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Vojtěch Jarník

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by

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The mathematical community lost an eminent man whose contributions to our science are of lasting value. On September 22, 1970 after a longer disease, Professor V. Jarník died.

Professor V. Jarník was born December 22, 1897 in Prague. After his studies at the Charles University in Prague (where he was influenced by Professor Karel Petr) he spent some time (from fall 1923 till spring 1925 and then again 1927/28) in Göttingen with Edmund Landau.

In 1928 he became professor at Charles University; he has been active here as assistent and professor for 47 years and retired in summer 1968.

He was ordinary member of the Czechoslovak Academy of Sciences and played an important role in the organization of university education and scientific research in Czechoslovakia.

For his outstanding achievements in the field of sciences he was awarded several distinctions, among others (in December 1967) the Order of Republic.

Jarník is known to the mathematical community by his original scientific work in analytic theory of numbers, in diophantine approximations, in geometry of numbers and in the theory of real functions.

In the following a few of his numerous results should be recalled.

One of the areas where his results are of decisive importance is the theory of lattice points in convex bodies.

Let K be a convex, bounded, body in the r-dimensional Euclidian space containing the origin as an interior point and $K(\sqrt{x})$ the body which arises in an obvious way by a homothetic transformation with respect to the origin (i.e. by multiplying all position vectors ending on K by the number $\sqrt{x}$). Denote by $A(x)$ the number of points with integral coordinates (lattice points) inside and on the boundary of $K(\sqrt{x})$ and by $V(x)$ the volume of $K(\sqrt{x})$. Beginning with about 1915 the difference $P(x) = A(x) - V(x)$ has been in the center of interest of many outstanding mathematicians. The question arose whether the estimate $P(x) = O(x^{1/3})$

for the circle $u_1^2+u_2^2 \leqslant x$ (proved by Sierpiński in 1906) can be essentially sharpened. Van der Corput proved (1923) that the true order is $< x^{1/3}$. But he also proved that $P(x) = O(x^{1/3})$ holds for a large class of closed curves in the plane, roughly speaking for such ones for which the radius of curvature does not increase more quickly than the radius of a circle.

Jarník proved in [9] the existence of convex closed curves (with the property just mentioned) for which $P(x) = \Omega(x^{1/3})$ holds. This result destroyed, so to say, many old expectations and conjectures.

The greatest number of Jarník's papers deals with the lattice points in ellipsoids.

Let $K(\sqrt{x})$ be the ellipsoid $Q(u_1, u_2, \ldots, u_r) \leqslant x, r \geqslant 2$, Q being a positive definite quadratic form. The fundamental result due to E. Landau (1915–1924) reads

$$(1) \qquad P(x) = O(x^{\frac{r}{2}-\frac{r}{r+1}}), \quad P(x) = \Omega(x^{\frac{r-1}{4}}).$$

In the case that the form Q is "rational" (i.e., its coefficients are integral multiples of the same real number), A. Walfisz (for $r \geqslant 8$) and E. Landau (for $5 \leqslant r < 8$) succeeded in improving the O-estimate (1) to

$$(2) \qquad P(x) = O(x^{\frac{r}{2}-1})$$

by means of a method due to Hardy, Ramanujan and Littlewood. Jarník (1925) proved by surprisingly simple arguments that this result is final since for "rational" ellipsoids we have $P(x) = \Omega(x^{\frac{r}{2}-1})$. This was the first final result in this domain.

The significance of Jarník's studies consists in elaborating very efficient O- and Ω-methods, not only for the investigation the function $P(x)$ but also for the investigation of the quadratic mean value $T(x) = \sqrt{M(x)/x}$, where $M(x) = \int_0^x P^2(t)\,dt$. His methods are suitable for a fairly wide class of "irrational" ellipsoids and in a number of cases they yield final results. In general, it is possible to say that all final results in this field are either due to Jarník, or based on his studies.

The method mentioned above makes use of the power series ($|z| < 1$)

$$\sum_{m_1,\ldots,m_r} z^{Q(m_1,\ldots,m_r)} \qquad (m_i = \text{integers})$$

as the generating function for $A(x)$. This method being obviously inapplicable to "irrational" forms Q, Jarník considers the Dirichlet series

$$\Theta(s) = \sum_{m_1,\ldots,m_r} e^{-sQ(m_1,\ldots,m_r)}$$

and expresses the function $A(x)$ in terms of the integral $(a > 0)$

$$\frac{1}{2\pi i}\int_{a-i\infty}^{a+i\infty}\frac{\Theta(s)\,e^{xs}}{s}\,ds. \tag{3}$$

After getting over some complications [$A(x)$ is expressed by (3) only for $x \neq Q(m_1, \ldots, m_r)$ with integers $m_1, \ldots, m_r$; the integral is not absolutely convergent] we can use (3) to find O-estimates of the function $P(x)$ if we have good enough O-estimates of the function $\Theta(s)$. For this purpose Jarník makes use of some transformation properties of the function $\Theta(s)$, which are available only for "diagonal" forms Q, i.e. for the forms

$$a_1Q_1(u_1, \ldots, u_{r_1}) + a_2Q_2(u_{r_1+1}, \ldots, u_{r_1+r_2}) + \ldots + \\ + a_\tau Q_\tau(u_{r_1+\ldots+r_{\tau-1}+1}, \ldots, u_{r_1+\ldots+r_\tau}) \tag{4}$$

($\tau \geqslant 1$, $r = r_1 + \ldots + r_\tau$, $a_1 > 0, \ldots, a_\tau > 0$, Q_i being positive definite quadratic forms with integral coefficients). Jarník improves the efficiency of these estimates by dividing the path of integration into a number of parts, in each of which $\Theta(s)$ may be very well estimated. At this point of Jarník's reasoning very fine and deep results from the theory of diophantine approximations appear as an auxiliary tool.

Let us present now at least some of Jarník's most important results. Let Q be a "diagonal" form (4). In the years 1928–29 Jarník showed ([17], [18], [19]) that even in this case (for $r \geqslant 5$) the estimate (2) holds. However, if the form Q is irrational, the estimate may be (for $r \geqslant 5$) improved to $P(x) = o(x^{\frac{r}{2}-1})$ ([19], [27]) and this result cannot be, in general, further improved. The question arises how the value

$$f(Q) = \limsup_{x\to\infty}\frac{\log|P(x)|}{\log x}$$

depends on the properties of the numbers $a_1, a_2, \ldots, a_\tau$. Jarník ([17]) shows that for almost all systems $a_1, \ldots, a_\tau$ we have $f(Q) \leqslant \frac{r}{2} - \lambda$, where $\lambda = \sum_{j=1}^{\tau}\min\left(1, \frac{r_j}{4}\right)$ and for all systems $a_1, \ldots, a_\tau$ (see [18]) we have $P(x) = \Omega(x^{\frac{r}{2}-\tau})$. If $r_j \geqslant 4$ $(j = 1, \ldots, \tau)$, we obtain the final result

$$f(Q) = \frac{r}{2} - \tau$$

which is valid for almost all systems $a_1, \ldots, a_r$. If $r_j = 1$, we then obtain (see relation (1) above) the estimate

$$\frac{r}{4} - \frac{1}{4} \leqslant f(Q) \leqslant \frac{r}{4},$$

which holds for almost all forms

$$a_1 u_1^2 + a_2 u_2^2 + \ldots + a_r u_r^2.$$

In the case $r_j \geqslant 4$ the above mentioned fact yields the inequality

$$\frac{r}{2} - \tau \leqslant f(Q) \leqslant \frac{r}{2} - 1$$

where the bounds cannot be, in general, further improved. Jarník achieved even deeper results in this direction. In [22] and [45] Jarník gives an affirmative answer to the subtle question whether the values of $f(Q)$ cover the whole interval $\left\langle \frac{r}{2} - \tau, \frac{r}{2} - 1 \right\rangle$. If $\tau \geqslant 2$, he gives a corresponding existence theorem (making use of Hausdorff measure). If $\tau = 2$, $r_j \geqslant 4$, the dependence of $f(Q)$ on a_1 and a_2 can be formulated briefly as

$$f(Q) = \frac{r}{2} - 1 - \frac{1}{\gamma(a_1, a_2)},$$

where $\gamma(a_1, a_2)$ is the supremum of all numbers $\beta > 0$ for which the inequality

$$|qa_1 - pa_2| \leqslant \frac{1}{q^\beta}$$

has infinitely many solutions in integers p, $q > 0$. (As B. Diviš proved in 1968, a similar relation can be found even for $\tau > 2$.) In a number of papers ([68], [70]) Jarník studied the case $\tau = 2$ in much more detail and to an unprecedented depth. This belongs—without any doubts—to the most remarkable results in the study of $P(x)$.

The last Jarník's paper from this field is very interesting as well. Put $P_0(x) = P(x)$ and let for $\varrho > 0$

$$P_\varrho(x) = \frac{1}{\Gamma(\varrho)} \int_0^x P(t)(x-t)^{\varrho-1} dt.$$

Evidently $P_{\varrho+1}(x) = \int_0^x P_\varrho(t) dt$. In a number of cases it is possible to find O- and Ω-estimates for the function $P_0(x)$ if satisfactory O- and Ω-estimates are known for $P_\varrho(x)$ (e.g. for ϱ large enough). Jarník in [89] studies the problem of dependence of O- and Ω-estimates on the parameter ϱ and shows that for rational forms we have

$$P_\varrho(x) = O(x^{\frac{r}{2}-1}), \quad P_\varrho(x) = \Omega(x^{\frac{r}{2}-1}),$$

when $0 \leqslant \varrho < \frac{r}{2} - 2$ and

$$P_\varrho(x) = O(x^{\frac{r-1}{4}+\frac{\varrho}{2}}), \quad P_\varrho(x) = \Omega(x^{\frac{r-1}{4}+\frac{\varrho}{2}}),$$

if $\varrho > \frac{r}{2} - \frac{1}{2}$. For $\frac{r}{2} - 2 \leqslant \varrho \leqslant \frac{r}{2} - \frac{1}{2}$ his results are not quite final; this means that the "classical" unsolved problems for $\varrho = 0$, $r = 2, 3, 4$ are "transfered" to the interval $2\varrho + 1 \leqslant r \leqslant 2\varrho + 4$.

Jarník's papers dealing with the function $M(x)$ are of independent importance. Very roughly speaking, Jarník proves for $M(x)$ results which would follow from the "expected" results concerning $P(x)$.

Let us consider again the diagonal forms (see, e.g. [33], [34], [38], [40], [69] and [71]). It is shown that it always holds

$$\liminf_{x \to \infty} \frac{T(x)}{x^{\frac{r}{4}-\frac{1}{4}}} > 0$$

(see [33]). On the other hand, we have (see [33], [71])

$$T(x) = \begin{cases} O(x^{\frac{1}{4}}) & \text{for} \quad r = 2, \\ O(x^{\frac{r}{2}-1} \log^\varepsilon x) & \text{for} \quad r > 2 \end{cases}$$

($\varepsilon = \frac{1}{2}$ for $r = 3$, $\varepsilon = 0$ for $r > 3$). At the same time, both estimates are very sharp since for rational forms we have, moreover, (see [69])

$$M(x) = \begin{cases} Kx^{r-1} + o(x^{r-1}) & \text{for} \quad r \geqslant 4, \\ Kx^2 \log x + O(x^2 \sqrt{\log x}) & \text{for} \quad r = 3, \end{cases}$$

with suitable positive constants K; and for almost all systems $\alpha_1, \ldots, \alpha_r$ we have (see [34], [71])

$$T(x) = O(x^{\frac{r}{4}-\frac{1}{4}} \log^\varepsilon x)$$

($\varepsilon = 0$ for $r = 2$, $\varepsilon = \frac{1}{2}$ for $r = 3$, $\varepsilon = (3r+3)/2$ for $r > 3$). In the papers [40] and [68] these problems are studied for the case $\tau = 2$ in much more detail. E.g., it is shown that ($\tau = 2, r_1, r_2 \geqslant 4$)

$$\limsup_{x\to\infty} \frac{\log T(x)}{\log x} = \frac{r}{2} - 1 - \frac{1}{\gamma(a_1, a_2)}.$$

The first result mentioned above is reproduced in the book of E. Landau *Vorlesungen über Zahlentheorie*, Bd. 2, Leipzig 1927. Some of the other results can be found in Walfisz's monograph *Gitterpunkte in mehrdimensionalen Kugeln*, Warszawa 1957.

Another field of Jarník's interest was the theory of diophantine approximations. His first paper in this area appeared in 1928, the last one only recently in 1969.

Recall the following notion. Let θ_{ik}, β_i be real numbers, $\varphi_i(t)$ functions defined for all sufficiently large positive t. Consider the linear inequalities

$$(5) \qquad \Big|\sum_{k=1}^{m} \theta_{ik}x_k - \beta_i - y_i\Big| < \varphi_i(t), \qquad i = 1, 2, \ldots, n.$$

We say that system (θ_{ik}, β_i) admits the approximation $\{\varphi_i(t)\}$ if for any $A > 0$ there exist integers x_i, y_k such that $A \leqslant \max_k |x_k| \leqslant t$ and (5) holds.

Take, e.g., the homogeneous case (i.e. $\beta_i = 0$) and $n = m = 1$. It follows from the results of Khintchin that almost all $\theta = \theta_{11}$ admit (do not admit) the approximation $\varphi(t)$ provided that $\int^{\infty} \varphi(t)\,dt$ diverges (converges). Even the study of this special case shows that the Lebesgue measure is giving only "rather rough" informations. Jarník ([28], [30], [35]) makes use of Hausdorff measure in order to obtain finer classification of the set of systems (θ_{ik}) admitting approximations by prescribed $\varphi(t) = \varphi_i(t)$, $i = 1, \ldots, n$. His results till 1936 are partly included in Koksma's monograph *Diophantische Approximationen*, Berlin 1936, where the connection with the results of other authors is presented.

Consider again the homogeneous case and suppose in the following $m > 1, n = 1$. Let $\theta_1, \ldots, \theta_m$ be m rationally independent numbers. Denote (x_i, p_i, q are integers)

$$\psi_1(t) = \min_{\substack{0<\max|x_i|\leqslant t\\ i=1,2,\ldots,m}} |x_1\theta_1 + \ldots + x_m\theta_m - x_0|,$$

$$\psi_2(t) = \min_{0<q\leqslant t} \big(\max_{1\leqslant i\leqslant m} |q\theta_i - p_i|\big).$$

It follows from the well known Dirichlet–Kronecker theorem that

$$\psi_1(t) < t^{-m}, \qquad \psi_2(t) < t^{-1/m}.$$

Denote by β_1 the upper bound of all λ for which $\liminf \psi_1(t)\cdot t^\lambda < \infty$ holds, and by β_2 the upper bound of those λ for which $\liminf \psi_2(t)\cdot t^\lambda < \infty$ holds. Khintchin proved the following so called "principle of transfer" (Übertragungsgesetz):

$$\frac{\beta_1-(m-1)}{m} \geqslant \beta_2 \geqslant \frac{\beta_1}{\beta_1(m-1)+m}. \tag{6}$$

Jarník first ([53], [54], [56]) examined the rather delicate question under what conditions the sign of equality in (6) holds.

In the important paper [62] he introduced the following numbers γ_1 and γ_2. We denote by γ_1 the upper bound of all λ for which $\limsup \psi_1(t)\cdot t^\lambda < \infty$ and by γ_2 the upper bound of those λ for which $\limsup \psi_2(t)\cdot t^\lambda < \infty$. Clearly: $m \leqslant \gamma_1 \leqslant \beta_1 \leqslant +\infty$, $1/m \leqslant \gamma_2 \leqslant \beta_2 \leqslant +\infty$. If $m = 2$, he proved $\gamma_2 = 1 - \dfrac{1}{\gamma_1}$, so that γ_2 is determined by γ_1. (This is not true for corresponding β_1, β_2, for $m = 2$.) For $m > 2$ he proved the formula (6) with γ_1, γ_2, instead of β_1, β_2. Here γ_1 is not determined by γ_2. Jarník gave in some cases an improvement of the interval where γ_2 can move (if γ_1 is fixed) and proved that his results are sharp.

In [83] Jarník examined the general case $m > 1, n > 1$. It follows from the Dirichlet–Kronecker theorem that any matrix $B = (\theta_{ik})$ admits the approximation $\varphi_i(t) = \varphi(t) = t^{-m/n}$.

Denote

$$\psi(t) = \min\left(\max_{1\leqslant j\leqslant n} |\theta_{j1}x_1 + \ldots + \theta_{jm}x_m + x_{m+j}|\right),$$

where $0 < \max|x_j| \leqslant t$ and suppose (in order to exclude some singular cases) $\psi(t) > 0$ for all $t \geqslant 1$. Denote by α and β respectively the upper bounds of all λ for which

$$\limsup_{\to\infty} t^\lambda \psi(t) < \infty \quad \text{and} \quad \liminf_{t\to\infty} t^\lambda \psi(t) < \infty$$

respectively. (Evidently, $m/n \leqslant \alpha \leqslant \beta \leqslant \infty$.) In [83] lower bounds for β in terms of α are given.

In [59] and [60] the connections between the homogeneous and non-homogeneous cases are treated.

The problems we mentioned above constitute, of course, only a small selection of questions investigated by Jarník in the area of diophantine approximations.

In the years 1939–49 Jarník published also a number of significant papers dealing with the geometry of numbers. Especially he studied

Minkowski's successive minima of convex bodies as well as non-convex bodies. We only mention his full characteristic of the cases for which Minkowski's inequality becomes an equality ([78]), its generalization for non-convex bodies in [74], an improvement of this inequality for some domains in the plane ([76]) and the study of various definitions of successive minima, which are important for non-convex bodies, in [77].

The last field in which Jarník published a great number of papers is the theory of functions of a real variable. His best known papers are those dealing with the study of properties (with regard to derivatives, Dini derivatives and approximative derivatives) of continuous functions in the sense of categories. These papers (mainly from the years 1933–36) are closely connected with the well known Polish mathematical school. Other results concerning the theory of function of a real variable are important as well. Thus, e.g., in [5] it is shown that the derivative of an (arbitrary, i.e. non-necessarily continuous) real function (defined on a bounded perfect set) is a function of the first Baire class. Paper [2] studies very thoroughly Bolzano's example of a continuous non-differentiable function. In [41] a construction is given of a continuous function which has an infinite derivative on a given set (having some natural properties) and finite Dini derivatives outside of this set. In [57] limit values of the function $f(x, y)$ of two real variables at $[\xi, \eta]$ are studied provided that $[x, y]$ approaches $[\xi, \eta]$ from various directions.

Jarník himself has written an Appendix in the book of E. Čech, *Bodové množiny*, where some of the results concerning the method of categories are presented.

But — and this should be underlined — Jarník beginning with his early years till the premature death published many papers dealing with different topics belonging to analysis. E.g., rearrangement of infinite series ([11], [13], [14]), superposition of functions ([50], [51]), problems connected with trigonometric series, etc.

Jarník work is in a close relation with the studies of several outstanding mathematicians. He closely cooperated with a number of them. In this connection let us mention at least the names of P. Erdös, H. Davenport, E. Landau, C. A. Rogers and A. Walfisz. On the other hand, his results inspired new investigations. A remarkable evidence of the depth and lasting value of Jarník's scientific work is the fact that even in the last few years essential results have been published which continue his works both in the theory of numbers and in the theory of functions of a real variable.

Jarník published 90 original papers, about two thirds of them belonging to the theory of numbers. Moreover, he published a four volume monograph on Differential and Integral calculus treated from a very modern point of view (1946–1955). Some of these volumes appeared in several editions. A number of critical studies (e.g. a study about B. Bolzano), expository papers, congress reports and a number of reviews conclude the list of his publications.

Jarník was a number of the Editorial Board of the Acta Arithmetica beginning with the first volume in 1937, and for 15 years the editor in chief of the mathematical section of the Časopis pro pěstování matematiky a fysiky.

Jarník was an outstanding teacher who was able to transmit his enthusiasm for mathematics to his students. A large number of contemporary Czechoslovak mathematicians may be considered as his students even if they work in various other disciplines of mathematics. In number theory he directly influenced the work of Vl. Knichal, K. Černý, J. Kurzweil, A. Apfelbeck, T. Šalát, Bř. Novák, B. Diviš and others.

In addition to all of this, his profound humane erudition, his tact and his pure human character resulted in an admiration and deep respect from all who have known him personally.

His work and his personality will remain for many years in the mind and heart of his friends and colleagues.