Matematika ve staré Indii

Summary

In: Irena Sýkorová (author): Matematika ve staré Indii. (English). Praha: Matfyzpress, Vydavatelství Matematicko-fyzikální fakulty Univerzity Karlovy v Praze, 2016. pp. 311–322.

Persistent URL: http://dml.cz/dmlcz/404230

SUMMARY

The main motive for writing this work was the fact that there is no comprehensive text written in Czech which deals with mathematics in ancient India. The aim of the monograph is to describe in detail the mathematical knowledge, computational procedures and arithmetic, algebraic and geometric methods which the ancient Indians knew and used. The text follows the development of Indian mathematics from the oldest mathematical knowledge contained in ancient Vedic texts to the knowledge originating from the classic medieval arithmetic and algebraic works.

The first chapter gives a short overview of Indian history. The oldest civilization of the Indian peninsula is briefly described in the second chapter. The ancient Indian geometrical knowledge is included in texts called *śulbasūtras* (the 1st millennium BC). These texts contain the most important rules used in the construction of sacrificial altars. Translations of the rules, mathematical analysis and commentary are the subject of the third chapter. The fourth chapter summarizes the mathematical knowledge from around the beginning of our era. A strong impulse for development of mathematics was the Jain cosmology, which used large numbers in calculations and motivated mathematicians to interesting considerations about infinity.

The classical era of medieval Indian mathematics was the period from the 5th century to the 14th century. The fifth chapter provides a chronological overview of the most significant scholars and their most important works. The sixth chapter analyses the expression of numbers and describes an important transformation of mathematical terminology. Detailed, commented description and interpretation of Indian basic arithmetic algorithms and methods, including calculations with fractions, are given in the seventh chapter. The eighth chapter deals with the medieval Indian algebra. In the field of algebra, Indian mathematicians probably achieved the greatest success. Noteworthy is the *kuṭṭaka* method, which was used to solve linear indeterminate equations with two unknowns, and the algorithm for solving so-called Pell's equation. The ninth chapter discusses the medieval Indian geometry.

Indian astronomical texts from the early years AD contain extensive trigonometric tables. However, Indian scholars considered trigonometry only as a special astronomical application of geometry and later mathematical texts do not contain it any more. Therefore, it is not included in this work.

The sources are mainly English translations of ancient Sanskrit texts and their commentaries, notably H. T. Colebrooke: *Algebra, with Arithmetic and Mensuration from the Sanscrit of Brahmegupta and Bhascara*, M. Rangacarya: *Ganita-sara-sangraha of Mahaviracarya with English Translation and Notes*, W. E. Clark: *The Āryabhaṭīya of Āryabhaṭa*, K. S. Shukla: *The Pāṭīgaṇita of Śrīdharācarya* and A. Bürk: *Das Āpastamba-Śulba-Sūtra* .

1 The Oldest Indian Mathematics

Mathematics related to Vedic Altars

The Vedic era (about 1500 – 500 BC) is the period in which the collections of sacred texts known as Vedas were created. Sacrificial rites were very important in those times. To make the sacrifice ritual successful the altar had to conform to very precise measurements. The *śulbasūtras* are appendices to the Vedas which give rules for the construction of altars. The most important *śulbasūtras* were written by Baudhāyana (about 800 BC), Āpastamba (about 600 BC), and Kātyāyana (about 200 BC). As the *śulbasūtras* deal with the science of geometry and its applications, the earliest Indian name for geometry was *śulba*.

The *śulbasūtras* contain:

1. construction of a line perpendicular to a given line,
2. description of constructions of geometrical shapes – triangles, squares, rectangles, isosceles trapeziums, circles,
3. an early form of the Pythagoras' theorem,
4. constructions of the same figures with double, triple or multiple areas,
5. constructions of a square equal to the sum or the difference of two unequal squares,
6. the solution of the problem of equivalence of area — squaring a circle and vice-versa, transformation of a rectangle into a square and vice-versa.

The *śulbasūtras* do not contain any proofs of the rules which they describe. Some of the rules, such as the method of constructing a square equal in area to a given rectangle, are exact. Others, such as constructing a square equal in area to a given circle, are only approximations.

However, the *śulbasūtras* contain also other problems. We can find fractions and calculations with them, surds and various expressions with them. A remarkable result of the mathematics of the *śulbasūtras* is a very close approximation of $\sqrt{2}$. We can express it by the formula

$$\sqrt{2} = 1 + \frac{1}{3} + \frac{1}{3 \cdot 4} - \frac{1}{3 \cdot 4 \cdot 34} \approx 1.414215686\ldots.$$

Mathematics in Jaina texts

The Jaina cosmological ideas influenced mathematics in many ways. The Jainas were fascinated with large numbers; their cosmology contained a time period of 2^{588} years. All numbers were divided into three classes – enumerable, innumerable and infinite. Moreover, Jaina mathematics distinguished five different types of infinity – infinite in one direction, infinite in two directions, infinite in area, infinite everywhere, and perpetually infinite.

The work *Sthānāṅga-sūtra* (3rd or 2nd century BC) contains ten topics of Jaina mathematics:

1. four operations of arithmetic – *parikarman*,
2. applications of the basic operations – *vyavahāra*,
3. geometry – *rajju*,
4. mensuration of solid bodies – *rāśi*,
5. operations with fractions – *kalā-savarṇa*,
6. linear equations – *yāvat-tāvat*,
7. quadratic equations – *varga*,
8. cubic equations – *ghana*,
9. biquadratic equations – *varga-varga*,
10. combinatorics – *vikalpa*.

Notable contribution of Jaina geometry is the measurement of a circle, because in Jaina cosmography the Earth was a large circular island. The value $\pi = \sqrt{10}$ was routinely used in Jaina texts.

Correct calculations for both permutations and combinations can be found in Jaina works. Indian rules correspond to contemporary formulas (for $n \in \mathbb{N}$)

$$C_1(n) = n, \quad C_2(n) = \frac{n(n-1)}{1 \cdot 2}, \quad C_3(n) = \frac{n(n-1)(n-2)}{1 \cdot 2 \cdot 3},$$

$$P_1(n) = n, \quad P_2(n) = n(n-1), \quad P_3(n) = n(n-1)(n-2).$$

The method of finding the number of combinations is called *Meru-prastāra* and it is a formation of an early Pascal triangle.

2 Medieval Indian Mathematics

The most important medieval authors and works are listed in the following overview. The authors were not only mathematicians, but often also astronomers or occasionally astrologers.

Āryabhaṭa I. (about 476–550) was an astronomer and the author of *Āryabhaṭīya* – an astronomical work, but the second chapter contains brief mathematical rules in 33 verses.

Brahmagupta (about 598–670) was the author of *Brāhma-sphuta-siddhānta*. It is an astronomical work consisting of 21 chapters, two of them deal with mathematics. The twelfth chapter contains arithmetic and calculating with numbers, the eighteenth one gives algebraic rules for finding unknown quantities.

Bhāskara I. (about 600–680) wrote a commentary to *Āryabhaṭīya*.

Anonymous manuscript **Bakhshālī** (probably the 7th or the 8th century) is written on birch bark. It is a collection of several rules and sample problems with explaining commentary and unique mathematical notation.

Mahāvīra (about 800–870) was a mathematician and he wrote *Gaṇita-sāra-saṃgraha* – the first known purely mathematical work. It contains more than 1100 verses.

Śrīdhara (about 870–930) was the author of the arithmetical work *Pāṭī-gaṇita* and its reduced version *Triśatikā*.

Āryabhaṭa II. (about 920–1000) wrote *Mahā-siddhānta* – an astronomical work with three mathematical chapters.

Śrīpati (1019–1066) was the author of arithmetical treatise *Gaṇita-tilaka*.

Bhāskara II. (1114–1185) was the great mathematician in medieval India. He wrote *Līlāvatī* – the most famous Sanskrit arithmetical work. It was composed in verses and divided into 13 chapters; its rules were explained by way of examples. Bhāskara II. was also the author of *Bījagaṇita*, the earliest extant independent treatise on algebra in Indian mathematics. It consists of 8 chapters devoted mainly to solving equations.

Nārāyaṇa (about 1340–1400) was the author of *Gaṇita-kaumudī*, which was influenced by Bhāskara II. The last chapter contains rules for construction of magic squares and magic figures.

Numbers

Ten has formed the basis of numeration in India from ancient times. In Sanskrit literature, there is no trace of extensive use of any other base of number systems. India is characterized by a very early use of large numbers and their names. While the Greeks did not have terminology for numbers greater than *myriad* (10^4), the Romans more than *mille* (thousand), the ancient Indians used terms for at least eighteen powers of ten. Initially, large numbers were described in words, but special symbols for small units existed very soon. Old numbers have been preserved in the inscriptions of Emperor Aśoka (the 3rd century BC); some of them were written in *kharoṣṭī* script, most of them in *brāhmī* script. However, these numbers were not yet in the positional system.

The most important feature of the Indian numeral system was a positional decimal notation. The good precondition for its invention was the existence of separate symbols for numbers from 1 to 9 and the symbol for zero. The earliest example of the decimal place value number is the inscription on a copper plate that contains the date 346 *Saṃvat era* corresponding to 595 AD. The old way of writing numbers without positional system was used in India to the 7th century AD, then the new way with positional notation began to prevail.

In the first centuries AD, the way of expressing numbers by special words was developed. In this system, numbers were expressed by the names of things or beings which connoted numbers. Thus number one was denoted by anything that is unique – the Earth, the Moon etc., the number two by any pair – eyes, hands and similarly others. The disadvantage of this notation, however, was considerable length. Some Indian scholars, who considered the verbal numbers too lengthy, replaced words by letters or rather syllables. Āryabhaṭa I. introduced an alphabetical system for expressing numbers in astronomy.

Arithmetic

Ancient Indian mathematicians distinguished twenty arithmetic operations called *parikarman* and eight determinations called *vyavahāra*. Determinations were a kind of procedure or method to solve a problem of given type, to determine the unknown quantity.

The twenty arithmetic operations included:

1. addition – *saṃkalita*,
2. subtraction – *vyavakalita*,
3. multiplication – *guṇana*,
4. division – *bhāga-hāra*,
5. square – *varga*,
6. square root – *varga-mūla*,
7. cube – *ghana*,
8. cube root – *ghana-mūla*,

9.-13. five rules for fractions– *pañca jāti*,

14. rule of three– *trai-rāśika*,
15. inverse rule of three – *vyasta-trai-rāśika*,
16. rule of five – *pañca-rāśika*,
17. rule of seven – *sapta-rāśika*,
18. rule of nine – *nava-rāśika*,
19. rule of eleven – *ekādaśa-rāśika*,
20. exchange, barter – *bhāṇḍa-prati-bhāṇḍa*.

The eight determinations included:

1. mixture – *miśraka*,
2. progressions – *średhī*,
3. plane figures – *kṣetra*,
4. excavation – *khāta*,
5. stock – *citi*,
6. saw – *krākacika*,
7. mound – *rāśi*,
8. shadows – *chāyā*.

The first eight operations were considered fundamental. Operations doubling and splitting, which were regarded as fundamental in Egypt and Greece, do not occur in the Indian treatises.

In the manuscript *Bakhshālī*, there were no special symbols for the basic arithmetic operations; they were expressed only by abbreviations: *yu* (*yuta*, added), *gu* (*guṇa*, multiplied), *bhā* (*bhāga*, divided). A special feature of the manuscript is the appearance of the symbol +, which meant that the number standing before this sign was to be subtracted.

Later, the symbol for subtraction was expressed by a dot or a small circle placed above the number, such as $\dot{5}$ or $\mathring{5}$. There were no symbols for other operations; numbers or expressions were recorded side by side.

Introductions of arithmetic texts contained definitions of decimal orders names and rules for operations with zero. Indian scholars distinguished several different methods of multiplication and used associative and distributive properties of multiplication.

The arithmetical algorithms were taken over by the Arab mathematicians, who learned decimal arithmetic from the Indians. Arabic manuscripts were later studied in Europe and the ancient Indian methods appeared in various modifications in the works of medieval mathematics.

Fractions

From about the 2nd century BC, fractions were written about the same way as today – the numerator over the denominator, but without the line between them. Mixed numbers have a whole number placed above the numerator. If the problem contained several fractions, they were separated by horizontal and vertical lines.

Due to the lack of appropriate symbolism, Indian expressions with fractions were ambiguous. For example the notation $\begin{array}{|c|c|}\hline 1 & 1 \\ 4 & 4 \\ \hline\end{array}$ could mean multiplication $\left(\frac{1}{4}\cdot\frac{1}{4}\right)$ as well as addition $\left(\frac{1}{4}+\frac{1}{4}\right)$; similarly the expression $\begin{array}{|c|}\hline 1 \\ 1 \\ 3 \\ \hline\end{array}$ could be read as $\left(1:\frac{1}{3}\right)$, but also as $\left(1\frac{1}{3}\right)$. The exact meaning of the notation could be understood only according to the assignment of the problem.

Hence expressions with fractions were divided into several classes called *jāti*. There were rules under which these classes could be expressed by using the appropriate fractions. The only common symbol was a dot, which marked a negative number or a number that was to be subtracted.

Arithmetic methods

The ancient Indian scholars considered as arithmetic some methods that we now include rather in algebra: one of them is *the rule of three* (*trai-rāśika*). The problem solves tasks of direct proportionality: if P yields F, what will I yield? Today, similar tasks are expressed as equality of ratios

$$x : F = I : P, \qquad \text{thence} \qquad x = \frac{F \cdot I}{P}.$$

There were three given terms – *pramāṇa* (argument), *phala* (fruit) and *icchā* (requisition), but they were sometimes called only the first, the second and the third respectively, because the terms were written in this order $P \mid F \mid I$. Most authors describing the rules pointed out that the first and the third term are of the same denomination. Except the rule of three, Indian mathematicians were skilled in the inverse rule of three, the rule of five, the rule of nine and the rule of eleven.

The rule of false position can be found in all Indian arithmetic works. The method was used to solve equations of the type

$$ax = b,$$

where it substituted direct division. The equation was solved using the method of false position by the choice of an arbitrary quantity x^*, then the product

$ax^* = b^*$ was calculated and if $b^* \neq b$, the solution of the given equation was expressed by the relation

$$x = \frac{b \cdot x^*}{b^*}.$$

When a suitable value x^* was chosen, the calculation was easier than using $x = \frac{b}{a}$.

Saṃkramaṇa was the name given to the method of solving the system of linear equations with unknowns x, y

$$x + y = a,$$
$$x - y = b,$$

where a, b were given numbers. According to the *saṅkramaṇa* method, the solution was

$$x = \frac{1}{2}(a + b), \qquad y = \frac{1}{2}(a - b).$$

Various problems concerning *interest* were also found in mathematical treatises. Simple problems were solved with the rule of three or the rule of five, some of the more complex examples led to a quadratic equation. In most examples simple interest prevailed, an indication of compound interest was in problems leading to quadratic equations.

Medieval Indian mathematicians had also rules for sums of arithmetic and geometric progressions, knew basic combinatorial rules and they also solved problems concerning barter and exchange.

Algebra

In ancient India, Algebra was considered more important than arithmetic. Unknown quantity was called *yāvat-tāvat* (as much as) abbreviated *yā*. If more unknowns were needed, the term *yāvat-tāvat* was used for the first of them and the remaining ones were denoted by colours or letters of the alphabet.

The second power was called *varga* (square), the third power *ghana* (cube). Names for the other powers were generated using these words multiplicatively, i.e. *varga-varga* was the fourth power, *varga-ghana* denoted the sixth power and so on. Powers whose exponents were not a multiple of two or three were expressed using the term *ghāta* to denote the summation of exponents. For example, the fifth power was expressed *varga-ghana-ghāta*, the seventh power was *varga-varga-ghana-ghāta*. These symbols for powers were placed after the unknowns. When it was necessary to express the product of more unknowns, it was indicated by the abbreviation *bhā* (*bhāvita*, product).

For example

yā va	(*yāvat varga*)	was	x^2,
yā va-gha-ghā	(*yāvat varga-ghana-ghāta*)	was	x^5,
yā gha kā va bhā	(*yāvat ghana kālaka varga bhāvita*)	was	x^3y^2.

Miscellaneous terms, which can be translated as a *number* or a *multiplier*, were used for the numbers representing the coefficients of unknowns. The absolute term in the equation was called *rūpa* (visible). There were no special symbols for arithmetic operations, expressions were written side by side. It was described by words what kind of operation should be performed. In several cases, the symbol + placed after the number denoted that the number was to be subtracted. Later the symbol for subtraction was a dot or a small circle placed above the quantity.

Indian algebra included computations with negative numbers and irrationals. Indian mathematicians knew quadratic irrationality with which they calculated very skilfully. The square root was calculated according to the formula

$$\sqrt{Q} = \sqrt{a^2 + b} \approx a + \frac{b}{2a},$$

where a^2 was the greatest square less than Q. Sometimes they used even more accurate relation

$$\sqrt{Q} = \sqrt{a^2 + b} \approx a + \frac{b}{2a} - \frac{\left(\frac{b}{2a}\right)^2}{2(a + \frac{b}{2a})}.$$

Equations

The main theme of medieval algebra was solving equations. First, it was necessary to form the equation. Two sides of the equation were written one below the other without any sign of equality. The same terms were usually placed one below the other and the absent terms were indicated by putting zeroes as their coefficients.

Equations were divided into several classes. This classification of individual scholars differed slightly. The following system was given by Bhāskara II.

1. equations in one unknown,
   a) linear equations,
   b) quadratic and higher degree equations,
2. equations in two or more unknowns,
   a) simultaneous linear equations,
   b) equations involving the second and higher powers of unknowns,
   c) equations involving a product of unknowns.

When solving linear equations, *the method of false position* was sometimes used to overcome the lack of appropriate symbolism. This method did not occur in later Indian algebraic works any more.

Brahmagupta gave two rules for solving a general quadratic equation

$$ax^2 + bx = c$$

with a positive coefficient of the quadratic term ($a \in \mathbb{Q}^+$, b, $c \in \mathbb{Q}$). This differed from earlier mathematicians who solved only equations with positive coefficients. Brahmagupta's procedure for solving a quadratic equation was called the *elimination of the middle term* (*madhyamāharaṇa*). Today we can express his rules by the formulas

$$x = \frac{\sqrt{4ac + b^2} - b}{2a}, \qquad x = \frac{\sqrt{ac + \left(\frac{b}{2}\right)^2} - \frac{b}{2}}{a}.$$

However, Brahmagupta did not mention the existence of two roots. Mahāvīra and Bhāskara II. calculated with two (positive) roots of a quadratic equation.

For simple systems of nonlinear equations with two unknowns, well-known identities were often used to express the sum and difference of unknowns and then the method *saṃkramaṇa* could be applied. For example, when solving a system of

$$x^2 + y^2 = a,$$
$$x + y = b,$$

Brahmagupta, using $(x - y)^2 = 2(x^2 + y^2) - (x + y)^2$, expressed the difference $x - y = \sqrt{2a - b^2}$ and with the help of *saṃkramaṇa* he obtained

$$x = \frac{1}{2}\left(b + \sqrt{2a - b^2}\right), \qquad y = \frac{1}{2}\left(b - \sqrt{2a - b^2}\right).$$

Indeterminate equations

Indian mathematicians reached important results in the study of indeterminate equations. A method for solution of an indeterminate linear equation (a, b, $c \in \mathbb{Q}$)

$$ax + c = by$$

was given by Aryābhata I., Brahmagupta, Mahāvīra and Bhāskara II. The method was called *kuṭṭaka* and Indian scholars considered it to be very important. This Indian method reminds a later method of using continued fractions.

Medieval Indian mathematicians achieved other remarkable results in solving so called Pell's equation, i.e. the equation ($a \in \mathbb{N}$, $\sqrt{a} \notin \mathbb{N}$, $b \in \mathbb{Z}$)

$$ax^2 + 1 = y^2,$$

Brahmagupta and Bhāskara II. introduced important rules for finding the smallest pair of natural numbers (x, y), which were the solution of Pell's equation.

Brahmagupta solved Pell's equation by finding a natural solution to the auxiliary equation

$$ax^2 + b = y^2, \qquad \text{where} \quad b \in \{\pm 1, \pm 2, \pm 4\}.$$

Indian scholars called this method *principle of composition* (*bhāvanā*).

Bhāskara II. supplemented his method and described the *cyclic method* (*cakravāla*). It is an algorithm which successively constructed equations (and also sought their integer solutions) $ax^2 + b_1 = y^2$, $ax^2 + b_2 = y^2$ and so on to obtain the equation where $b_k \in \{\pm 1, \pm 2, \pm 4\}$. Bhāskara knew, though probably only on the basis of experience, that the procedure described in his method ends. After a finite number of steps, he obtained equations $ax^2 + b_k = y^2$, where $b_k \in \{\pm 1, \pm 2, \pm 4\}$. Then he could continue according to Brahmagupta's *principle of composition*.

In his work *Bījagaṇita*, we find some interesting examples in which the author had to solve systems with more than two equations. These were problems where he was looking for mostly two natural numbers whose sum, difference, product, etc. was the second or the third power of a natural number.

Some of the Indian problems and methods of their solution are similar to Diophantus' *Arithmetica*, significant difference is that Indians, with only minor exceptions, were looking for solutions only in the domain of natural numbers.

Geometry

Indian mathematics devoted much less attention to geometry than to arithmetic and algebra. There did not exist any separate geometric work, basic knowledge of geometry is contained in six of the eight determinations which, however, were usually part of arithmetic. The determinations were particularly concerned with plane geometry (measurement of the perimeter and the area of basic plane figures), spatial geometry (calculations of excavation, pile of bricks, pile of sand) and measurements with shadows.

Determination devoted to plane shapes included measurements of a triangle, a rectangle, a circle, an arc, an ellipse and an annulus. Some authors distinguished the "rough" size of the area (sufficient for practical purposes) and the "accurate" area.

Triangles, Pythagorean triples

Ancient Indian mathematicians knew general relations for the construction of right-angled triangles with integer or rational sides, for instance

$$(m^2 - n^2, 2mn, m^2 + n^2), \qquad \left(m, \frac{1}{2}\Big(\frac{m^2}{n} - n\Big), \frac{1}{2}\Big(\frac{m^2}{n} + n\Big)\right)$$

or

$$\left(m, \frac{2mn}{n^2 - 1}, \frac{n^2 + 1}{n^2 - 1} m\right).$$

Knowledge of Pythagoras' theorem was practised in many different problems. Some Indian examples were almost identical to Chinese problems.

The area of a triangle was computed according to the “rough” relation

$$S_p = \frac{a+b}{2} \cdot \frac{c}{2}.$$

The “accurate” area of a triangle was given by (v is the altitude, c is the base-line)

$$S = v\frac{c}{2} \quad \text{or} \quad S = \sqrt{s(s-a)(s-b)(s-c)}, \quad \text{where} \quad s = \frac{1}{2}(a+b+c).$$

Quadrilaterals

Ancient Indians distinguished five different types of quadrilaterals – square, rectangle, isosceles trapezium, trapezium with three equal sides and scalene quadrilateral. Isosceles trapeziums had already a significant role in the construction of Vedic altars.

Brahmagupta constructed cyclic quadrilaterals with perpendicular diagonals from right-angled triangles with integer sides. Such quadrilaterals are sometimes called “Brahmaguptan quadrilaterals”. He considered two right-angled triangles, for instance with sides (a, b, c) and (x, y, z). Then he multiplied each of them by the legs of the second. Thus he obtained four right-angled triangles (xa, xb, xc), (ya, yb, yc), (ax, ay, az) and (bx, by, bz). He put together equal sides of the triangles and built a quadrilateral whose sides were hypotenuses of the four above mentioned triangles.

The “rough” area of a quadrilateral with sides a, b, c, d was given by

$$S_p = \frac{a+c}{2} \cdot \frac{b+d}{2}.$$

The “accurate” area of a quadrilateral was calculated according to the formula

$$S = \sqrt{(s-a)(s-b)(s-c)(s-d)}, \qquad \text{where} \quad s = \frac{1}{2}(a+b+c+d).$$

The formula, however, gives the exact result only for cyclic quadrilaterals.

Circle

An area of a circle was described by the formula

$$S = \frac{o}{2} \cdot \frac{d}{2}, \qquad \text{where } o \text{ was the circumference and } d \text{ was the diameter.}$$

Āryabhaṭa I. claimed, that the circle with a diameter equal to 20 000 had a circumference 62 832. It gives a very accurate value $\pi = 3,1416$. Later scientists expressed rules for "rough" and "accurate" area and circumference of a circle depending on its diameter as

$$o_p = 3d, \qquad S_p = 3\left(\frac{d}{2}\right)^2, \qquad o = \sqrt{10}d, \qquad S = o\frac{d}{4} = \sqrt{10}\,\frac{d^2}{4}.$$

The "rough" calculations used $\pi = 3$ while the value $\pi = \sqrt{10}$ occured in the "accurate" formulas. Bhāskara II. used the formulas

$$o_p = \frac{22}{7}d, \qquad\qquad o = \frac{3\,927}{1\,250}d$$

for the "rough" and the "accurate" circumference of a circle. The Indians knew the relationship between the diameter of a circle and a chord, now known as Euclid's theorem on height

$$v^2 = ab,$$

where v is the half chord, a, b are the segments of the perpendicular diameter divided by the chord.

Solids

Sections dealing with solids were also contained in Indian arithmetic texts. They included determinations *khāta*, *citi* and *rāśi*. All these determinations calculated volume. The first of them dealt with excavations, which were usually in the shape of a truncated pyramid, the second one focused on piles of bricks, which had the shape of a pyramid or a truncated pyramid and the last one aimed at heaps of grain, which were in the shape of a cone. These formulas were often only approximate, but sufficient for practical needs.

Conclusion

Writing numbers in the decimal positional system strongly influenced performing of arithmetic operations. The most of todays arithmetic operations are similar to the ones used by the Indians, since today, the numbers are expressed in the same way. However, ancient Indian mathematicians calculated skilfully not only with integers, but also with fractions. Due to the lack of appropriate symbolism, some expressions with fractions were divided into several classes.

Indian arithmetic was divided into operations and determinations. In addition to the basic operations with integers and fractions, the operations included also some methods, e.g. the rule of three.

Indian mathematicians were the first who systematically indicated unknowns by letters and used abbreviations to express the powers of the unknowns. They used a dot placed above a figure to indicate negative numbers. Thus, it was also possible to write equations with negative coefficients, which greatly simplified the classification of equations. Indian scholars came to very interesting results in solving indeterminate equations, especially the equation today known as Pell's equation.