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Vojtěch Jarník's work in combinatorial optimization ☆

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Abstract

We discuss two papers of Vojtěch Jarník from 1930 and 1934 which are devoted to the Minimal Spanning Tree Problem and the Euclidean Steiner Tree Problem. These papers are historical milestones in combinatorial optimization. © 2001 Elsevier Science B.V. All rights reserved.

MSC: 01A60; 05C05; 90C27

Keywords: History of mathematics; History of computing; Graphs

0. Introduction

Jarník's status as one of the foremost mathematicians of his time is well known, see e.g. [28], [30]. With respect to his lasting achievements in number theory and analysis the aim of this note may seem to be very modest: we want to discuss two lesser known papers [1,2] which belong to an area different from the major part of Jarník's œuvre, namely to the area which much later became known as combinatorial or discrete optimization. These are the only papers by Jarník related to such problems and in fact the only papers which do not belong to the main line of his work (i.e. number theory, analysis and its foundations). Perhaps this would only be enough to justify a shorter note. But there is much more here than meets the eye. Papers [1,2] were overlooked for a long time, and, as we shall demonstrate, they are even now little known. But they are important and, as we wish to demonstrate, Jarník deserves much more credit for these truly pioneering works. In both of these papers Jarník was lucky to have dealt with problems which have since proved to be cornerstone pieces of Combinatorial Optimization developed in full in the fifties and sixties in the context of Linear Programmming and Computer Science.

☆ This paper is a modified version of a paper included in: B. Novák (Ed.), Life and Work of Vojtěch Jarník, Prometheus, Praha, 1999, pp. 37–54.

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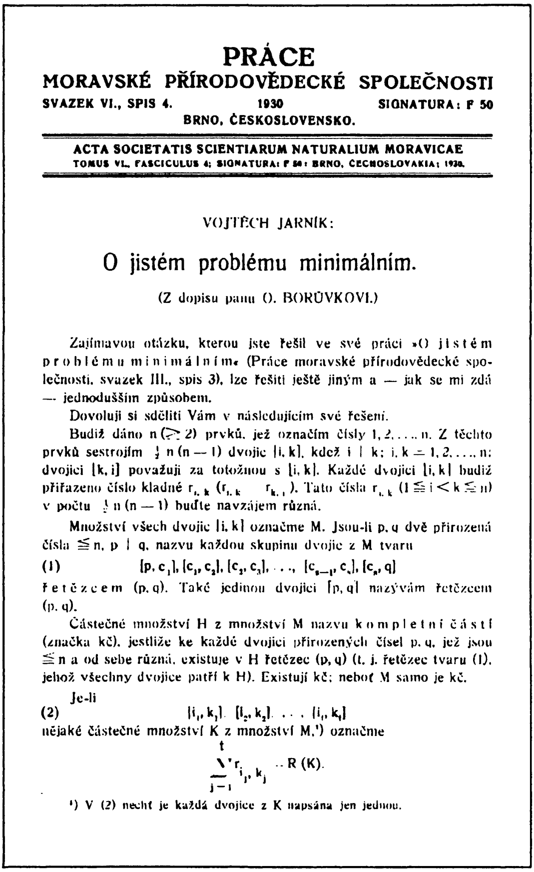


PRÁCE
MORAVSKÉ PŘÍRODOVĚDECKÉ SPOLEČNOSTI
SVAZEK VI., SPIS 4. 1930 SIGNATURA: F 50
BRNO, ČESKOSLOVENSKO.

ACTA SOCIETATIS SCIENTIARUM NATURALIUM MORAVICAE
TOMUS VI., FASCICULUS 4; SIGNATURA: F 50; BRNO, ČECHOSLOVAKIA; 1930.

VOJTĚCH JARNÍK:

O jistém problému minimálním.

(Z dopisu panu O. BORŮVKOVI.)

Zajímavou otázku, kterou jste řešil ve své práci »O jistém problému minimálním« (Práce moravské přírodovědecké společnosti, svazek III., spis 3), lze řešiti ještě jiným a — jak se mi zdá — jednodušším způsobem.

Dovoluji si sděliti Vám v následujícím své řešení.

Budiž dáno $n (\geq 2)$ prvků, jež označím čísly $1, 2, \ldots, n$. Z těchto prvků sestrojím $\frac{1}{2} n (n-1)$ dvojic $[i, k]$, kdež $i \neq k$; $i, k = 1, 2, \ldots, n$; dvojici $[k, i]$ považuji za totožnou s $[i, k]$. Každé dvojici $[i, k]$ budiž přiřazeno číslo kladné $r_{i,k}$ $(r_{i,k} = r_{k,i})$. Tato čísla $r_{i,k}$ $(1 \leqq i < k \leqq n)$ v počtu $\frac{1}{2} n (n-1)$ buďte navzájem různá.

Množství všech dvojic $[i, k]$ označme M. Jsou-li p, q dvě přirozená čísla $\leqq n$, $p \neq q$, nazvu každou skupinu dvojic z M tvaru

$$(1) \qquad [p, c_1], [c_1, c_2], [c_2, c_3], \ldots, [c_{s-1}, c_s], [c_s, q]$$

řetězcem (p, q). Také jedinou dvojici $[p, q]$ nazývám řetězcem (p, q).

Částečné množství H z množství M nazvu kompletní částí (značka kč), jestliže ke každé dvojici přirozených čísel p, q, jež jsou $\leqq n$ a od sebe různá, existuje v H řetězec (p, q) (t. j. řetězec tvaru (1), jehož všechny dvojice patří k H). Existují kč; neboť M samo je kč.

Je-li

$$(2) \qquad [i_1, k_1], [i_2, k_2], \ldots, [i_t, k_t]$$

nějaké částečné množství K z množství M,[1]) označme

$$\sum_{j=1}^{t} r_{i_j, k_j} = R(K).$$

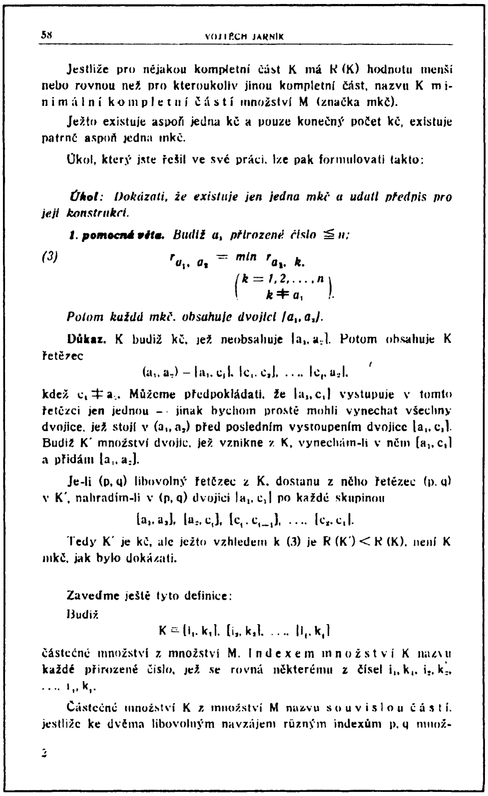
[1]) V (2) nechť je každá dvojice z K napsána jen jednou.

Fig. 1.

1. On a minimal problem

Jarník's paper [1] is a very short one and we can include a translation of most of it (the original two pages are given in Figs. 1 and 2).

One should see the original and look at a translation of [1]. The problem is stated and treated with a rigour and clarity which is missing in many later additions to this area. So we consider this as a good opportunity to present parts of Jarník's paper in full (we include a translation of about two thirds of [1]). We found no mistakes or even misprints in [1]! The paper [1] also has an interesting form: it is written in the "first person"-form and the reason for this is explained by its subtitle. We have tried to preserve Jarník's style as closely as possible. In particular, all symbols and notations are preserved. While a longer discussion will follow, we have included a few comments within the translation (we use square brackets [ ] for these; the translation itself is in italics).



58 VOJTĚCH JARNÍK

Jestliže pro nějakou kompletní část K má R(K) hodnotu menší nebo rovnou než pro kteroukoliv jinou kompletní část, nazvu K minimální kompletní částí množství M (značka mkč).

Ježto existuje aspoň jedna kč a pouze konečný počet kč, existuje patrně aspoň jedna mkč.

Úkol, který jste řešil ve své práci, lze pak formulovati takto:

Úkol: *Dokázati, že existuje jen jedna mkč a udati předpis pro její konstrukci.*

1. pomocná věta. *Budiž a_1 přirozené číslo $\leqq n$;*

$$r_{a_1, a_2} = \min_{\substack{k=1,2,\dots,n \\ k \neq a_1}} r_{a_1, k}. \tag{3}$$

Potom každá mkč. obsahuje dvojici $[a_1, a_2]$.

Důkaz. K budiž kč, jež neobsahuje $[a_1, a_2]$. Potom obsahuje K řetězec

$$(a_1, a_2) - [a_1, c_1], [c_1, c_2], \dots, [c_t, a_2],$$

kdež $c_1 \neq a_2$. Můžeme předpokládati, že $[a_1, c_1]$ vystupuje v tomto řetězci jen jednou — jinak bychom prostě mohli vynechat všechny dvojice, jež stojí v (a_1, a_2) před posledním vystoupením dvojice $[a_1, c_1]$. Budiž K′ množství dvojic, jež vznikne z K, vynechám-li v něm $[a_1, c_1]$ a přidám $[a_1, a_2]$.

Je-li (p, q) libovolný řetězec z K, dostanu z něho řetězec (p, q) v K′, nahradím-li v (p, q) dvojici $[a_1, c_1]$ po každé skupinou

$$[a_1, a_2], [a_2, c_t], [c_t, c_{t-1}], \dots, [c_2, c_1].$$

Tedy K′ je kč, ale ježto vzhledem k (3) je $R(K') < R(K)$, není K mkč, jak bylo dokázati.

Zaveďme ještě tyto definice:

Budiž

$$K = [i_1, k_1], [i_2, k_2], \dots, [i_t, k_t]$$

částečné množství z množství M. Indexem množství K nazvu každé přirozené číslo, jež se rovná některému z čísel $i_1, k_1, i_2, k_2, \dots, i_t, k_t$.

Částečné množství K z množství M nazvu souvislou částí, jestliže ke dvěma libovolným navzájem různým indexům p, q množ-

2

Fig. 2.

Vojtěch Jarník
On a certain minimal problem
(From a letter to O. Borůvka)

In your article 'On a certain minimal problem' (*which appeared in 'Práce moravské přírodovědecké společnosti'*, *vol. III*, *No.* 3) *you solved an interesting problem. It seems to me that there is a simpler solution of this problem. Allow me to state my solution here.*

[Thus Jarník decided to use the same title for his paper as Borůvka [3]. Borůvka was the first to solve the Minimal Spanning Tree problem, see [20] and comments below.]

Let n elements be given, I denote them as numbers 1,2,...,n. From these elements I form $\frac{1}{2}n(n-1)$ *pairs* $[i,k]$ *where* $i \neq k$, $i,k = 1,2,\dots,n$. *I consider the pair* $[k,i]$ *identical with the pair* $[i,k]$. *To every pair* $[i,k]$ *let there be associated a positive number* $r_{i,k}$ $(r_{i,k} = r_{k,i})$. *Let these numbers* $r_{i,k} (1 \leqslant i < k \leqslant n)$ *be pairwise different.*

[It is interesting to note that Jarník denotes the unordered pair by $[i,k]$, which is standard usage in graph theory today. This is also a departure from Borůvka's paper

[3] where the numbers $r_{i,k}$ are denoted by $[i,k]$. The fact that the numbers $r_{i,k}$ — i.e. in later terminology weights of edges — are supposed to be distinct is neither discussed nor justified. It seems that both Borůvka and Jarník were aware — as classical mathematicians — of "perturbation arguments". Certainly applications that they had in mind clearly suggest this, see [5,6] and the discussion of the concluding remarks of Jarník's paper below.]

We denote by M the set of all pairs $[i,k]$. *For two distinct natural numbers* $p,q \leqslant n$, *I call a chain* (p,q) *any set of pairs from M of the following form:*

$$[p,c_1],[c_1,c_2],\ldots,[c_{s-1},c_s],[c_s,q]. \tag{1}$$

Also, a single pair $[p,q]$ *I call a chain* (p,q).

[Even today the terminology is not unique — a set of the form (1) is called a path, trail, walk; Jarník considers (1) as a family — repetitions are allowed.]

A subset H of M I call a <u>*complete subset*</u> (*kč in short*), *if for any pair of distinct natural numbers* $p,q \leqslant n$ *there exists a chain* (p,q) *in H* (*i.e. a chain of form* (1) *all of whose pairs belong to H*). *There are kč; M itself is a kč.*

[Jarník's lucid Czech mathematical style became famous and standard; he may well be a bit playful here: kč is close to Kč — an abbreviation of Czech currency ('koruna česká').]

If

$$[i_1,k_1],[i_2,k_2],\ldots,[i_t,k_t] \tag{2}$$

is a subset K of M, we put

$$\sum_{j=1}^{t} r_{i_j,k_j} = R(K).$$

If for a complete set K the value R(K) is smaller than or equal to the values for all other complete sets, then I call K a minimal complete set in M (*symbolically mkč*). *As there exists at least one kč and there are only finitely many kč, there exists at least one mkč. The problem, which you* [i.e. O. Borůvka] *solved in your paper, can be formulated as follows*:

<u>*Problem*</u>: *Prove that there exists a unique mkč and give a formula* [i.e. an algorithm] *for its construction.*

[Of course *mk*č is the unique minimum spanning tree. There is no mention of trees in this paper.]

<u>*First Lemma*</u>: *Let* a_1 *be a natural number* $\leqslant n$ *with*

$$r_{a_1,a_2} = \min\{r_{a_1,k};\ k=1,2,\ldots,n,\ k \neq a_1\}. \tag{3}$$

Then every mkč contains a pair $[a_1,a_2]$.

[Summary of proof: The First Lemma is proved by a textbook argument: if K is a *k*č not containing $[a_1,a_2]$, then consider a chain $(a_1,a_2)=[a_1,c_1],[c_1,c_2],\ldots,[c_t,a_2]$

and form a new set K' by removing $[a_1,c_1]$ from K while adding $[a,a_2]$. Then K' is again a $k\check{c}$ and $R(K') < R(K)$.]

We introduce the following: Let $K \equiv [i_1,k_1],[i_2,k_2],\dots,[i_t,k_t]$ be a subset of M. An index of K I call any natural number from among $i_1,k_1,i_2,k_2,\dots i_t,k_t$. A subset K of M I call a connected subset if for any two distinct indices p,q of K it is possible to find in K a chain (p,q) (i.e. a chain (p,q) consisting of pairs from K only).

2. Lemma: Let S be a connected subset; let $h_1,h_2,\dots,h_s$ be all the indices of S; let $s < n$.

Let $l_1,l_2,\dots,l_t$ be numbers from $1,2,\dots,n$ which fail to be indices of S, let

$$r_{a,b} = \min\{r_{h_i,l_j};\ i=1,2,\dots,s,\ j=1,2,\dots,t\}. \tag{4}$$

Then I claim: every mkč containing S contains $[a,b]$ as well.

[We do not translate the proof but just summarize it. The Second Lemma is proved again by a textbook argument: let K be a $k\check{c}$ containing S and not containing $[a,b]$. Let a be an index of S. Then there exists in K a chain $(a,b)=[c_o,c_1],[c_1,c_2],\dots,[c_v,c_{v+1}]$ with $c_o=a$, $c_{v+1}=b$, $v\geqslant 1$. Let c_w be the last of the numbers $c_o,c_1,\dots,c_v$ which is an index of S. Then define subset K' by removing $[c_w,c_{w+1}]$ and adding $[a,b]$. K' is again a $k\check{c}$. Here Jarník considers two cases: $c_w=a$ and $c_w \neq a$. But $R(K') < R(K)$ and thus K fails to be an $mk\check{c}$.

Jarník does not mention that Lemma 1 is a special case of Lemma 2. Indeed, in his setting Lemma 1 is not a special case of Lemma 2 as a single vertex does not correspond to the index set of any $k\check{c}$.]

Let us now introduce a certain subset J of M [J for Jarník?] *as follows:*

Definition of set J:

$J \equiv [a_1,a_2],[a_3,a_4],\dots,[a_{2n-3},a_{2n-2}]$ where $a_1,a_2,\dots$ are defined as follows:

First Step:

Choose as a_1 any of the elements $1,2,\dots,n$. Let a_2 be defined by the relation $r_{a_1,a_2} = \min r_{a_1,l}$ $(l=1,2,\dots,n; l \neq a_1)$

kth Step:

Having defined

$$a_1,a_2,a_3,\dots,a_{2k-3},a_{2k-2} \quad (2 \leqslant k < n) \tag{5}$$

we define a_{2k-1},a_{2k} by $r_{a_{2k-1},a_{2k}} = \min r_{i,j}$ where i ranges over all the numbers $a_1,a_2,\dots,a_{2k-2}$ and j ranges over all the remaining numbers from $1,2,\dots,n$. Moreover, let a_{2k-1} be one of the numbers in (5) such that a_{2k} is not among the numbers in (5). It is evident that in this procedure exactly k of the numbers in (5) are different, so that for $k < n$ the kth step can be performed.

The solution to our problem is then provided by the following

Proposition

1. *J is an mkč.*
2. *There is no other mkč.*
3. *J consists of exactly $n-1$ pairs.*

[Summary of Proof: The proof is by induction on n. Jarník defines $J_2 \equiv [a_1, a_2]$ by the First Lemma. Given a connected set J_k with k indices Jarník uses the Second Lemma to define J_{k+1}. He proves carefully that J_{k+1} is connected. He then puts $J = J_n$.]

Remark:
The following is a visual interpretation of the solved problem:
We are given n balls numbered $1, 2, \ldots, n$ *which are joined pairwise by* $\frac{1}{2}n(n-1)$ *sticks. Let* $r_{a,b}$ *be the mass of the stick joining balls a and b. Let the sticks be bent if necessary so that they do not touch. From this system we want to remove some of the sticks so that the n balls hold together and the mass of the remaining sticks is as small as possible.*

In Prague, Feb. 12, 1929.
[It is interesting to note how tempting it was for both Borůvka and Jarník to formulate an application of the problem. Borůvka was led to the problem by his friends from the Electric Power Company of Western Moravia in Brno, cf. [5], and indeed published a note in an electrotechnical journal [4]. Jarník added a geometric interpretation — in $\mathbb{R}_3$.]

2. Jarník's paper in a historical perspective

A noncombinatorialist may wonder why we have discussed Jarník's paper [1] in such detail, and why it is worth translating. The reason is very simple as the following problem is perhaps the central problem of combinatorial optimization and a cradle of many key notions:

Minimal spanning tree (MST). Given a set V and a weight function $w\colon \binom{V}{2} \to \mathbb{R}$, find a tree (V, E) such that $\sum_{e \in E} w(e)$ is minimal.

MST was first solved by Borůvka [3] and [4]. Jarník quickly realized the novelty of this problem and immediately contributed his elegant solution [1]. Borůvka never returned to this problem although he lectured about his solution in Paris [5]. Other early contributions were illustrious too: by G. Choquet [7], by K. Florek, J. Lukasiewicz, J. Perkal, H. Steinhaus, S. Zubrzycki [9]. And after 1955 progress has been very fast and a number of general procedures and special algorithms were formulated. A rich spectrum of these results and a history of the problem is described in [20], [26] and [27]. Let us just note that O. Borůvka is quoted by both standard early references: J. Kruskal [23] and R. C. Prim [29]. Vojtěch Jarník's article only began to be quoted later, see e.g. K. Čulík, V. Doležal, M. Fiedler [17], despite the fact that his treatment was very precise (like all his mathematical work) and modern. This should be clear from the above translation. His algorithm is identical with Prim's algorithm [29] and his argument is a standard proving argument even now after 65 years. Perhaps it is time to do justice to this elegant procedure and call it the Jarník–Prim algorithm. Jarník returned to this topic only once more in his second paper [2], which we will discuss below. We believe that the geometrical interpretation given in the final lines of [1] provided his definitely nonplanar motivation for [2].

3. On minimal graphs containing *n* given points

We proceed as in Section 1: First we provide a translation of the key parts of the Jarník–Kössler paper [2]. We have decided (mainly because of space limitations) to translate only the first two sections of this paper. They are devoted to general properties of "Steiner trees". It appears that virtually all general properties of Steiner trees have already been explicitly stated in [2]. Even today they are attributed to others and even today one can find in [2] arguments superior to those in common use (such as the local planarity of k-dimensional Steiner trees; cf. Theorem 3(c) of [2] and p. 77 of [21]). We hope to return to this paper in the near future and give a critical version of the whole paper [2]. Below we give a brief discussion of its context and later development. Let us note that what follows may be the first translation of the essential parts of [2]. Such a translation is badly needed. Even the recent papers and books (such as [21]) are not aware of what a rich source of ideas is provided by [2]. Some of the main misquotations will be discussed below.

[2] is a paper with 13 pages, numbered 223–235. We include a translation of p. 223–229. The first and last pages are reproduced in Figs. 3 and 4.

On minimal graphs containing n given points
Vojtěch Jarník and Miloš Kössler
(*received Feb.* 10, 1934)

In this paper we consider the following problem: given n points $C_1, C_2, \ldots, C_n$, we want to find a connected set consisting of finitely many segments, which contains the points $C_1, C_2, \ldots, C_n$, so that 'the total length' of this set is the least possible (of course for $n=2$ such a "shortest connection" is a line segment joining points C_1 and C_2). In Section 2 *we prove the existence of such a 'minimal graph', and in* Section 3 *we consider the case when the points $C_1, C_2, \ldots, C_n$ form the vertices of a regular n-gon.*

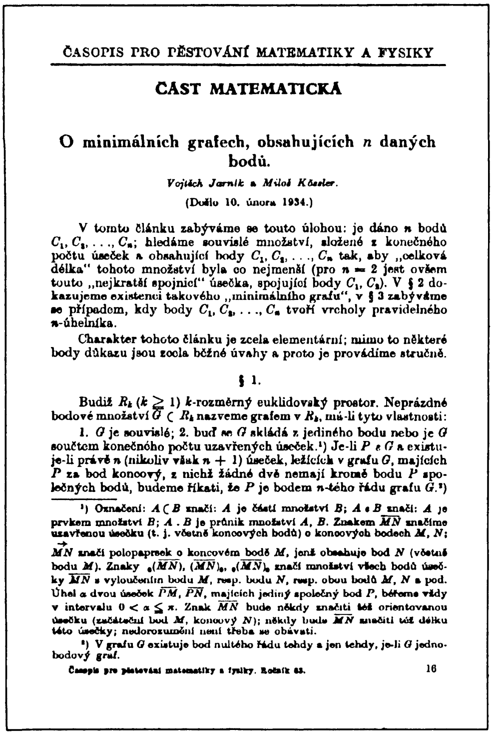
The nature of this article is completely elementary. Also some of the steps in the proof are routinely known and thus we are brief there.

[The reader should bear in mind that this paper was published before e.g. König's book [11] and no references are given.]

1.

Let R_k ($k \geqslant 1$) be the k-dimensional Euclidean space.

[So already this first line contradicts the common belief that, while Jarník–Kössler pioneered the Euclidean Steiner problem for the plane, the k-dimensional case was



ČASOPIS PRO PĚSTOVÁNÍ MATEMATIKY A FYSIKY

ČÁST MATEMATICKÁ

O minimálních grafech, obsahujících n daných bodů.

Vojtěch Jarník a Miloš Kössler.

(Došlo 10. února 1934.)

V tomto článku zabýváme se touto úlohou: je dáno n bodů $C_1, C_2, \ldots, C_n$; hledáme souvislé množství, složené z konečného počtu úseček a obsahující body $C_1, C_2, \ldots, C_n$ tak, aby „celková délka" tohoto množství byla co nejmenší (pro $n = 2$ jest ovšem touto „nejkratší spojnicí" úsečka, spojující body C_1, C_2). V § 2 dokazujeme existenci takového „minimálního grafu", v § 3 zabýváme se případem, kdy body $C_1, C_2, \ldots, C_n$ tvoří vrcholy pravidelného n-úhelníka.

Charakter tohoto článku je zcela elementární; mimo to některé body důkazu jsou zcela běžné úvahy a proto je provádíme stručně.

§ 1.

Budiž R_k ($k \geq 1$) k-rozměrný euklidovský prostor. Neprázdné bodové množství $G \subset R_k$ nazveme grafem v R_k, má-li tyto vlastnosti:

1. G je souvislé; 2. buď se G skládá z jediného bodu nebo je G součtem konečného počtu uzavřených úseček.[1]) Je-li $P \in G$ a existuje-li právě n (nikoliv však $n + 1$) úseček, ležících v grafu G, majících P za bod koncový, z nichž žádné dvě nemají kromě bodu P společných bodů, budeme říkati, že P je bodem n-tého řádu grafu G.[2])

[1]) Označení: $A \subset B$ značí: A je částí množství B; $A \in B$ značí: A je prvkem množství B; $A \,.\, B$ je průnik množství A, B. Znakem $\overline{MN}$ značíme uzavřenou úsečku (t. j. včetně koncových bodů) o koncových bodech M, N; $\overrightarrow{MN}$ značí polopaprsek o koncovém bodě M, jenž obsahuje bod N (včetně bodu M). Znaky ${}_0(\overline{MN})$, $(\overline{MN})_0$, ${}_0(\overline{MN})_0$ značí množství všech bodů úsečky $\overline{MN}$ s vyloučením bodu M, resp. bodu N, resp. obou bodů M, N a pod. Úhel α dvou úseček $\overline{PM}$, $\overline{PN}$, majících jediný společný bod P, béřeme vždy v intervalu $0 < \alpha \leq \pi$. Znak $\overline{MN}$ bude někdy značiti též orientovanou úsečku (začáteční bod M, koncový N); někdy bude $\overline{MN}$ značiti též délku této úsečky; nedorozumění není třeba se obávati.

[2]) V grafu G existuje bod nultého řádu tehdy a jen tehdy, je-li G jednobodový graf.

Časopis pro pěstování matematiky a fysiky. Ročník 63. 16

Fig. 3.

considered only by Gilbert and Pollack in [19]. In fact the whole paper [2] is written for k dimensions.]

A nonempty point set $G \leqslant R_k$ *is called a graph in* R_k *if it has the following properties*:

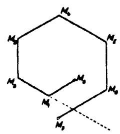
1. *G is connected,*
2. *either G contains one point only or G is a sum of finitely many closed segments.*

[From now on we use the word union instead of sum. Now follows a footnote where Jarník in his characteristic style clearly defines all used symbols starting with $A \in B$ and ending with ${}_0(\overline{MN})$, $(\overline{MN})_0$, ${}_0(\overline{MN})_0$ for half-open and open line segments; $\overline{MN}$ denotes a line segment, an oriented line segment or the length of this segment; '*one does not have to be afraid of a misunderstanding*'.]

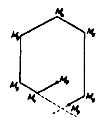
If $P \in G$ *and there exist exactly* n (*and not* $n+1$) *segments of* G *for which* P *is an end-vertex and which do not have common points except for* P, *then we say that* P *is a point of nth order* [or degree] *of* G. *The points of order one are called endpoints, points of higher order are called branching points* (*in every graph there are finitely many of both types of points*). *If* P *is a point of nth order in* G, *then we put* $V(P) = n - 2$, *and we further put* $V(G) = \sum V(P)$. $V(P)$ *is called the weight of point* P.

235

však bod M_7 zřejmě neleží v $\Lambda(\Gamma'')$, jest $\Lambda(\Gamma'') \neq \Lambda(\Gamma)$, jak bylo dokázati. Případ B) je symetrický s případem A).



Obr. 8.



Obr. 9.

Z tvrzení 9 plyne ihned tvrzení 8: Existuje-li typický graf Γ_1, plyne z tvrzení 9 existence posloupnosti typických grafů Γ_1, Γ_2, Γ_3, ... takové, že pro $i < l$ jest $\Lambda(\Gamma_l)$ pravou částí bodového množství $\Lambda(\Gamma_i)$; tedy $\Gamma_i \neq \Gamma_l$ pro $i < l$, jak bylo dokázati.

•

Sur les graphes minima, contenant n points donnés.

(Extrait de l'article précédent.)

Soient $C_1, C_2, \ldots, C_n$ n points d'un espace euclidien. Considérons tous les ensembles connexes G, satisfaisant aux conditions suivantes: 1. G contient les points $C_1, C_2, \ldots, C_n$. 2. G est la somme d'un nombre fini de segments tels que deux quelconques entre eux n'aient qu'un point commun tout au plus. Soit $l(G)$ la somme des longueurs de ces segments. Dans cet article, on démontre l'existence d'un G_0, pour lequel $l(G_0)$ atteint la valeur minimum; ensuite, on démontre quelques propriétés de l'ensemble G_0 et on détermine G_0 complètement dans le cas particulier où les points $C_1, C_2, \ldots, C_n$ sont les sommets d'un polygone régulier ($n \geqq 13$.)

Fig. 4.

A cycle is a graph which is a closed, simple, continuous curve. A graph, no part of which is a cycle, is called a tree. Now the following well-known theorem holds:

Theorem 1: *If G is a tree, then $V(G) = -2$.*

[A note is added, stating that any tree with at least 2 points has at least 2 end-vertices. A typical proof by induction on the number of vertices is given. The authors take care in defining vertices of G.]

2.

Let n $(n \geqslant 2)$ points $C_1, C_2, \ldots, C_n$ in the space $\mathbb{R}_k$ $(k \geqslant 1)$ be given. These points are called basic points. Let G be a graph in $\mathbb{R}_k$ containing points $C_1, C_2, \ldots, C_n$.

[Recall that a graph is defined as a topological realization of a graph in the usual sense and that it is always connected.]

By a vertex of graph G we shall understand:

1. *basic points*
2. *all points of G of order > 2*
3. *all points of G of order 2 in which two noncollinear line segments meet.*

A segment $\overline{MN} \subset G$ *is called a 'side of graph G'* [i.e. an edge] if ${}_0\overline{MN}_0$ *does not contain a vertex and both M and N are vertices. The graph G is then a union of its sides. Obviously there are only finitely many vertices and sides in a graph; if two sides have a common point, then this point is endpoint of both sides. The sum of all side-lengths is called the length of G and denoted by* $l(G)$.

Let $\mathscr{M}$ *denote the set of all graphs in* $\mathbb{R}_k$ *containing* $C_1,\ldots,C_n$. *In what follows let us fix a lower bound d for all graph lengths in* $\mathscr{M}$. *If* $l(G)=d$, *then G is called a 'minimal graph in* $\mathbb{R}_k$ *with respect to the points* $C_1,\ldots,C_n$'. *First we prove*

<u>*Theorem 2:*</u> *Let* $C_1, C_2, \ldots, C_n$ *be points of* $\mathbb{R}_k$ $(k \geqslant 1, n \geqslant 2)$. *Then there exists at least one minimal graph in* $\mathbb{R}_k$ *with respect to the points* $C_1, C_2, \ldots, C_n$.

We first introduce some notation. Let $G \in \mathscr{M}$. *A free end of G is an endpoint of G which is not a basic point. A free corner of G is a vertex of order* 2 *which is not a basic point. Let* $\mathscr{N}$ *be the set of all* $G \in \mathscr{M}$ *which are trees and which have no free ends. Let* $\mathscr{P}$ *be the set of all* $G \in \mathscr{N}$ *which have no free corners. First we prove the following statements:*

<u>*Proposition 1:*</u> *Let* $G \in \mathscr{M} - \mathscr{N}$. *Then there exists* $G_1 \in \mathscr{N}$ *such that* $l(G_1) < l(G)$.

<u>*Proposition 2:*</u> *Let* $k \geqslant 3$ *and* $G \in \mathscr{N} - \mathscr{P}$. *Then there exists* $G_1 \in \mathscr{P}$ *such that* $l(G_1) < l(G)$.

<u>*Proposition 3:*</u> *Let* d_1 *be a lower bound for all lengths of graphs* $G \in \mathscr{P}$. *Then there exists at least one graph* $G_\circ \in \mathscr{M}$ *with* $l(G_\circ) \leqslant d_1$.

<u>*Proposition 4:*</u> *If G is a minimal graph in* $\mathbb{R}_k$ *with respect to the points* $C_1, C_2, \ldots, C_n$, *and if K is the smallest convex set in* $\mathbb{R}_k$ *containing* $C_1, C_2, \ldots, C_n$, *then* $G \subset K$ [i.e. the convex hull contains all the Steiner points].

Theorem 2 *follows from Propositions* 1 – 4 *as follows:*

A) *If* $k \geqslant 3$, *then Propositions* 1 *and* 2 *yield* $d_1 = d$ *and Theorem* 2 *follows from Proposition* 3.
B) *If* $k \leqslant 2$, *then we embed* $\mathbb{R}_k$ *in* $\mathbb{R}_3$. *From* A) *we get a minimal graph G in* $\mathbb{R}_3$ *with respect to the points* $C_1, C_2, \ldots, C_n$. *But Proposition* 4 *implies* $G \subset \mathbb{R}_k$.

Thus it suffices to prove Propositions 1 – 4.

[Note again that for Jarník the k-dimensional case is essential.]

[Proof of Proposition 1 is by deleting endpoints together with the corresponding sides. The proofs of the remaining Propositions are elegant and more interesting, and we outline the Jarník–Kössler arguments in a greater detail.]

<u>*Proof of Proposition 2:*</u> *Let* $k \geqslant 3$ *and* $G \in \mathscr{N} - \mathscr{P}$, *i.e.* $G \in \mathscr{M}$ *is a tree without free ends containing at least one free corner* M_1 *in which two non-collinear sides* $\overline{M_1M_2}$ *and* $\overline{M_1M_3}$ *meet.* M_1 *is not a basic point. We prove: there exists a graph* $G' \in \mathscr{N}$ *with less free corners satisfying* $l(G') < l(G)$.

[It now follows that by repeating this argument one obtains Proposition 2.]

We shall distinguish two cases:

Case 1: *Both* M_2 *and* M_3 *are basic points. Then the set* $G - [_0(\overline{M_2M_1}) + \overline{(M_1M_3})_0]$ *is the union of two disjoint trees* $G_2, G_3, M_2 \in G_2$, $M_3 \in G_3$. *The segment* $\overline{M_2M_3}$ *contains at least one point of* G_2 (*say* M_2) *and at least one point of* G_3 (*say* M_3). *Thus let* P_2, P_3 *be points of the segment* $\overline{M_2M_3}$ *such that* $P_2 \in G_2, P_3 \in G_3$ *and no point of the segment* $_0(\overline{P_2P_3})_0$ *belongs to either* G_2 or G_3. *Then the graph* $G' = \{G - [_0((\overline{M_2M_1}) + ((\overline{M_1M_3})_0]\} + \overline{P_2P_3}$ *is in* $\mathcal{N}$ *and has less free corners than* G.

[This is justified in detail.]
Obviously $l(G') < l(G)$.

Case 2: *One of the points* M_2, M_3 — *say* M_2 — *is not a basic point. Let* S *be a* [$(k-1)$- *dimensional*] *hyperplane containing* M_2 *but not* M_3. *If* M_2' *is any point of* S, *then we denote by* $G(M_2')$ *the graph obtained from* G *by replacing all sides* $\overline{M_iM_2}$ *of* G *by segments* $\overline{M_iM_2'}$. *Put* $\overline{M_2M_1} + \overline{M_1M_3} - \overline{M_2M_3} = a > 0$. *It is clear that there exists* $\delta > 0$ *such that every graph* $G(M_2')$ *for which* $\overline{M_2M_2'} < \delta$ *satisfies*:

1. $l(G(M_2')) < l(G) + \frac{1}{2}a$, $\overline{M_2'M_1} + \overline{M_1M_3} - \overline{M_2'M_3} > \frac{1}{2}a$,
2. *the graph* $G(M_2')$ *has the same vertices* (*of the same order*) *and the same sides as* G *with the exception that instead of the vertex* M_2 *and sides* $\overline{M_2M_i}$ *we have* M_2' *and* $\overline{M_2'M_i}$.

[This may be seen as follows:]
Let us consider all lines through M_3 *and some other point of* G. *These lines intersect* S *in a set* $\sum$ *which consists of finitely many points, segments and half-lines. As* $k \geqslant 3$ [*and thus* S *is at least* 2-*dimensional*] *there exists at least one* $M_2' \in S - \sum$ *such that* $\overline{M_2M_2'} < \delta$. *This graph then has properties* 1. *and* 2. *Moreover, the graph* $G(M_2')$ *has the following property*: *no point of* $G(M_2')$ *belongs to the segment* $_0(\overline{M_2'M_3})_0$.

[This is justified in a detailed footnote.]
Now define the graph $G' = \{G(M_2') - [\overline{M_2'M_1} + \overline{M_1M_3}]\} + \overline{M_2'M_3}$. *Clearly* $G' \in \mathcal{N}$, G' *has less free corners than* G, *and finally from Condition* 1 *it follows that* $l(G') < l(G)$.

Proof of Proposition 3: *This is a routine limit argument. Let* $G_1, G_2, \ldots$ *be a sequence of graphs from* $\mathcal{P}$ *and let* $\lim_{r-\infty} l(G_r) = d_1$.

[We preserve as before all the notation of [2]].

As $C_1 \in G_r$, *all graphs* G_r *lie in a closed ball with centre* C_1 *and diameter equal to the upper bound of the numbers* $l(G_r)$ ($r = 1, 2, \ldots$). *All vertices of the graph* G_r *are basic or branching points. By Theorem* 1 *it follows that* $V(G_r) = -2$. *As all the endpoints* (*with weight* -1) *are basic points, we have at most* n *of them. Thus the number of branching points* (*with weight at least* 1) *is at most* $n-2$ *and the graph* G_r *has at most* $2n-2$ *points. Hence there exists a subsequence* $G_1', G_2', \ldots$ *of* $G_1, G_2, \ldots$ *such that all* G_r' *have the same number of vertices. We denote the vertices of* G_r' *by*

$X_1^r, X_2^r, \ldots, X_z^r$ *such that* $X_i^r = C_i$ *for* $1 \leqslant i \leqslant n$. *For graph* G_r' *define the matrix*

$$\begin{pmatrix} 0 & a_{12}^r & a_{13}^r & \cdots & a_{1z}^r \\ a_{21}^r & 0 & a_{23}^r & \cdots & a_{2z}^r \\ a_{31}^r & a_{32}^r & 0 & \cdots & a_{3z}^r \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ a_{z1}^r & a_{z2}^r & a_{z3}^r & \cdots & 0 \end{pmatrix}$$

where $a_{il}^r = 1$ *or* 0 *according to whether or not* $\overline{X_i^r X_l^r}$ *is a side of the graph* G_r'.

[So this is the adjacency matrix of G_r'.]

As there are only finitely many such matrices, there is a subsequence $G_{s_1}', G_{s_2}', \ldots$ *such that the same matrix*

$$\begin{pmatrix} 0 & a_{12} & a_{13} & \cdots & a_{1z} \\ a_{21} & 0 & a_{23} & \cdots & a_{2z} \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ a_{z1} & a_{z2} & a_{z3} & \cdots & 0 \end{pmatrix}$$

corresponds to every graph of the subsequence. Finally, as the sequences $X_i^1, X_i^2, X_i^3, \ldots$ $(i = 1, 2, \ldots, z)$ *are bounded, we can find a subsequence* $G_{t_1}', G_{t_2}', \ldots$ *such that all the limits* $\lim_{p=\infty} X_i^{tp} = X_i$ $(i = 1, 2, \ldots, z)$ *exist. Let* G_0 *denote the union of segments* $\overline{X_i X_l}$ $(1 \leqslant i < l \leqslant 2)$ *for which* $a_{il} = 1$.

[Footnote: *Of course some of these segments may degenerate to points.*]

Obviously $G_\circ \in \mathscr{M}$ *and the following holds*:

$$l(G_{t_p}') = \sum_{1 \leqslant i < l \leqslant z} a_{il} \overline{X_i^{t_p} X_l^{t_p}},$$

$$l(G_0) \leqslant \sum_{1 \leqslant i < l \leqslant z} a_{il} \overline{X_i X_l} = \lim_{p=\infty} l(G_{t_p}') = d_1.$$

This completes the proof.

[This is a word-for-word, symbol-preserving translation. And even today the most elegant argument!]

Proof of Proposition 4: Let $G \in \mathscr{M}$ *be a graph which violates* $G \subset K$. *Then there exists a hyperplane* S [$(k-1)$-*dimensional*] *such that all basic points lie on one side of* S *and a nonempty subset* G' *of* G *lies on the other side of* S. *Define a graph* G_1 *by replacing the subset* G' *by an orthogonal projection of* G' *onto the hyperplane* S. *Obviously* $G_1 \in \mathscr{M}$ *and* $l(G_1) < l(G)$, *which completes the proof.*

[k dimensions are essential again.]

Now we can easily prove Theorem 3 which describes the structure of minimal graphs in greater detail.

Theorem 3: Let G be a minimal graph in $\mathbb{R}_k$ $(k \geqslant 1)$ *with respect to points* $C_1, C_2, \ldots,$ C_n $(n \geqslant 2)$. *Then G has the following properties*:

a) *G is a subset of the smallest convex set containing* $C_1, C_2, \ldots, C_n$.

b) *G is a tree without free ends and free corners.*

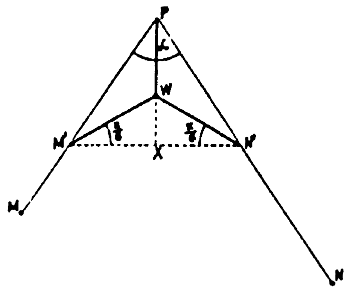


Fig. 5.

c) *If two sides of G have a common point, then their angle is at least* $\frac{2}{3}\pi$.
d) *Every branching point of G has degree* 3. *The three sides of the graph incident to a branching point lie in a* (*2-dimensional*) *plane and any two have angle* $\frac{2}{3}\pi$.

[Here as elsewhere k dimensions are essential. We have not found d) in later literature. This yields a better and stronger argument than e.g. in [21] p. 77.]

Proof of Theorem 3: Property a) *follows from Proposition* 4. *To prove* b) *we can assume* (*by* a)) *that* $k \geqslant 3$ (*if* $k < 3$ *then we can embed* $\mathbb{R}_k$ *into* $\mathbb{R}_3$). *Then* b) *follows from Propositions* 1 *and* 2. *Property* c) *we prove as follows*: *let* $G \in \mathcal{M}$ *and let* $\overline{PM}$, $\overline{PN}$ *be two sides of G with angle* $\alpha < \frac{2}{3}\pi$. *We construct a point* M' *in the interior of side* $\overline{PM}$ *and a point* N' *in the interior of side* $\overline{PN}$ *such that* $\overline{PM'} = \overline{PN} = h$. *Then we have* (*see Fig.* 5)

$$\overline{M'W} = \overline{N'W} = \overline{M'W}\frac{2}{\sqrt{3}} = \frac{2}{\sqrt{3}}h \sin \tfrac{1}{2}\alpha$$

$$\overline{PW} = \overline{PX} - \overline{WX} = h \cos \tfrac{1}{2}\alpha - \frac{1}{\sqrt{3}}h \sin \tfrac{1}{2}\alpha$$

and thus

$$\overline{M'W} + \overline{N'W} + \overline{PW} = h(\sqrt{3}\sin \tfrac{1}{2}\alpha + \cos \tfrac{1}{2}\alpha) < 2h = \overline{PM'} + \overline{PN'}.$$

[This step is justified in a detailed and characteristic footnote: *We have* $(\mathrm{d}/\mathrm{d}x)(\sqrt{3}\sin x + \cos x) = \sqrt{3}\cos x - \sin x = \cos x(\sqrt{3} - \tan x) > 0$ *for* $0 < x < \frac{1}{3}\pi$ *and thus* $\sqrt{3}\sin x + \cos x$ *is an increasing function for* $0 \leqslant x \leqslant \frac{1}{3}\pi$, *hence we have for* $0 < x < \frac{1}{3}\pi$ (Fig. 5):
$\sqrt{3}\sin x + \cos x < \sqrt{3}\sin\frac{1}{3}\pi + \cos\frac{1}{3}\pi = 2.$]

Define graph $G_1 = [G - (\overline{M'P} + \overline{N'P})] + \overline{M'W} + \overline{N'W} + \overline{PW}$. *Obviously* $G_1 \in \mathcal{M}$, $l(G_1) < l(G)$ *and thus G is not a minimal graph.*

Property d) *follows immediately from c*): *three line segments incident in a point and not lying in a plane form angles whose sum is less than* 2π.

Remark: From Theorem 3 *we obtain the following for the minimal graph G*: *if P is a branching point, then* $V(P) = 1$, *whereas* $V(P) = -1$ *for every endpoint P*.

From $V(P) = -2$ it follows that the number of branching points equals the number of endpoints -2.

This is the end of the first two sections of the Jarník–Kössler paper. This is a remarkable text in both its clarity and contents. This part deals with general properties of Steiner trees, and these properties are generally attributed to later contributors although they are explicitly stated in the Jarník–Kössler paper. Here is a sample of such instances, mostly taken from a recent monograph [21] devoted to 'the Steiner Tree Problem'.

The fact that for a Steiner tree all branching points are of degree 3, as well as the angle condition, the number of branching points, the convex hull result (i.e. Theorem 1.1, Theorem 1.2 of [21]) are attributed to Courant and Robbins [8], Corollary 1.1, Corollary 1.5 of [21] are attributed to Gilbert and Pollak [19]. These results are all explicitly contained in [2] as various parts of Propositions 1–4 and Theorems 2–3.

Moreover, the generalization to k dimensions treated in [21], Section 6.1 is not only mentioned but instrumental in [2]. In fact the whole paper is written in k dimensions. And the complicated argument of [21], p. 77 is replaced by the pleasant Jarník–Kössler argument that three sides incident with a branching point are coplanar.

Even after all these years the Jarník–Kössler paper in its general part (i.e. Sections 1 and 2) is an example of clear style and elegance, and it is worth studying even today. The clarity of the introduction to the problem is not shared by many later texts.

No wonder, the 'Steiner problem' is due to Jarník and Kössler and was elaborated by them to a degree surpassed only 30 years later.

The Jarník–Kössler paper [2] continues with the treatment of regular n-gons. They solve the cases $n = 3, 4, 5$ explicitly and carefully with all details (without referring to any earlier work for $n = 3$) and remark that for $n = 6$ the situation is entirely different: the solution is given by 5 sides of a regular hexagon. By an elegant argument they solve the case of all regular n-gons for all $n \geqslant 13$. They leave open cases $7 \leqslant n \leqslant 12$ and remark that this is a finite problem *which could be directly solved with a certain amount of effort.* Indeed, their method of solution for $n = 3, 4, 5$ suggests that they were aware of the finiteness of the problem (proved much later by Melzak [25]).

4. Jarník–Kössler's paper in a historical perspective

The problem of finding a shortest connection between n given points in the plane has a long history. Indeed, it is one of the oldest optimization problems and it was, and is, frequently used as an example of maximality (and minimality) arguments. However, for most of the time in the long history of the problem, only the case $n=3$ was considered. This goes back to a question posed by Fermat, was considered by Mersenne and solved by Torricelli and Cavalieri. The elegant solution of this problem of elementary geometry of course attracted many researchers such as Simpson and Steiner who also considered a generalization of the 3-point problem in a different direction: given n points in the plane, find a *single* vertex with the smallest sum of distances.

The history is involved and there are several sources available, such as [24] and [14], and also early industrial applications such as the book [13] and the thorough mathematical treatment in [12].

K.F. Gauss came close to Steiner tree problem when he modified a question posed to him by H.C. Schumacher and wrote [10]:

'If one considers a version of rectangle problem where one speaks about shortest connecting system then one has to consider more individual cases and one gets an interesting mathematical problem. This problem is close to my interests as I had several times an opportunity to consider it in connection with the railroad connecting Hamburg, Bremen, Hanover and Brauschweig. I got an idea that this could be a nice problem for our students.' So Gauss had 4-point problem clearly in mind.

Gauss continues by drawing four possibilities for Steiner trees on 4 points (there are four possibilities in his handwriting and only three in the printed version [10]: one of the possibilities seems to be not clearly relevant and two possibilities are in fact rotations of each other). Gauss closes by saying that he has no more time that day. He does not seem to return to this later in his correspondence.

(We thank R. L. Graham and H. Harborth who informed me about the Gauss contribution.)

However, prior to 1934 the problem of the shortest connection of n points was not considered. It was first considered by Jarník and Kössler [2], with a clarity and rigour which we hope is clear from the translation of the first two sections.

It is difficult to speculate why the authors considered this problem. In Jarník's œuvre the papers [1] and [2] present the only singularity. As a possible solution to this puzzle one could perhaps stress the fact that Jarník instantly recognized the novelty of Borůvka's problem and saw it as an n-point minimization problem. His interpretation of the minimal spanning tree problem given at the end of [1] (Section 1 of this paper contains a translation of this) may suggest how naturally he may have arrived at the problem considered in [2]. That could also suggest why Jarník considered *essentially* the k-dimensional problem. He did not arrive at it from the geometry of the plane but from spatial geometry (see again the Remark at the end of [1], translated in Section 1).

Like Borůvka, Jarník never returned to this problem again.

The 3-point problem was a classical optimization problem and it found its way into the Courant–Robbins book [8] where the problem for $n = 3$ (i.e. the Fermat–Torricelli–Cavalieri–Simpson–Steiner problem) is called the Steiner problem and the problem of the nearest point to a given set of points (i.e. the problem considered by Steiner) is called a 'mathematically sterile generalization'. The problem of the shortest interconnection between n points is called the generalized Steiner problem [8]. This is clearly Jarník's problem or the Jarník–Kössler problem or Gauss-Jarník problem.

These attributions (and some stylistic expressions) suggest that Courant and Robbins were motivated by [12,14]. (Neither Gauss nor Jarník and Kessler are mentioned in [8].)

In the thirties Jarník was an internationally famous mathematician (a speaker at both the Zürich 1932 and the Oslo 1936 Congress of the International Mathematical Union) and thus the main reason for the omission probably was that Courant and Robbins did not know about his work outside number theory and analysis. The 'Steiner' problem was then dormant for another 30 years until it was revived by Melzak [25], Gilbert and Pollack [19] and others with the vigour and confidence of the newly developing fields of combinatorial (discrete) optimization and the theory of algorithms, see [16]. The problem is hard both theoretically [18] and practically, and for its direct applications in VLSI [22] and other fields (see, e.g. [21]) it is still intensively studied. (The euclidean problem however may be approximated by a recent result of Arora [15].) The problem is far from being solved.

Summarizing, let us just say that with these combinatorial papers [1,2] Jarník was very lucky. Single handedly (with the help of Borůvka and Kössler) he started important branches of fields which in his time were not born yet. The style and rigour of his contributions have lasting value. Jarník's contribution is widely unrecognized (e.g. neither the recent Handbook of Combinatorics nor the Handbook of Computational Geometry mention him).

It is not a marginal contribution by a passerby. It is rather an important contribution by a major mathematician. Combinatorics was gaining strength while slowly emerging from the 'slums of topology', through the expertise and brilliance of mathematicians from other fields. From number theory these were Erdös and Turán and Jarník.

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