_0 \in S$ and an arbitrary neighborhood V of a_0 in the space S. All we have to prove is that $HV \neq 0$. There exists a neighborhood U_0 of a_0 in the space K such that $V = SU_0$. Since the set G_1 is dense in K, there exists a point $a_1 \in G_1 U_0 = S \cdot \Gamma_1 U_0 \subset \Delta_1 \Gamma_1 U_0$. Hence $\Delta_1 \Gamma_1 U_0$ is a neighborhood of a_1 in the space K. Since K is regular, there exists a neighborhood U_1 of a_1 (in the space K) such that $\bar{U}_1 \subset \Delta_1 \Gamma_1 U_0$. Generally, let there be given for a certain value of n a point $a_n \in G_n$ and its neighborhood U_n (in the space K) such that $\bar{U}_n \subset \Delta_n \Gamma_n U_{n-1}$. Then $a_n \in G_n \subset S$ and SU_n is a neighborhood of a_n in the space S; since G_{n+1} is dense in S, there exists a point $a_{n+1} \in G_{n+1} U_n = S \cdot \Gamma_{n+1} U_n \subset \Delta_{n+1} \Gamma_{n+1} U_n$. Hence $\Delta_{n+1} \Gamma_{n+1} U_n$ is a neighborhood of a_{n+1} in the regular space K, so that there exists a neighborhood U_{n+1} of a_{n+1} (in the space K) such that $\bar{U}_{n+1} \subset \Delta_{n+1} \Gamma_{n+1} U_n$. Thus we construct a sequence $\{a_n\}$ of points and a sequence $\{U_n\}$ of open sets so that $a_n \in G_n U_n$, $\bar{U}_{n+1} \subset \Delta_{n+1} \Gamma_{n+1} U_n$. Since $a_n \in U_n$, we have $U_n \neq 0$. Since K is compact and $\bar{U}_{n+1} \subset U_n$, there exists a point $b \in \prod U_n = \prod \bar{U}_n$. Since $\bar{U}_{n+1} \subset \Delta_{n+1} \Gamma_{n+1} U_n$, we have $b \in \prod \Delta_n$. $\prod \Gamma_n = S \cdot \prod \Gamma_n = \prod G_n = H$. Moreover $b \in U_0$, so that $b \in HU_0 = HV$.

Let S be a metric space. A *Cauchy sequence* in S is a sequence $\{x_n\} \subset S$ such that, given a number $\varepsilon > 0$, there exists a number p such that the distance of x_m and x_n is less than ε, whenever both m and n are greater than p. A metric space S is called *metrically complete* if, given any Cauchy sequence $\{x_n\}$ in S, there exists a point $x \in S$ such that $\lim x_n = x$. A topological space is called *completely metrizable*, if it is homeomorphic with a metrically complete space.

We next prove our principal theorem: *A metrizable space S is topologically complete if and only if it is completely metrizable.*

Let S be a metrically complete space and let ρ be its distance function. We may suppose that $\rho(x, y) \leqq 1$ for every pair of points, since otherwise we may replace ρ by ρ_1, putting $\rho_1(x, y) = \rho(x, y)$ if $\rho(x, y) \leqq 1$, $\rho_1(x, y) = 1$ if $\rho(x, y) > 1$ Since S is metric, it is completely regular, so that $\beta(S)$ exists. For any given $a \in S$, $\rho(a, x)$ is a bounded continuous function in the domain S so that there exists a continuous function $\varphi_a(x)$ in the domain $\beta(S)$ such that $\varphi_a(x) = \rho(a, x)$ for each $x \in S$. If $a \in S$, $b \in S$, then the set $T(a, b)$ of all points $x \in \beta(S)$ such that $\varphi_a(x) + \varphi_b(x) \geqq \rho(a, b)$ is closed in $\beta(S)$ and contains S. Since S is dense in $\beta(S)$, we must have $T(a, b) = \beta(S)$, i.e. $\varphi_a(x) + \varphi_b(x) \geqq \rho(a, b)$ for each $x \in \beta(S)$.

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[18] It is evident from the proof that it is possible to replace this by the weaker assumption that S is a G_δ in some regular compact space.

For $a \in S$ and $n = 1, 2, 3, \cdots$ let $\Gamma(a, n)$ be the set of all points $x \in \beta(S)$ such that $\varphi_a(x) < n^{-1}$. Since the function $\varphi_a(x)$ is continuous, $\Gamma(a, n)$ is an open subset of $\beta(S)$. Therefore

$$G_n = \sum_{a \in S} \Gamma(a, n)$$

is an open set. We shall prove that $S = \prod G_n$, so that the set S is a G_δ in $\beta(S)$ and thus topologically complete. Evidently $\prod G_n \supset S$. Conversely let $b \in \prod G_n$. We have to prove that $b \in S$. According to the definition of G_n, there exist points $a_n \in S$ such that $\varphi_{a_n}(b) < n^{-1}$. Therefore

$$\rho(a_n, a_m) \leqq \varphi_{a_n}(b) + \varphi_{a_m}(b) < \frac{1}{n} + \frac{1}{m},$$

so that $\{a_n\}$ is a Cauchy sequence in S. Since S is metrically complete, there exists a point $a \in S$ such that $a = \lim a_n$. It is sufficient to prove that $a = b$. Suppose that $a \neq b$. Since $\beta(S)$ is a Hausdorff space, there exist two open subsets U and V of $\beta(S)$ such that $a \in U, b \in V, UV = 0$. Since US is a neighborhood of a in the metric space S, there exists an integer $n > 0$ such that U contains every point $x \in S$ such that $\rho(a, x) < 2 \cdot n^{-1}$. This can be written in the form $SW \subset U$, W being the set of all points $x \in \beta(S)$ such that $\varphi_a(x) < 2 \cdot n^{-1}$. Since φ_a is continuous, W is an open subset of $\beta(S)$. Since S is dense in $\beta(S)$ and U, V and W are open in $\beta(S)$, we have $W \subset \overline{W} = \overline{SW} \subset \overline{U} \subset \beta(S) - V$, or $WV = 0$. Hence for each $x \in V$ we have $\varphi_a(x) \geqq 2 \cdot n^{-1}$; in particular $\varphi_a(b) \geqq 2 \cdot n^{-1}$. Since $\rho(a_n, a_m) < n^{-1} + m^{-1}$ and $\lim a_n = a$, we have $\rho(a, a_n) \leqq n^{-1}$. Hence for each $x \in S$ we have $\rho(a, x) \leqq \rho(a, a_n) + \rho(a_n, x) \leqq n^{-1} + \rho(a_n, x)$, whence it easily follows that for each $x \in \beta(S)$ we have $\varphi_a(x) \leqq \varphi_{a_n}(x) + n^{-1}$, in particular $\varphi_a(b) \leqq \varphi_{a_n}(b) + n^{-1} < n^{-1} + n^{-1} = 2 \cdot n^{-1}$, which is a contradiction.

Now suppose that the metric space S is topologically complete. Let ρ denote the distance function of S; again, we shall suppose that $\rho(x, y) \leqq 1$ for every couple of points. Since S is topologically complete, there exists a sequence $\{F_n\}$ of closed subsets of $\beta(S)$ such that $\beta(S) - S = \sum F_n$. If $S = \beta(S)$, then S is a bicompact metric space, and then it is well known that S is metrically complete. Hence let us suppose that $S \neq \beta(S)$; we may then assume that $F_n \neq 0$ for every n. Given any point $a \in S$, $\rho(a, x)$ is a bounded continuous function in the domain S, which admits of a continuous extension φ_a to the domain $\beta(S)$. If the point $b \in \beta(S)$ is different from a, then there exist open subsets U and V of $\beta(S)$ such that $a \in U, b \in V, UV = 0$. Since SU is a neighborhood of a in the metric space S, there exists a number $\varepsilon > 0$ such that U contains every point $x \in S$ such that $\rho(a, x) < \varepsilon$. Since S is dense in $\beta(S)$, it easily follows that $\overline{U}$ contains every point $x \in \beta(S)$ such that $\varphi_a(x) < \varepsilon$. Since $U \subset \beta(S) - V = \overline{\beta(S) - V}$, we have $\overline{U} \subset \beta(S) - V$ so that $b \in \beta(S) - \overline{U}$, whence $\varphi_a(b) \geqq \varepsilon$. Thus we proved that $\varphi_a(b) > 0$ for every $b \in \beta(S)$ except for $b = a$. Since the set $F_n \neq 0$ is closed in the bicompact space $\beta(S)$, it is easy to see that the function $\varphi_a(x)$, x running over F_n, admits of a minimum value $\sigma(a, F_n)$. Since $a \in S, F_n S = 0$, we have $\sigma(a, F_n) > 0$.

If $a \in S$, $b \in S$, then we have $\rho(a, x) \leqq \rho(a, b) + \rho(b, x)$ for every $x \in S$, whence $\varphi_a(x) \leqq \rho(a, b) + \varphi_a(x)$ for every $x \in \beta(S)$. Therefore $\sigma(a, F_n) \leqq \rho(a, b) + \sigma(b, F_n)$, and similarly $\sigma(b, F_n) \leqq \rho(a, b) + \sigma(a, F_n)$. Hence

$$| \sigma(a, F_n) - \sigma(b, F_n) | \leqq \rho(a, b).$$

Now let us put for $x \in S$, $y \in S$

$$f_n(x, y) = \rho(x, y) + \sigma(x, F_n) + \sigma(y, F_n),$$

$$g_n(x, y) = \frac{\rho(x, y)}{f_n(x. y)},$$

$$\rho_0(x, y) = \rho(x, y) + \sum_1^{\infty} 2^{-n} \cdot g_n(x, y).$$

Since $\rho(x, y) \geqq 0$, $\sigma(x, F_n) > 0$, $\sigma(y, F_n) > 0$, we have $f_n(x, y) > 0$. Hence $g_n(x, y)$ exists and $0 \leqq g_n(x, y) \leqq 1$, so that the series $\sum 2^{-n} \cdot g_n(x, y)$ is convergent. It is evident that $\rho_0(x, y) = \rho_0(y, x)$ and that $\rho_0(x, x) = 0$, whereas $\rho_0(x, y) > 0$ if $x \neq y$. Next we shall prove that $\rho_0(x, z) \leqq \rho_0(x, y) + \rho_0(y, z)$ for $x \in S$, $y \in S$, $z \in S$. Since

$$\frac{t_1}{c + t_1} \leqq \frac{t_2}{c + t_2} \quad \text{for } c > 0,\ 0 \leqq t_1 \leqq t_2$$

and since $0 \leqq \rho(x, z) \leqq \rho(x, y) + \rho(y, z)$, we have

$$g_n(x, z) \leqq \frac{\rho(x, y) + \rho(y, z)}{\rho(x, y) + \rho(y, z) + \sigma(x, F_n) + \sigma(z, F_n)}.$$

Since

$$\sigma(y, F_n) \leqq \rho(x, y) + \sigma(x, F_n),$$

$$\sigma(y, F_n) \geqq \rho(y, z) + \sigma(z, F_n),$$

we have

$$\rho(x, y) + \rho(y, z) + \sigma(x, F_n) + \sigma(z, F_n) \geqq \begin{cases} \rho(x, y) + \sigma(x, F_n) + \sigma(y, F_n), \\ \rho(y, z) + \sigma(y, F_n) + \sigma(z, F_n), \end{cases}$$

whence

$$g_n(x, z) \leqq g_n(x, y) + g_n(y, z),$$

so that indeed

$$\rho_0(x, z) \leqq \rho_0(x, y) + \rho_0(y, z).$$

Hence ρ_0 has all the properties of a distance function. Next we prove that ρ and ρ_0 are equivalent metrics in S, i.e. that for $x \in S$ and $\{x_n\} \subset S$ we have

$$\lim \rho(x_n, x) = 0 \text{ if and only if } \lim \rho_0(x_n, x) = 0.$$

If $\lim \rho_0(x_n, x) = 0$, then $\lim \rho(x_n, x) = 0$, since $0 \leqq \rho(x_n, x) \leqq \rho_0(x_n, x)$. Conversely suppose that $\lim \rho(x_n, x) = 0$. Choose a number $\varepsilon > 0$ and an integer $k > 0$ such that $2^{-k+1} < \varepsilon$. Then we have for all values of n

$$\sum_{i=k+1}^{\infty} 2^{-i} g_i(x_n, x) \leqq \sum_{i=k+1}^{\infty} 2^{-i} = 2^{-k} < \tfrac{1}{2}\varepsilon,$$

whence

$$\rho_0(x_n, x) < \rho(x_n, x) + \sum_{i=1}^{k} 2^{-i} g_i(x_n, x) + \tfrac{1}{2}\varepsilon$$

$$\leqq \rho(x_n, x) + \sum_{i=1}^{k} 2^{-i} \frac{\rho(x_n, x)}{\rho(x_n, x) + \sigma(x, F_i)} + \tfrac{1}{2}\varepsilon.$$

Since $\lim \rho(x_n, x) = 0$, we must have

$$\lim_{n \to \infty} \sum_{i=1}^{k} 2^{-i} \frac{\rho(x_n, x)}{\rho(x_n, x) + \sigma(x, F_i)} = 0,$$

so that there exists an integer p such that for $n > p$ we have

$$0 \leqq \sum_{i=1}^{k} 2^{-i} \frac{\rho(x_n, x)}{\rho(x_n, x) + \sigma(x, F_i)} < \tfrac{1}{2}\varepsilon.$$

Therefore

$$\rho_0(x_n, x) < \rho(x_n, x) + \varepsilon$$

for every $n > p$. Since $\lim \rho(x_n, x) = 0$ and the number $\varepsilon > 0$ was arbitrary, we have indeed $\lim \rho_0(x_n, x) = 0$. Thus we proved that ρ and ρ_0 are equivalent metrics in S, i.e. that the metric spaces $S = (S, \rho)$ and (S, ρ_0) are homeomorphic.

It remains to be shown that the metric space (S, ρ_0) is metrically complete. Hence suppose that $\{x_n\}$ is a Cauchy sequence in (S, ρ_0). We have to prove that there exists a point $x \in S$ such that $\lim \rho_0(x_n, x) = 0$, or, what we already know to be equivalent, that $\lim \rho(x_n, x) = 0$. Since the space $\beta(S)$ is bicompact, it is easy to see that there exists a point $x \in \beta(S)$ such that, given any neighborhood U of x (in the space $\beta(S)$), we have $x_n \in U$ for an infinite number of values of n. It is sufficient to prove that $x \in S$, for then, since $\{x_n\}$ is a Cauchy sequence, it is easy to show that $\lim \rho(x_n, x) = 0$. Suppose, on the contrary, that the point x belongs to the set $\beta(S) - S = \sum F_n$. Hence there exists an integer $k > 0$ such that $x \in F_k$.

We shall prove that $\sigma(x_n, F_k) \to 0$ for $n \to \infty$. Choose a number $\varepsilon > 0$. There exists an integer $p > 0$ such that for $n > p$, $m > p$ we have $\rho(x_n, x_m) \leqq \rho_0(x_n, x_m) < \varepsilon$. Let n be greater than p. The number $\sigma(x_n, F_k)$ is the minimum value of $\varphi_{x_n}(y)$ for $y \in F_k$. Since $x \in F_k$, we must have $0 < \sigma(x_n, F_k) \leqq \varphi_{x_n}(x)$. There exists a neighborhood Ω_n of x in $\beta(S)$ such that $| \varphi_{x_n}(z) - \varphi_{x_n}(x) | < \varepsilon$ for every $z \in \Omega_n$. There exists an integer $m_n > p$ such that $x_{m_n} \in \Omega_n$, whence $| \varphi_{x_n}(x_{m_n}) - \varphi_{x_n}(x) | < \varepsilon$, i.e. $| \rho(x_n, x_{m_n}) - \varphi_{x_n}(x) | < \varepsilon$. Since $n > p$, $m_n > p$, we must have $\rho(x_n, x_{m_n}) < \varepsilon$, whence $\varphi_{x_n}(x) < 2\varepsilon$. Therefore $0 < \sigma(x_n, F_k) < 2\varepsilon$ for $n > p$, so that indeed $\sigma(x_n, F_k) \to 0$ for $n \to \infty$.

Since $\{x_n\}$ is a Cauchy sequence in (S, ρ_0), there exists an integer p such that $\rho_0(x_n, x_p) < 2^{-k-2}$ for each $n > p$. But

$$\rho_0(x_n, x_p) \geqq 2^{-k} g_k(x_n, x_p) = 2^{-k} \frac{\rho(x_n, x_p)}{\rho(x_n, x_p) + \sigma(x_n, F_k) + \sigma(x_p, F_k)}.$$

Since

$$\sigma(x_p, F_k) \leqq \rho(x_n, x_p) + \sigma(x_n, F_k),$$

it follows that

$$\rho_0(x_n, x_p) \geqq 2^{-k-1} \frac{\rho(x_n, x_p)}{\rho(x_n, x_p) + \sigma(x_n, F_k)} \geqq 0,$$

so that for every $n > p$ we have

$$0 \leqq \frac{\rho(x_n, x_p)}{\rho(x_n, x_p) + \sigma(x_n, F_k)} < \tfrac{1}{2},$$

whence $\rho(x_n, x_p) < \sigma(x_n, F_k)$. But $\sigma(x_n, F_k) \to 0$ f ∞. Therefore $\rho(x_n, x_p) \to 0$ for $n \to \infty$. Hence there exists an integer $q >$ h that for every $n > q$ we have $\rho(x_n, x_p) < \frac{1}{2} \varphi_{x_p}(x)$. [Since $x_p \in S$, $x \in \beta(S) - S$, we know that $\varphi_{x_p}(x) > 0$.] There exists a neighborhood U of x in the space $\beta(S)$ such that $\varphi_{x_p}(z) > \frac{1}{2}\varphi_{x_p}(x)$ for any $z \in U$. There exists an integer $n > q$ such that $x_n \in U$, whence $\rho(x_n, x_p) = \varphi_{x_p}(x_n) > \frac{1}{2}\varphi_{x_p}(x)$, which is a contradiction.

IV

Let S be a completely regular space. Let $\lambda(S)$ be the set of all points $x \in \beta(S)$ such that x possesses a neighborhood U(in the space $\beta(S)$) such that $S \cdot \overline{U}$ is a normal space. [$\overline{U}$ is the closure of U in $\beta(S)$]. It is easy to see that $\lambda(S)$ is an open subset of $\beta(S)$.

Let F_1 and F_2 be two closed subsets of a completely regular space S such that $F_1F_2 = 0$. Then

$$\bar{F}_1 \cdot \bar{F}_2 \cdot \lambda(S) = 0,$$

the bars indicating closures in $\beta(S)$. Supposing the contrary, there exists a point $a \in \bar{F}_1 \cdot \bar{F}_2 \cdot \lambda(S)$. Since $a \in \lambda(S)$, there exists a neighborhood U of a (in the space $\beta(S)$) such that $S \cdot \overline{U}$ is a normal space. There exists a neighborhood V of a such that $\bar{V} \subset U$. Put

$$\Phi_1 = \bar{V} \cdot F_1, \Phi_2 = \overline{U} \cdot F_2 + S(\overline{U} - U).$$

Then Φ_1 and Φ_2 are two closed subsets of $S\overline{U}$ such that $\Phi_1\Phi_2 = 0$. Moreover, it is easy to see that $a \in \bar{\Phi}_1 \cdot \bar{\Phi}_2$. Since $S\overline{U}$ is a normal space, there exists a bounded continuous function f in the domain $S\overline{U}$ such that $f(x) = 0$ for each $x \in \Phi_1$ and $f(x) = 1$ for each $x \in \Phi_2$. For $x \in S$ put (i) $g(x) = f(x)$ if $x \in SU$, (ii) $g(x) = 1$ if $x \in S - U$. Then it is easy to see that g is a bounded continuous extension of f to the domain S. According to the definition of $\beta(S)$, there exists a continuous extension φ of g (hence of f) to the domain $\beta(S)$. We have

$\varphi(x) = f(x) = 0$ for each $x \in \Phi_1$ and $\varphi(x) = f(x) = 1$ for each $x \in \Phi_2$. Since φ is continuous, we must have $\varphi(x) = 0$ for each $x \in \bar{\Phi}_1$ and $\varphi(x) = 1$ for each $x \in \bar{\Phi}_2$, so that $\bar{\Phi}_1\bar{\Phi}_2 = 0$, which is a contradiction.

The topological space S will be called *locally normal* if each point $x \in S$ possesses a neighborhood U such that $\bar{U}$ is a normal space. Any normal space is locally normal; more generally, any open subset of a locally normal space is locally normal.

A locally normal Riesz space S is completely regular. Let a be a given point of a locally normal space S and let V be a given neighborhood of a. There exists a neighborhood U of a such that $\bar{U}$ is a normal space. Also $\overline{UV}$ is a normal space, since it is a closed subset of $\bar{U}$. Since (a) and $\overline{UV} - UV$ are two closed subsets of the normal space $\overline{UV}$ without a common point, there exists a continuous function f in the domain $\overline{UV}$ such that $f(a) = 0$ and $f(x) = 1$ for each $x \in \overline{UV} - UV$. For $x \in S$ put (i) $g(x) = f(x)$ if $x \in UV$, (ii) $g(x) = 1$ if $x \in S - UV$. Then it is easy to see that g is a continuous function in the domain S such that $g(a) = 0$ and $g(x) = 1$ for each $x \in S - V$. Therefore S is completely regular.

A completely regular space S need not be locally normal. Let ω be the first infinite ordinal number. Let ω_1 be the first uncountable ordinal number. Let S_1 be the space of all ordinal numbers $\leqq \omega$. Let S_2 be the space of all ordinal numbers $\leqq \omega_1 \cdot \omega$. The topology in S_1 and in S_2 is defined in the usual way by means of intervals. Let S_{12} be the cartesian product of the two spaces S_1 and S_2. Let T be the set of all points $(x, y) \in S_{12}$, for which $x = \omega$ and $y = \omega_1 \cdot n (n = 1, 2, 3, \cdots)$. Let $S = S_{12} - T$. Then S is a completely regular space, but it is not locally normal.

It is easy to see that a completely regular space S is locally normal if and only if $S \subset \lambda(S)$. I do not know whether there exists a completely regular space $S \neq 0$ such that $S \cdot \lambda(S) = 0$.

A Riesz space S is locally normal if and only if it is homeomorphic with an open subset of a normal Riesz space.[19] We know that an open subset of a normal Riesz space is a locally normal Riesz space. Conversely let S be a locally normal Riesz space. Let S_0 be a new space consisting of all points of S and of a single new point ω. The topology of S_0 is defined as follows. If $\omega \in A \subset S_0$, then A is closed in S_0 if and only if $A - (\omega)$ is closed in S. If $A \subset S_0 - (\omega) = S$, then A is closed in S_0 if and only if (i) A is closed in S, (ii) $\bar{A} \subset \lambda(S)$, the bar indicating closure in $\beta(S)$. It is easy to see that S_0 is a Riesz space and that S is an open subset of S_0. It remains to be shown that the space S_0 is normal. Let F_1 and F_2 be two closed subsets of S_0 such that $F_1F_2 = 0$. Since the point ω belongs at most to one of the two sets F_1 and F_2, we may suppose that $F_1 \subset S$. Since F_1 is closed in S_0, the closure $\bar{F}_1$ of F_1 in the space $\beta(S)$ is a subset of $\lambda(S)$. Put $F_3 = F_2 - (\omega)$. Then F_1 and F_3 are two closed subsets of S and $F_1F_3 = 0$. We know that $\bar{F}_1 \cdot \bar{F}_3 \cdot \lambda(S) = 0$ (the closures being formed again in $\beta(S)$). But

[19] I do not know whether the restriction to Riesz spaces is really necessary in this theorem.

$\bar{F}_1 \subset \lambda(S)$ so that $\bar{F}_1$ and $\bar{F}_3 + \beta(S) - \lambda(S)$ are two closed subsets of $\beta(S)$ without a common point. Since $\beta(S)$ is a bicompact Hausdorff space, it is normal, so that there exists a continuous function φ in the domain $\beta(S)$ such that $\varphi(x) = 0$ for each $x \in \bar{F}_1$ and $\varphi(x) = 1$ for each $x \in \bar{F}_3$ and for each $x \in \beta(S) - \lambda(S)$. Let us define a function f in the domain S_0 in the following way. If $x \in S$, then $f(x) = \varphi(x)$; moreover $f(\omega) = 1$. Then it is easy to see that f is a continuous function in the domain S_0 such that $f(x) = 0$ for each $x \in F_1$ and $f(x) = 1$ for each $x \in F_2$.

I conclude with two more unsolved questions. A topological space S is called *completely normal* if every subset of S is a normal space. S may be called *locally completely normal* if every point $x \in S$ possesses a neighborhood U such that $\bar{U}$ is a completely normal space. S may be called *completely locally normal* if every subset of S is a locally normal space. It is easy to see that a locally completely normal space is completely locally normal. I do not know whether the converse holds true. Any open subset of a completely normal space is a locally completely normal space. I do not know whether a locally completely normal space must be homeomorphic with an open subset of a completely normal space.

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