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On pseudomanifolds

Lectures at the Inst. Adv. St., Princeton, (mimeographed) (1935), 17 pp.

Persistent URL: http://dml.cz/dmlcz/501043

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Notes on lecture by

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Fall Term 1935

The Institute for Advanced Study

Princeton, N. J.

ON PSEUDOMANIFOLDS

by

Eduard Čech

Before passing to the proper content of these lectures, I shall give a brief survey of a few fundamental facts of the homology theory, in such form as I shall apply it later.

A complex K is a finite set ($\neq 0$) of elements (called vertices of the complex), in which some subsets are distinguished (and called simplices of the complex); two conditions must be satisfied: (1) each vertex is distinguished, (2) each subset of a distinguished set is distinguished. If there is given a fixed abelian group $\mathfrak{R}$, then we can form in a known manner K-chains (with coefficients taken from $\mathfrak{R}$) and their boundaries, which leads to the notion of cycles and homologies. We shall consider also relative cycles and homologies in the sense of Lefschetz. A subcomplex K_1 of a complex K is a complex such that (not only each vertex of K_1 is a vertex of K but also) every K_1-simplex is a K-simplex. Let $K_2 \subset K_1 \subset K$. Let $C^n(K)$ be an (n, K)-chain. We say that $C^n(K)$ is situated in K_1 [and we write $C^n(K) \subset K_1$] if $C^n(K)$ is a K_1-chain. We say that $C^n(K)$ is an (n, K)-cycle mod K_2 in K_1, if $C^n(K) \subset K_1$, $FC^n(K) \subset K_2$, where the letter F signifies the boundary. We say that $C^n(K)$ is homologous to zero mod K_2 in K_1 (and we write $C^n(K) \sim 0$ mod K_2 in K_1), if there exists an (n+1, K)-chain $D^{n+1}(K) \subset K_1$ such that $FD^{n+1}(K) = C^n(K) + E^n(K)$, where $E^n(K) \subset K_2$.

For the later purposes it is essential that the coefficient group $\mathfrak{R}$ be a field. Therefore, we assume it now. If $\alpha \in \mathfrak{R}$ and if $C^n(K)$ is an (n, K)-chain, then we can form the chain $\alpha C^n(K)$ in an obvious manner.

Now let R be a topological space, that is to say, an abstract set (whose elements are called points) in which certain sets (called closed sets) are distinguished in such a manner as to have the following properties: (1) 0 and R are closed, (2) the sum of two closed sets is closed, (3) the intersection of any number of closed sets is closed, (4) any set consisting of a single point is closed. A set $U \subset R$ is called <u>open</u>, if $R - U$ is closed.

A <u>covering</u> $\mathfrak{U}$ of the space R is a finite set of open subsets $\neq 0$ of R whose sum is the whole R. A covering is a complex by virtue of the following definition: if $U_0, U_1, \ldots, U_n$ are different vertices (= elements) of $\mathfrak{U}$, then $(U_0, U_1, \ldots, U_n)$ is a $\mathfrak{U}$-simplex if and only if $\prod_0^n U_i \neq 0$.

If $S \subset R$ and if $\mathfrak{U}$ is a covering of R, then $\mathfrak{U}(S)$ will be the subcomplex of $\mathfrak{U}$ defined as follows: a $\mathfrak{U}$-simplex $(U_0, U_1, \ldots, U_n)$ belongs to $\mathfrak{U}(S)$ if and only if $S \cdot \prod_0^n U_i \neq 0$. This definition is useful essentially only for <u>closed</u> subsets S of R, because we have always $\mathfrak{U}(S) = \mathfrak{U}(\bar{S})$ (the bar always denotes the <u>closure</u>). If $S = \bar{S} \subset T = \bar{T} \subset R$ and if $C^n(\mathfrak{U})$ is an $(n, \mathfrak{U})$-chain, then we shall write $C^n(\mathfrak{U}) \subset S$ instead of $C^n(\mathfrak{U}) \subset \mathfrak{U}(S)$; we shall say that $C^n(\mathfrak{U})$ is an $(n, \mathfrak{U})$-cycle mod S in T if $C^n(\mathfrak{U})$ is an $(n, \mathfrak{U})$-cycle mod $\mathfrak{U}(S)$ in $\mathfrak{U}(T)$ and in a similar way we interpret a homology $C^n(\mathfrak{U}) \sim 0$ mod S in T. If $S = 0$, we speak of <u>absolute</u> cycles; if $T = R$, we leave out the words "in T".

Now let $\mathfrak{U}$ and $\mathfrak{V}$ be two coverings (of the space R; we shall consider <u>only</u> coverings of R). We say that $\mathfrak{V}$ is a <u>refinement</u> of $\mathfrak{U}$, if it is possible to attach to each vertex V of the covering $\mathfrak{V}$ a vertex $U = \pi V$ of the covering $\mathfrak{U}$ such that $V \subset U$. The operation π is called <u>projection</u> (of $\mathfrak{V}$ into $\mathfrak{U}$); in general, there exist many such projections.

If $(V_0, V_1, \ldots, V_n) = \tau_n$ is an $(n, \mathfrak{V})$-simplex, there are two possibilities; either the $\pi V_0, \pi V_1, \ldots, \pi V_n$ are not all different from each other and we put $\pi \tau_n = 0$; or they are, and then $(\pi V_0, \pi V_1, \ldots, \pi V_n)$ is an $(n, \mathfrak{U})$-simplex σ_n and we write $\pi \tau_n = \sigma_n$.

This operation of projecting a simplex is to be understood in such a sense that if τ_n is oriented, then $\pi\,\tau_n$ also has a definite orientation (obviously describable).

Let π_1 and π_2 be two projections of $\mathfrak{W}$ into $\mathfrak{V}$ and let $C^n(\mathfrak{W})$ be an $(n, \mathfrak{W})$ -cycle mod S in T. Then $\pi_1\, C^n(\mathfrak{W})$ and $\pi_2\, C^n(\mathfrak{W})$ are two $(n, \mathfrak{V})$ -cycles mod S in T, homologous to each other mod S in T. Hence, although the projection is not determined without ambiguity, it becomes so if applied to cycles of a definite type (mod S in T) provided that we identify cycles which are homologous to each other (again mod S in T).

We retain the notation $S = \overline{S} \subset T = \overline{T} \subset R$. An (n, R)-cycle mod S in T is a function C^n attaching to each covering $\mathfrak{V}$ of R (as <u>$\mathfrak{V}$ -coordinate of C^n</u>) a definite $(n, \mathfrak{V})$ -cycle $C^n(\mathfrak{V})$ mod S in T, but supposing that the following condition be verified: If $\mathfrak{W}$ is a refinement of $\mathfrak{V}$, then $\pi\, C^n(\mathfrak{W}) \sim C^n(\mathfrak{V})$ mod S in T (of course π is a projection of $\mathfrak{W}$ into $\mathfrak{V}$). The definition of a sum $C^n_1 + C^n_2$ of two (n, R)-cycles and of the product $r\, C^n$ $(r \in \mathfrak{R})$ is obvious. $C^n \sim 0$ signifies of course $C^n(\mathfrak{V}) \sim 0$ for each covering $\mathfrak{V}$.

Although our fundamental assumptions are extremely general (at the present stage of the game, it is not very essential that R is a topological space), we have an important and by no means trivial theorem. It is convenient to start with a definition: A <u>linear family</u> $\Lambda^n(\mathfrak{V})$ of $(n, \mathfrak{V})$ -cycles mod S in T is a non empty family of such cycles having the following property: if $C^n_1(\mathfrak{V}) \in \Lambda^n(\mathfrak{V})$, $C^n_2(\mathfrak{V}) \in \Lambda^n(\mathfrak{V})$, $r_1 \in \mathfrak{R}$, $r_2 \in \mathfrak{R}$, $r_1 + r_2 = 1$, $C^n(\mathfrak{V}) \sim r_1 C^n_1(\mathfrak{V}) + r_2 C^n_2(\mathfrak{V})$ then $C^n(\mathfrak{V}) \in \Lambda^n(\mathfrak{V})$. Now we can state the following <u>fundamental existence theorem</u>:

Let there be given, for each covering $\mathfrak{V}$, a linear family $\Lambda^n(\mathfrak{V})$ of $(n, \mathfrak{V})$ -cycles mod S in T such that, if $\mathfrak{W}$ is a refinement of $\mathfrak{V}$, $\pi\Lambda^n(\mathfrak{W}) \subset \Lambda^n(\mathfrak{V})$. Then there exists an (n, R)-cycle C^n mod S in T such that $C^n(\mathfrak{U}) \in \Lambda^n(\mathfrak{U})$ for every $\mathfrak{U}$.

The following three lemmas, which we will find very useful, are immediate corollaries of the fundamental existence theorem. Each lemma will be preceded by a quite obvious remark (independent of the existence theorem).

If C^n is an (n, R)-cycle mod S in T and if we set $\Gamma^{n-1}(\mathfrak{U}) = FC^n(\mathfrak{U})$ for every covering $\mathfrak{U}$, then Γ^{n-1} is an absolute $(n-1, R)$-cycle in S, which we shall denote by FC^n.* Evidently $\Gamma^{n-1} \sim 0$ in T. But conversely:

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* The following remark is quite useful: If D^n is another (n, R)-cycle mod S in T, then $C^n \sim D^n$ mod S implies $FC^n \sim FD^n$ in S.

<u>Lemma I</u>. If Γ^{n-1} is an absolute $(n-1, R)$-cycle in S, which is ~ 0 in T, then there exists an (n, R)-cycle C^n mod S in T such that $FC^n \sim \Gamma^{n-1}$ in S.

If C^n is an (n, R)-cycle mod S in T and if there exists an absolute (n, R)-cycle Γ^n such that $C^n \sim \Gamma^n$ mod S, then $FC^n \sim 0$ in S. But conversely:

<u>Lemma II</u>. If C^n is an (n, R)-cycle mod S in T such that $FC^n \sim 0$ in S, then there exists an absolute (n, R)-cycle Γ^n in T such that $C^n \sim \Gamma^n$ mod S.

If C^n is an (n, R)-cycle mod S, if D^n is an (n, R)-cycle mod S in T and if $C^n \sim D^n$ mod S, then $C^n \sim 0$ mod T. But conversely:

<u>Lemma III</u>. If C^n is an (n, R)-cycle mod S such that $C^n \sim 0$ mod T, then there exists an (n, R)-cycle D^n mod S in T such that $C^n \sim D^n$ mod S.

Naturally, very few theorems on homology may be proved without introducing more particular spaces R. We shall, from this point on, suppose that the space R is <u>normal</u>. This signifies: If S_1 and S_2 are two <u>closed</u> sets such that $S_1S_2 = 0$, then there exists two <u>open</u> sets G_1 and G_2 such that $S_1 \subset G_1$, $S_2 \subset G_2$, $G_1G_2 = 0$. In a normal space R, the following lemmas IV-VI are true. (The importance of lemma IV is immediately obvious.)

If $S_1 \subset R$, $S_2 \subset R$, then $\mathfrak{N}(S_1 S_2) \subset \mathfrak{N}(S_1) \cdot \mathfrak{N}(S_2)$ but in general $\mathfrak{N}(S_1 S_2) \neq \mathfrak{N}(S_1) \cdot \mathfrak{N}(S_2)$. Therefore, $C^n(\mathfrak{N}) \subset S_1$, $C^n(\mathfrak{N}) \subset S_2$ <u>does not</u> imply $C^n(\mathfrak{N}) \subset S_1 S_2$. But still:

<u>Lemma IV.</u> Given a covering $\mathfrak{N}$ and given the closed sets S_1 and S_2 there exists a refinement $\mathfrak{W}$ and a projection π such that $C^n(\mathfrak{W}) \subset S_1$, $C^n(\mathfrak{W}) \subset S_2$ implies $\pi C^n(\mathfrak{W}) \subset S_1 S_2$.

In close connection with this is the following

<u>Lemma V.</u> Given $S = \overline{S}$ and a covering $\mathfrak{N}$, there exist an open set $G \supset S$ and a refinement $\mathfrak{W}$ such that $C^n(\mathfrak{W}) \subset \overline{G}$ implies $C^n(\mathfrak{W}) \subset S$.

<u>Lemma VI.</u> If $S = \overline{S} \subset T = \overline{T} \subset R$, $T - S = \sum P_k$ with mutually separated P_k (in finite number) and if C^n is an (n, R)-cycle mod S in T, then there exist (n, R)-cycles C_k^n mod $S\overline{P}_k = \overline{P}_k - P_k$ in $\overline{P}_k$ such that $C^n \sim \sum C_k^n$ mod S in T.

Given a closed subset S of R, we shall denote by $\mathcal{M}$ the family of all absolute $(n-1, R)$-cycles Γ^{n-1} in S that are ~ 0 in R, each such Γ^{n-1} being regarded as equal to zero if it is ~ 0 in S.* $\mathcal{M}$ is a <u>modulus</u>; by this

* n is a given integer; later, n will be the dimension of R.

we mean that it is an additive abelian group having multipliers (operators) $r \in \mathfrak{R}$ (each of which determines an automorphism of $\mathcal{M}$). Since $\mathfrak{R}$ is a <u>field</u>, $\mathcal{M}$ always possesses an independent basis; the number of the elements of a basis (which is the same for all bases) will be called the <u>rank</u> of $\mathcal{M}$.

If R is an n-manifold (in the classical sense), the following theorem is well known: If $S = \overline{S} \subset R \neq S$, then the number p of components of $R - S$ is $= g + 1$, g being the rank of the modulus $\mathcal{M}$. The statement $p = g+1$ may be decomposed into two halves: $p \leqq g+1$ and $p \geqq g+1$. It is remarkable that the first half may be proved in a surprising general case:

Theorem I. Let there exist absolute (n, R)-cycles Ω_i^n $(1 \leq i \leq m)$ having the following property: If T_1 and T_2 are two closed sets such that $T_1 \neq R \neq T_2$ and if Δ_1^n is an absolute (n, R)-cycle in T_1 and similarly Δ_2^n for T_2, then the homology $\sum_1^m r_i \Omega_i^n \sim \Delta_1^n + \Delta_2^n$ implies $r_1 = \dots = r_m = 0$. Let $S = \overline{S} \subsetneq R$. If R-S has at least p+1 components, then the rank of the modulus $\mathcal{M}$ is $\geq pm$.

Proof. We have $R-S = \sum_0^p P_k$ with separated $P_k \neq 0$. By lemma VI, there exist (n, R)-cycles C_{ik}^n mod $S\overline{P}_k$ in $\overline{P}_k$ $(1 \leq i \leq m,\ 0 \leq k \leq p)$ such that $\Omega_i^n \sim \sum_0^p C_{ik}^n$ mod S and, therefore $\Omega_i^n \sim C_{ik}^n$ mod $R-P_k$. Let $\Gamma_{ik}^{n-1} = F C_{ik}^n$ (here and in what follows k runs over the values 1,2,...,p only, k = 0 being left out). Evidently $\Gamma_{ik}^{n-1} \in \mathcal{M}$. Let $\sum r_{ik} \Gamma_{ik}^{n-1} \sim 0$ in S. Precisely, we have to prove that all $r_{ik} = 0$. Let us assume that, on the contrary, $r_{11} \neq 0$. Now $\sum r_{ik} C_{ik}^n$ is an (n, R)-cycle mod S in $R-P_0$ and $F \sum r_{ik} C_{ik}^n \sim 0$ in S. By lemma II, it follows that there exists an absolute (n, R)-cycle Δ_0^n in $R-P_0$ such that $\sum r_{ik} C_{ik}^n \sim \Delta_0^n$ mod S. If $k \geq 2$, then $C_{ik}^n \subset R-P_1$; therefore $\sum r_{ik} C_{ik}^n \sim \sum r_{i1} C_{i1}^n$ mod $R-P_1$. Since $S \subset R-P_1$, we have $\sum r_{i1} C_{i1}^n \sim \Delta_0^n$ mod $R-P_1$. But $C_{i1}^n \sim \Omega_i^n$ mod $R-P_1$. Therefore $\sum r_{i1} \Omega_i^n - \Delta_0^n \sim 0$ mod $R-P_1$. By lemma III it follows that there exists an absolute (n, R)-cycle $\Delta_1^n \subset R-P_1$ such that $\sum r_{i1} \Omega_i^n \sim \Delta_0^n + \Delta_1^n$. Since $\Delta_0^n \subset R-P_0 \neq R$, $\Delta_1^n \subset R-P_1 \neq R$, we have $r_{i1} = 0$, in particular $r_{11} = 0$, which is a contradiction.

Corollary. Let R be a compact subset of the euclidean E_{n+1} and let there exist m+1 complementary domains of R (rel. E_{n+1}) having the whole R as their boundary. Let S be a closed subset of R and let g be the $(n-1)^{th}$ Betti number of S. Then the set R - S has at most $\left[\frac{g}{m}\right] + 1$ components.

This corollary was given by Wilder, but (in the case $m \geqq 2$) with g instead of $\left[\frac{g}{m}\right] + 1$, which is weaker except when $g \leqq 1$ or $g = m = 2$.

Now we shall assume that R has the following two properties:

(1) R is bicompact, i.e. if any family $\emptyset$ of open sets covers R, then a finite subfamily of $\emptyset$ covers R.

(2) dim R = n. That signifies: (i) every covering $\mathfrak{U}$ has a refinement $\mathfrak{V}$ such that $\dim \mathfrak{V} \leqq n$ ($\dim \mathfrak{V}$ being the largest dimension of a $\mathfrak{V}$ simplex), (ii) not every covering $\mathfrak{U}$ has a refinement $\mathfrak{V}$ such that $\dim \mathfrak{V} < n$.

These assumptions imply the following statement: If C^n is an (n, R)-cycle mod S, then there exists a uniquely determined minimal closed $T \supset S$ such that $C^n \sim 0$ mod T. The existence of T is a consequence of (1), the uniqueness follows from (2). We shall call T the carrier of the cycle C^n and we shall apply it in the following form: If $C^n \sim 0$ mod $T_0 = \overline{T}_0$, then the set T_0 must contain the carrier T.

The space R will be called an n-pseudomanifold,* if it has the following properties: (1) R is a bicompact normal space. (2) dim R = n(= 1,2,3,...). (3) There exists some absolute (n, R)-cycle Ω^n which is not ~ 0. (4) If $S = \overline{S} \subset R \neq S$, and if Δ^n is an absolute (n, R)-cycle in S, then $\Delta^n \sim 0$. (5) Given a point $a \in R$ and a neighborhood ** U of a, there exists a neighborhood $V \subset U$ of a having the following property: If C^n is any (n, R)-cycle mod R-U, then there exists an absolute (n, R)-cycle Ω^n such that $C^n \sim \Omega^n$ mod R-V.

* A more proper name would be an orientable pseudomanifold, but I shall not give here the more general definition.

** All my neighborhoods are open.

Everywhere in the sequel, R is a given pseudomanifold and S is a given closed subset of R.

Theorem II. R is a locally connected continuum.

Proof. That R is a continuum, is quite trivial.* U being a given

* As a matter of fact, a far more general property of R is a corollary of theorem I.

neighborhood of a point $a \in R$, let V be a smaller neighborhood of a as in property (5) in the definition of an n-pseudomanifold. It is sufficient to prove that the whole set V is a part of one quasicomponent of U. Let us assume the contrary. Then we have $U = P+Q$ with separate summands such that $PV \neq 0 \neq QV$. Let Ω^n be an absolute (n, R)-cycle which is not ~ 0. Since Ω^n may be regarded as an (n, R)-cycle mod R-U, by lemma VI there exist two (n, R)-cycles: C^n mod $\overline{P}$-U in $\overline{P}$ and D^n mod $\overline{Q}$-U in $\overline{Q}$ such that $\Omega^n \sim C^n + D^n$ mod R-U. By property (5) of a pseudomanifold, there exists an absolute (n, R)-cycle Ω^n_0 such that $C^n \sim \Omega^n_0$ mod R-V. Since $C^n \subset \overline{P}$, we have $\Omega^n_0 \sim 0$ mod $R-V+\overline{P} \subset R-QV$. By lemma III it follows that there exists an absolute (n, R)-cycle Δ^n in R-QV such that $\Omega^n_0 \sim \Delta^n$. Since $R-QV \neq R$, $\Delta^n \sim 0$ by property (4) of a pseudomanifold. It follows that $\Omega^n_0 \sim 0$ and, therefore, $C^n \sim 0$ mod R-V. Similarly we have $D^n \sim 0$ mod R-V. Since $\Omega^n \sim C^n + D^n$ mod $R-U \subset R-V$, we have $\Omega^n \sim 0$ mod $R-V \neq R$. By lemma III and by property (4) of a pseudomanifold, this implies that $\Omega^n \sim 0$ which is a contradiction.

Lemma VII. Let T be the carrier of the (n, R)-cycle C^n mod S. Then the set T-S is open.

Proof. Let there exist, on the contrary, a point

$a \in (T-S).\overline{R-T}.$

Since U = R-S is a neighborhood of a, we may determine a smaller neighborhood V of a by property (5) of a pseudomanifold. Then there exists an absolute (n, R)-cycle Δ^n such that $C^n \sim \Delta^n$ mod R-V. Since $C^n \sim 0$ mod T, we have $\Delta^n \sim 0$ mod R-V+T. But R-V+T is closed and $\neq$ R, so that $\Delta^n \sim 0$ by lemma II and property (4) of a pseudomanifold. It follows that $C^n \sim 0$ mod R-V. Since T is the carrier of C^n, we must have T $\subset$ R-V, which is evidently wrong.

Now T-S is open, therefore open in R-S, and T-S is also closed in R-S. Therefore:

<u>Lemma VIII.</u> The carrier T of an (n, R)-cycle C^n mod S is the sum of (some of the) components of R-S.

<u>Lemma IX.</u> Let P be a component of R-S. Let C^n be an (n, R)-cycle mod S. Then there exists an absolute (n, R)-cycle Ω^n such that $C^n \sim \Omega^n$ mod R-P.

<u>Proof.</u> Choose a point a $\in$ P. Since R is locally connected and S is closed, P = U is open and, therefore, it is a neighborhood of a. Let V be a smaller neighborhood determined by property (5) of a pseudomanifold. It follows that there exists an absolute (n, R)-cycle Ω^n such that $C^n \sim \Omega^n$ mod R-V. Therefore the carrier T of $C^n - \Omega^n$ is contained in R-V. By lemma VIII, it follows that T $\subset$ R-P. But $C^n \sim \Omega^n$ mod T by definition of T. Since T $\subset$ R-P, we have $C^n \sim \Omega^n$ mod R-P.

Now let us recall that $\mathcal{M}$ was the modulus of all absolute (n-1, R)-cycles Γ^{n-1} in S such that $\Gamma^{n-1} \sim 0$ in R, such a cycle Γ^{n-1} being regarded as zero if it is ~ 0 in S.

We shall consider submoduli $\mathcal{N}$ of the modulus $\mathcal{M}$ (called <u>moduli</u> briefly). If $\mathcal{N}$ is such a modulus, then $\overline{\mathcal{N}}$ (the "closure" of $\mathcal{N}$) is, by definition, the family of all those $\Gamma^{n-1} \in \mathcal{M}$ having the following property:

Given any covering $\mathfrak{U}$, there exists a $\Delta^{n-1} \in \mathcal{M}$ (depending on $\mathfrak{U}$) such that

$$\Gamma^{n-1}(\mathfrak{U}) \sim \Delta^{n-1}(\mathfrak{U}) \qquad \text{in } S.$$

Evidently $\bar{\mathcal{N}}$ is a modulus $(\mathcal{N} \subset \bar{\mathcal{N}} \subset \mathcal{M})$.

Everywhere in the sequel, Ψ denotes the family of <u>all</u> components of R-S. If $\phi \subset \Psi$, then $\mathcal{H}(\phi)$ will denote the point set which is the sum of all the sets belonging to the family ϕ. E.g. $\mathcal{H}(0) = 0$, $\mathcal{H}(\Psi) = R-S$ and generally $\mathcal{H}(\phi) + \mathcal{H}(\Psi - \phi) = R-S$, $\mathcal{H}(\phi) \cdot \mathcal{H}(\Psi - \phi) = 0$.

Everywhere in the sequel, if $\phi \subset \Psi$, $\mathcal{M}(\phi)$ is the set of all those $\Gamma^{n-1} \in \mathcal{M}$, for which $\Gamma^{n-1} \sim 0$ mod $R - \mathcal{H}(\phi)$. So $\mathcal{M}(0) = \mathcal{M}$, $\mathcal{M}(\Psi) = 0$. In general, $\phi_1 \subset \phi_2 \subset \Psi$ implies $\mathcal{M}(\phi_1) \supset \mathcal{M}(\phi_2)$.

If $\phi \subset \Psi$, then $\mathcal{M}(\phi)$ is a modulus and

$$\overline{\mathcal{M}(\phi)} = \mathcal{M}(\phi).$$

From this point on, we shall assume that the n^{th} Betti number of R (= the rank of the modulus of all the (n, R)-cycles) is <u>finite.</u> We shall denote it by m and shall choose, once for all, a fixed Betti basis Ω_i^n $(1 \leq i \leq m)$ for the absolute (n, R)-cycles. By property (3) of a pseudomanifold, $m > 0$. We shall see later that in the case $n = 1$ we must have $m = 1$. But for $n > 1$, every value of m is actually possible. Indeed, Wilder gave an example in the euclidean E_{n+1}, of a compact set R such that R is the boundary of all components of E_{n+1}-R, the number $m+1 = 2, 3, \ldots$ or $m = \infty$ of those components being given, and each such component being uniformly locally connected. It is easy to prove (as a corollary of our following theorems) that such an R is an n-pseudomanifold, for which the number m has the signification given above.

Now we have the following general theorem regarding the separation of a pseudomanifold by an arbitrary closed subset:

<u>Theorem III.</u> Let $\phi_1 \subset \phi_2 \subset \Psi$. Let p be the number of the components forming the family $\phi_2 - \phi_1$. Let g be the rank of the modulus

$$\mathcal{M}(\phi_1) \bmod \mathcal{M}(\phi_2)$$

(= the max. number of cycles $\Gamma_i^{n-1} \in \mathcal{M}(\phi_1)$ such that $\sum r_i \Gamma_i^{n-1} \in \mathcal{M}(\phi_2)$ implies $r_i = 0$). Let

$c = 1$ if both $p > 0$ and $\phi_1 = 0$,

$c = 0$ if either $p = 0$ or $\phi_1 \neq 0$.

Then

$$g = m(p-c).$$

<u>Proof.</u> Let us assume that $g < m(p-c)$, so that g is finite. Let P_k $(0 \leqq k < p)$ be all the components of R-S belonging to the family $\phi_2 - \phi_1$. By lemma VI, there exist (n, R)-cycles C_{ik}^n mod $S\overline{P}_k$ in $\overline{P}_k$ such that $C_{ik}^n \sim \Omega_i^n$ mod $R - P_k$. Let $\Gamma_{ik}^{n-1} = FC_{ik}^n$ so that evidently $\Gamma_{ik}^{n-1} \in \mathcal{M}(\phi_1)$. Since $g < m(p-c)$, there must exist numbers r_{ik} which are not all null and such that $\sum_{i=1}^m \sum_{k=c}^p r_{ik} \Gamma_{ik}^{n-1} \sim 0$ in $R - \mathcal{H}(\phi_2)$. By lemma I it follows that there exists an (n, R)-cycle D^n mod S in $R - \mathcal{H}(\phi_2)$ such that

$$FD^n \sim \sum_{i=1}^m \sum_{k=c}^p r_{ik} \Gamma_{ik}^{n-1} \text{ in } S.$$

It follows that $D^n - \sum_{i=1}^m \sum_{k=c}^p r_{ik} C_{ik}^n$ is an (n, R)-cycle mod S in $\sum_{k=c}^p \overline{P}_k + R - \mathcal{H}(\phi_2)$, whose boundary is ~ 0 in S. By lemma II, it follows that there exists an absolute (n, R)-cycle $\Delta^n \subset \sum_{k=c}^p \overline{P}_k + R - \mathcal{H}(\phi_2)$ such that $D^n - \sum_{i=1}^m \sum_{k=c}^p r_{ik} C_{ik}^n \sim \Delta^n$ mod S. Now, if $c = 1$, we have $\sum_{k=c}^p \overline{P}_k + R - \mathcal{H}(\phi_2) \subset R - P_0 \neq R$, and if $c = 0$, we have $\phi_1 \neq 0$ and $\sum_{k=c}^p \overline{P}_k + R - \mathcal{H}(\phi_2) \subset R - \mathcal{H}(\phi_1) \neq R$; by property (4) in the definition of a pseudomanifold, it follows that $\Delta^n \sim 0$ and, therefore,

$$D^n \sim \sum_{i=1}^m \sum_{k=c}^p r_{ik} C_{ik}^n \quad \text{mod } S.$$

Let us choose the value of

k $(c \leqq k \leqq p)$. We have $D^n \subset R - \mathcal{H}(\phi_2) \subset R - P_k$, $C^n_{ik} \sim \Omega^n_i$ mod $R - P_k$, $S \subset R - P_k$, $D^n \sim \sum_{i=1}^{m} \sum_{k=c}^{p} r_{ik} C^n_{ik}$ mod S. Therefore $\sum_{i=1}^{m} r_{ik} \Omega^n_i \sim 0$ mod $R - P_k$ By lemma III and by property (4) of a pseudomanifold, this implies $\sum_{i=1}^{m} r_{ik} \Omega^n_i \sim 0$ and, therefore, $r_{ik} = 0$, which is a contradiction.

II. Let $g > m(p-c)$, so that p is finite. There exist cycles $\Gamma^{n-1}_\lambda \in \mathcal{M}(\phi)$ $(0 \leqq \lambda \leqq m(p-c))$ such that

$$\sum s_\lambda \Gamma^{n-1}_\lambda \in \mathcal{M}(\phi_2) \quad \text{implies} \quad s_\lambda = 0.$$

Let P_k $(0 \leqq k \leqq p-1)$ be all components forming the family $\phi_2 - \phi_1$. By lemma I, there exist (n, R)-cycles C^n_λ mod S in $R - \mathcal{H}(\phi_1)$ such that $FC^n_\lambda \sim \Gamma^{n-1}_\lambda$ in S. By lemma IX, there exist numbers $r_{ik\lambda} \in \mathcal{R}$ such that

$$C^n_\lambda \sim \sum_{i=1}^{m} r_{ik\lambda} \Omega^n_i \text{ mod } R - P_k .$$

Let us consider the system of linear equations

$$\sum_{\lambda=0}^{m(p-c)} r_{ik\lambda} s_\lambda = t_i \qquad (1 \leqq i \leqq m,\ 0 \leqq k \leqq p-1)$$

where $t_1 = \dots = t_m = 0$ in the case $c = 0$. The number of the equations of our system is less than the number of unknowns; $\mathcal{R}$ being a field, there follows the existence of a solution s_λ, t_i such that not every s_λ is $= 0$. Evidently

$$\sum s_\lambda C^n_\lambda \sim \sum t_i \Omega^n_i \quad \text{mod } R - P_k ;$$

therefore the carrier T of $\sum s_\lambda C^n_\lambda - \sum t_i \Omega^n_i$ satisfies the inclusion $T \subset R - P_k$, whence $T \subset R - \sum_0^{p-1} P_k = R - \mathcal{H}(\phi_2 - \phi_1)$. In the case $c = 0$ we have $t_i = 0$, $C^n_\lambda \subset R - \mathcal{H}(\phi_1)$, whence $T \subset R - \mathcal{H}(\phi_1)$. The same thing is true if $c = 1$, because this implies $\mathcal{H}(\phi_1) = 0$. Therefore

$$T \subset R - [\mathcal{H}(\phi_2 - \phi_1) + \mathcal{H}(\phi_1)] = R - \mathcal{H}(\phi_2),$$

whence

$$\sum s_\lambda C^n_\lambda \sim \sum t_i \Omega^n_i \text{ mod } R - \mathcal{H}(\phi_2)$$

and, therefore

$$\sum \lambda_\lambda \Gamma_\lambda^{n-1} \sim \sum \lambda_\lambda F C_\lambda^n \sim 0 \text{ in } R-\mathcal{H}(\phi_2),$$

which implies the contradiction $\lambda_\lambda = 0$.

Now we shall determine the modulus $\mathcal{M}(\phi)$ in a very general case.

Theorem IV. Let Ξ be a family of closed subsets of S. Let ϕ be the family of all those components P of R-S whose boundary $\overline{P}$-P does not belong to the family Ξ. Let us suppose that Ξ has the following property: for any set $B \in \Xi$ the set $\mathcal{H}(\phi)$ is a subset of a connected subset of R-B. Let $\mathcal{N}$ be the submodulus of $\mathcal{M}$ generated by all $\Gamma^{n-1} \in \mathcal{M}$ such that $\Gamma^{n-1} \subset B$, B being some set of the family Ξ. Then we have $\mathcal{M}(\phi) = \overline{\mathcal{N}}$.

Proof. I. Let $\Gamma^{n-1} \subset B \in \Xi$, $\Gamma^{n-1} \sim 0$ in R. By lemma I, there exists an (n, R)-cycle C^n mod B such that $FC^n \sim \Gamma^{n-1}$ in B. According to the property assumed of Ξ, there exists a component Q of R-B such that $\mathcal{H}(\phi) \subset Q$. By lemma IX, there exist numbers r_i such that $C^n \sim \sum r_i \Omega_i^n$ mod R-Q, so that, by lemma III, there exists an (n, R)-cycle D^n mod B in R-Q such that $C^n - \sum r_i \Omega_i^n \sim D^n$ mod B, whence $FC^n \sim FD^n$ in B and, therefore, $\Gamma^{n-1} \sim FD^n$ in B. But $D^n \subset$ R-Q, so that

$$\Gamma^{n-1} \sim 0 \text{ in } R-Q \subset R-\mathcal{H}(\phi),$$

i.e., $\Gamma^{n-1} \in \mathcal{M}(\phi)$. It follows that $\mathcal{N} \subset \mathcal{M}(\phi)$. Since $\mathcal{M}(\phi) = \overline{\mathcal{M}(\phi)}$, we must have $\overline{\mathcal{N}} \subset \mathcal{M}(\phi)$.

II. It remains to be proved that $\mathcal{M}(\phi) \subset \overline{\mathcal{N}}$. Let $\Gamma^{n-1} \in \mathcal{M}(\phi)$ and let $\mathfrak{U}$ be a given covering. We have to prove the existence of a $\Delta^{n-1} \in \mathcal{N}$ such that $\Gamma^{n-1}(\mathfrak{U}) \sim \Delta^{n-1}(\mathfrak{U})$ in S. By lemma V, there exists a neighborhood G of S and a refinement $\mathfrak{W}$ of $\mathfrak{U}$ such that, for any $(n, \mathfrak{W})$-chain $E^n(\mathfrak{W})$, $E^n(\mathfrak{W}) \subset \overline{G}$ implies $E^n(\mathfrak{W}) \subset$ S. Since $\Gamma^{n-1} \in \mathcal{M}(\phi)$, by lemma I there exists an (n, R)-cycle C^n mod S in R-$\mathcal{H}(\phi)$

such that $FC^n \sim \Gamma^{n-1}$ in S. Since R-G is bicompact and R is locally connected, R-S has only a finite number of components P such that both $P \in \Psi - \emptyset$ and $P-G \neq 0$. Let P_k $(1 \leq k \leq p)$ be all those components and let

$$Q = \mathcal{H}(\Psi - \emptyset) - \sum P_k \quad \text{whence } Q \subset G.$$

Since $[R - \mathcal{H}(\emptyset)] - S = \sum P_k + Q$ with separate summands, by lemma VI there exist (n, R)-cycles D^n mod$(S \cdot \sum \overline{P}_k)$ in $\sum \overline{P}_k$ and E^n mod $S\overline{Q}$ in $\overline{Q}$ such that $C^n \sim D^n + E^n$ mod S, whence $C^n \sim D^n$ mod $\overline{Q}$, wherefore $FD^n \sim FC^n \sim \Gamma^{n-1}$ in $\overline{Q} \subset \overline{G}$; by definition of G and $\mathcal{W}$ it follows that $FD^n(\mathcal{W}) \sim \Gamma^{n-1}(\mathcal{W})$ in S, whence $FD^n(\mathcal{V}) \sim \Gamma^{n-1}(\mathcal{V})$ in S, since $\mathcal{W}$ is a refinement of $\mathcal{V}$ and both FD^n and Γ^{n-1} are absolute (n-1, R)-cycles in S. Since $D^n \subset \sum \overline{P}_k$ and $\sum \overline{P}_k - S = \sum P_k$ with separate summands in the righthand side, lemma VI implies the existence of (n, R)-cycles D^n_k mod $\overline{P}_k - P_k$ in $\overline{P}_k$ such that $D^n \sim \sum D^n_k$ mod S, whence $FD^n \sim \sum FD^n_k$ in S. Since $P_k \in \Psi - \emptyset$, we have $\overline{P}_k - P_k \in \Xi$. Since D^n_k is a cycle mod $\overline{P}_k - P_k$ in $\overline{P}_k$, it follows that $FD^n_k \in \mathcal{N}$ and, therefore $\Delta^{n-1} = \sum FD^n_k \in \mathcal{N}$. But we had $FD^n(\mathcal{V}) \sim \Gamma^{n-1}(\mathcal{V})$ in S and $FD^n \sim \Delta^{n-1}$ in S, which implies that $\Gamma^{n-1}(\mathcal{V}) \sim \Delta^{n-1}(\mathcal{V})$ in S.

The significance of theorem IV will appear clearly if we consider some special cases of it, which, still, are very general.

Case I. Let A be a given subset of S. (There would be no loss of generality in assuming A closed.) Let the family Ξ consist of all closed subsets B of S such that A is not a subset of B. The family $\emptyset$ consists of all components P of R-S whose boundary contains A. It is easy to verify that, given $B \in \Xi$, $\mathcal{H}(\emptyset)$ is a subset of a connected subset of R-B. Therefore, $\mathcal{M}(\emptyset) = \overline{\mathcal{N}}$, where the modulus $\mathcal{N}$ is generated by all absolute (n-1, R)-cycles Γ^{n-1}, ~ 0 in R, such that $\Gamma^{n-1} \subset B \in \Xi$. Let us introduce the following notations:

(1) g is the rank of $\mathcal{M}$ mod $\mathcal{M}(\phi)$,

g^* is the rank of $\mathcal{M}(\phi)$;

(2) $\left.\begin{matrix} p \\ p^* \end{matrix}\right\}$ is the number of components P of $R-S$ such that $A \left\{\begin{matrix} \text{is} \\ \text{is not} \end{matrix}\right.$ a subset of the boundary of P.

We may apply theorem III in two manners, putting first $\phi_1 = 0$, $\phi_2 = \phi$ and second $\phi_1 = \phi$, $\phi_2 = \Psi$ and we have the following two statements:

(3) If $p = 0$, then $g = 0$; if $p > 0$, then $g = m(p-1)$.

(4) If either $p^* = 0$ or $p > 0$, then $g^* = mp^*$; if both $p^* > 0$ and $p = 0$, then $g^* = m(p^*-1)$.

<u>Case II</u>. Let there be given a <u>connected</u> subset A of S (not necessarily closed). Ξ is the family of all those closed subsets of S which do not meet A. ϕ is the family of all components of $R-S$ whose boundary meets A. As in case I, it is easy to verify that, given a $B \in \Xi$, the set $\mathcal{H}(\phi)$ is a subset of a connected subset of $R-B$. Therefore, $\mathcal{M}(\phi) = \overrightarrow{\mathcal{N}}$, where the modulus $\mathcal{N}$ is generated by all $\Gamma^{n-1} \in \mathcal{M}$ such that $\Gamma^{n-1} \subset B \in \Xi$. Let us introduce again the notation (1), and, instead of (2):

(2') $\left.\begin{matrix} p \\ p^* \end{matrix}\right\}$ is the number of components P of $R-S$ whose boundary $\left\{\begin{matrix} \text{meets} \\ \text{does not meet} \end{matrix}\right.$ the set A.

Then we have again the statements (3) and (4).

The case II may be generalized as follows. Let there be given a subset A of S and a family $\Gamma \neq 0$ of subsets of A such that: (i) if $C \in \Gamma$ and $C^* \subset C$, then $C^* \in \Gamma$, (ii) if $C \in \Gamma$, then $A-C$ is connected. (In particular A must be connected, since $0 \in \Gamma$.) Ξ will be the family of all $B = \overline{B} \subset S$ such that the set AB belongs to Γ. ϕ will be the family of all components of R-S whose boundary meets A in a set <u>not</u> belonging to Γ. If we have (1) and

(2") $\left.\begin{matrix} p \\ p^* \end{matrix}\right\}$ is the number of components of R-S whose boundary meets A in a set $\left\{\begin{matrix}\text{not belonging} \\ \text{belonging}\end{matrix}\right\}$ to the family Γ,

then we have again (3) and (4).

It is easy to describe the most general 1-pseudomanifold R. If S consists of two points, then the modulus $\mathcal{M}$ evidently has rank g = 1. But if R-S has p components, it follows from theorem III that g = m(p-1). Therefore m = 1, as was announced above, and p = 2. It follows that R has the property that any two points decompose it in precisely two parts. Therefore, as is well known, R is the sum of two simply ordered continua having only the terminal points in common. If R is separable, it is a circle.

I shall finish with a very quick summary of further results.

If $\{\phi_\iota\}$ is an arbitrary collection (finite, countable or uncountable) of subfamilies of ψ, then $\mathcal{M}(\prod \phi_\iota) = \overline{\sum \mathcal{M}(\phi_\iota)}$ where $\sum \mathcal{M}(\phi_\iota)$ is the minimum modulus containing all $\mathcal{M}(\phi_\iota)$. If the collection $\{\phi_\iota\}$ is finite, then $\overline{\sum \mathcal{M}(\phi_\iota)} = \sum \mathcal{M}(\phi_\iota)$.

It is more difficult to describe $\mathcal{M}(\sum \phi_\iota)$. The result is that $\mathcal{M}(\sum \phi_\iota)$ may be determined by means of the moduli $\mathcal{M}(\phi_\iota)$ only if we know, for each couple (ι, κ) whether $\phi_\iota \phi_\kappa$ is or is not vacuous. In particular we have simply $\mathcal{M}(\sum \phi_\iota) = \prod \mathcal{M}(\phi_\iota)$ if always $\phi_\iota \phi_\kappa \neq 0$.

My further remarks are here stated only for separable (= metrizable) pseudomanifolds. In theorem III we have p = ∞ if and only if g = ∞. But we can obtain more precise statements. The simplest case is when p is "weakly infinite", i.e. for every $\epsilon > 0$ there exists only a finite number of components

$P \in \phi_2 - \phi_1$ having diameter $> \epsilon$. The necessary and sufficient condition is that the rank of

$$\mathcal{M}(\phi_1) \bmod \mathcal{M}(\phi_2)$$

be, too, "weakly infinite" in the following sense. Given an $\epsilon > 0$, the rank of

$$\mathcal{M}(\phi_1) \bmod [\mathcal{M}(\phi_2) + \mathcal{N}_\epsilon]$$

is finite, where $\mathcal{N}_\epsilon$ is the modulus generated by all $\Gamma^{n-1} \in \mathcal{M}(\phi_1)$ such that $\Gamma^{n-1} \subset B \subset S$, the diameter of B being less than ϵ.

Let us suppose that the family Ξ in the theorem IV has the following property: If A_n and A are closed subsets of S such that no A_n belongs to Ξ, and if $\lim A_n = A$ (in Hausdorff's sense), then A does not belong to Ξ. Then (in the notations of theorem IV) we have $\mathcal{N} = \bar{\mathcal{N}}$, if and only if the following statement is true: If $P_k \in \Psi - \Phi$, $A = \lim P_k$, then $A \in \Xi$. The assumed property of Ξ is true in both cases I and II treated above as illustrations of theorem IV, but not necessarily in the above generalization of case II.