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Geometric elements in the theory of transformations of ordinary second-order linear differential equations

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In particular, a decomposition theorem which gives the conditions for the state-determined, causal, representation of a system is presented, with its application in dynamical systems theory and automata theory.

(9) Mathematical theory of multi-level systems M.D. Mesarović

A control (or decision-making) system which consists of a family of hierarchically arranged control subsystems is termed a multi-level system. It has been shown that many complex industrial, biological and organizational systems can be modelled mathematically in such a way.

A central problem in the theory of multi-level systems is the interrelationship between levels, and in particular exchange of information and decisions. To develop a mathematical theory of interlevel relationship the simplest type of multi-level system, namely a two-level system with a single unit on the first level, has been considered; the decision problem of the second level unit in such a system is termed co-ordination.

A mathematical theory of co-ordination for systems described in Banach spaces is presented. Theorems giving necessary and sufficient conditions for optimal co-ordination of abstract dynamical systems are presented and their application to systems described by differential equations is discussed.

(10) Geometric elements in the theory of transformations of ordinary second-order linear differential equations

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The basic problem in the theory of transformations of second-order linear differential equations formulated by Kummer in 1834 is the following:

Given two such equations (1) $y'' = q(t)y$, (2) $Y'' = Q(T)Y$

defined on some intervals of the real line, how can one find functions $w(t)$, $X(t)$ (with $w(t) \not\equiv 0$, $X'(t) \not\equiv 0$) such that $y(t) = w(t).Y[X(t)]$ is a solution of (1) whenever $Y(T)$ is a solution of (2)?

In fact, it turns out that (under some supplementary conditions) $w,X$ satisfy the above if, and only if, $w \equiv C/\sqrt{X'}$ ($C$ = constant) and $X$ is a solution of the equation

$$- \{X,t\} + Q(X)X'^2 = q(t)$$

where $\{X,t\}$ denotes the Schwarzian derivative of $X$ at $t$.

Some important notions which occur in the theory of transformations are the following. Let $u,v$ be two linearly independent solutions of (1); then the first and second phases with respect to the basis $(u,v)$ are any real functions $\alpha(t)$, $\beta(t)$ continuous where (1) is defined and satisfying (for $v,v' \neq 0$) $\tan \alpha = u/v$, $\tan \beta = u'/v'$ respectively. The function $\theta = \beta - \alpha$ is called a polar function of the basis $(u,v)$.

Let $q(t) < 0$ for all $t$. Take an arbitrary $t$ and suppose $u(t) = 0$. Define $\phi_n(t) = n^{th}$ zero of $u$ following $t$, $\chi_n(t) = n^{th}$ zero of $u'$ following $t$. Now instead suppose $v'(t) = 0$, and let $\psi_n(t)$, $\omega_n(t)$ be the $n^{th}$ zeros of $v'$, $v$ (respectively) following $t$. Define also $\phi_{-n}(t)$ etc. by replacing 'following' by 'preceding'. Then the $\phi_\nu$ ($\nu = \pm 1, \pm 2, \ldots$) etc. are called the central dispersions with index $\nu$, and do not depend on the choices of $u$ and $v$. When $\nu = +1$ they are called basic central dispersions. There exist numerous formulae which relate all these quantities, and give an effective analytic apparatus for solving various problems in the theory of differential equations.

An equation (3) $Y''+AY'+BY = 0$ defines (up to non-singular homogeneous linear transformation) a planar curve, by the locus of $(U(t), V(t))$ where $U,V$ are two independent integrals. Conversely, if $U,V$ are $C^2$ functions with $UV'-VU' \neq 0$ then there is an equation (3) with $U,V$ as independent integrals. The curve $\mathcal{C}$ given by $(U,V)$ can be reparametized so that (3) takes the form (1), and assuming $q < 0$ the central dispersions can be interpreted geometrically: the points on $\mathcal{C}$ given by $t$ and $\phi_\nu(t)$ lie on the same straight line through the origin, the tangents at points given by $t$ and $\psi_\nu(t)$ are parallel, and so on.

The above can be used to attack problems of the following type:

(a) Which differential equations satisfy $\phi_1(t) = \psi_1(t)$?

(b) Which satisfy $\phi_1(t) = \Phi(t)$, for given $\Phi$?

Problem (a) has been thoroughly studied, and it is known that $\phi_1(t)$ must be of the form $ct+k$ $(c>0)$, and $\mathcal{C}$ has polar equation of the form $r = C^\alpha F(\alpha)$ where $C > 0$ and $F(> 0)$ is a $C^2$ periodic function satisfying a certain differential inequality. For (b) it is known, for instance, that if $\Phi$ is $C^3$ with $\Phi(t) > t$, $\Phi'>0$ and $\Phi \to \pm\infty$ as $t \to \pm\infty$ (resp.) then there exists an uncountable set of equations (1) with $\phi_1 = \Phi$. The corresponding functions $q$ can be given quite explicitly, especially in the case $\Phi(t) = t+k$.

More detailed information is available in [1]; further geometric applications of the theory of transformations have been made in [2].

REFERENCES:

[1] Borůvka, O. Lineare Differentialtransformationen 2. Ordnung, Deutscher Verlag der Wissenschaften, Berlin, 1967.

[2] Guggenheimer, H. (To appear in Archivum Mathematicum, (Brno)).