Ladislav Svante Rieger (1916–1963)

Resumé

In: Eliška Pecinová (author): Ladislav Svante Rieger (1916–1963). (Czech). Praha: Matfyzpress, 2008. pp. 259–272.

Persistent URL: http://dml.cz/dmlcz/400768

Kapitola 8

Resumé

Ladislav Svante Rieger was an eminent figure in mathematics of the 20th century whose scientific work has gained an international response. Primarily, he gets credit for the Czechoslovak science. With his pioneering research, he set foundations of the study of mathematical logic and axiomatic set theory in both the Czech Republic and Slovakia. However, he is not generally known to wider mathematical public, mainly due to his untimely death. This monograph is devoted to L.S. Rieger's life and scientific work in mathematics, attention is also paid to his popular and philosophical publications. It consists of eight chapters plus rich factual and illustrated appendices.

The first chapter describes Rieger's family background, his personal life, pedagogical and scientific activities, and his mathematical legacy. The next four chapters discuss the major results of Rieger's mathematical treatises in individual domains of his scientific interest. Chapter two gives a summary of Rieger's early research in algebra, focused on group theory. The third chapter deals with Rieger's achievements in lattice theory with application to mathematical logic. Chapter four is dedicated to Rieger's research in axiomatic set theory. Other scientific works, namely Rieger's monograph on algebraic logic, are put near in the fifth chapter. Each chapter is concluded by a brief historical overview of the evolution of the related mathematical area, a special attention is paid to depiction of the conditions in the Czechoslovak mathematics. The sixth chapter introduces Rieger's popular and philosophical works and gives an overview of Czechoslovak mathematical literature in 1950's. The chapter seven concludes the book.

The factual appendix comprises the list of L.S. Rieger's publications, reviews in the journal *Mathematical Reviews*, and an outline of his teaching activities together with his talks for wider mathematical community. The illustrated appendix contains copies of documents and photographs from Rieger's estate.

8.1 Life of Ladislav Svante Rieger

8.1.1 Family

Ladislav Svante Rieger was a member of the famous Rieger family. Let us thus start with his family background.

L.S. Rieger's great-grand father was the well known and influential figure of the era of the Czech National Revival, František Ladislav Rieger (1818–1903). F.L. Rieger was one of the leading characters of the Czech patriotic movement. Together with František Palacký (1796–1876), he was in the lead of the Czech politics for 30 years. F.L. Rieger was also an editor of the first Czech universal encyclopedia *Riegrův slovník naučný* and played a significant role in the foundation of the Czech National Theatre.

F.L. Rieger's son and L.S. Rieger's grandfather, Bohuslav Rieger (1857–1907), was a prominent lawyer and historian. In 1899, he was appointed a Full Professor of Austrian history at the Czech university in Prague. In his scientific and pedagogical activity, B. Rieger focused on constitutional and court law. Among others, he was the founder and an editor of the journal *Sborník věd státních a správních.*

L.S. Rieger's father, Ladislav Rieger (1890–1958), was one of eminent Czech philosophers of his period. He was a representative of Marxist philosophy – dialectical materialism. L. Rieger originally studied physics and chemistry; however, from 1920, he fully devoted himself to philosophy. From 1945, he lectured at the Faculty of Arts, Charles University, and in the same year was appointed a Full Professor of philosophy. From 1952, L. Rieger was the head of the division of philosophy *Kabinet pro filosofii* (from 1957 Institute of Philosophy) of the Czechoslovak Academy of Sciences. Apart from other things, he was the first editor of the philosophical journal *Filosofický časopis.*

8.1.2 Childhood and studies

L.S. Rieger's mother, Alžběta Jarešová (1890–1951), had worked as a teacher before she married L. Rieger in 1915. The married couple moved to Sweden very shortly after their wedding. Ladislav Svante Rieger was born on June 25, 1916 in Malmö. After the end of the World War I, the family moved back to Prague where Alžběta gave birth to the daugter Věra (*1919). L.S. Rieger's childhood was, we dare say, quite dramatic. He had to cope with unpleasant atmosphere caused by financial problems of the family, parents' disputes which finally ended in divorce, and changes of housing. Having attended several schools, in June 1935, L.S. Rieger graduated from the state grammar school *Státní československé reformní reálné gymnasium* in Prague XIX.

L.S. Rieger was very active in public and political life already as a grammar school student. He became a member of several student and other politically left-wing oriented organizations. His inclination to left-wing politics grew even stronger during the World War II when L.S. Rieger actively participated in illegal communist activities. In 1945, he officially joined the Communist Party of Czechoslovakia.

Besides mathematics, L.S. Rieger was also concerned with philosophy (perhaps under influence of his father), first with dialectical materialism. In this period, he published his first works, all of them philosophically oriented. In the subsequent years, L.S. Rieger was appealed by ideas of the *Vienna Circle*. The philosophical movement associated with this group is called logical positivism. At that time, L.S. Rieger started studying logic intensively.

In October 1935, L.S. Rieger became a student of mathematics and physics at the Faculty of Science, Charles University. His lecturers were, above all, Vojtěch Jarník (1897–1970), Vladimír Kořínek (1899–1981), Karel Petr (1868–1950) and František Záviška (1879–1945). In February 1938, L.S. Rieger passed the first state exam in both the subjects. Unfortunately, he did not have an opportunity to take the second one until 1945 due to the wartime difficulties.

8.1.3 Professional life

Very shortly after the end of the World War II, in September 1945, L.S. Rieger passed the second state exam which enabled him to teach at Czechoslovak secondary schools. As early as March 1946, Rieger submitted his dissertation, passed two rigorosum examinations, and was conferred the Doctor of Natural Sciences degree (RNDr.). One year later, L.S. Rieger married the neurologist Helena Holingerová (*1918). In their marriage two daughters were born, Jitka (*1950) and Alena (*1956).

Already in August 1945, L.S. Rieger accepted a position of an assistant at the Mathematical Institute (with František Rádl (1876–1957) as the head) of the Czech Technical University in Prague. In October 1951, he was appointed an Associate Professor and in the subsequent academic years 1951/52–1957/58 was the head of the *Mathematical Institute* of the Faculty of Engineering. Rieger's activity at the technical university was mainly of a pedagogical character. In this period, he gave mainly compulsory lectures for the first and second year students. This activity was, considering a wide range of lectures and a great amount of students, very time-consuming. Besides, he lead some optional courses, e.g. in vector analysis, operator calculus, and statistics.

At the end of 1958, L.S. Rieger moved to the *Mathematical Institute of the Czechoslovak Academy of Sciences* where he spent the rest of his life. He worked in the department of numerical methods. At the beginning of the following year, Rieger submitted his doctoral dissertation whose opponents were V. Alda (*1923), M. Katětov (1918–1995) and A. Mostowski (1913–1975). On the grounds of its successful defence, which took part in December that year, L.S. Rieger received the Doctor of Physical and Mathematical Sciences degree (DrSc.).

Apart from the main pedagogical and research workplaces, L.S. Rieger marginally lectured also at the Charles University. From the academic year 1951/52 (with short breaks) until his death in 1963, he lead some optional and recommended courses, first at the Faculty of Science, and from 1952 at the Faculty of Mathematics and Physics. He usually had one two-hour course each semester, namely lectures and seminars in mathematical logic and axiomatic set theory.

Participants of Rieger's seminars became later his followers in these areas. The persons in question were Jiří Bečvář (1926–1999), Petr Vopěnka (*1935) whose seminars followed up with Rieger's seminar after his death, and the former Rieger's postgraduate student Petr Hájek (*1940).

In the period 1954–1960, L.S. Rieger was an editor of the mathematical journal *Časopis pro pěstování matematiky*; in the following year, he went over to the editorial board of *Czechoslovak Mathematical Journal*. In the years 1959–1962, Rieger also wrote reviews to the international reference journal *Mathematical Reviews*. L.S. Rieger died untimely at the age of 46. On February 14, 1963 he succumbed to brain cancer.

8.2 Characteristics of Rieger's scientific work

The complete list of Rieger's publications is comprised of 51 papers (6 of them, published by 1948, are of a philosophical character), 22 of them are original scientific treatises. Predominantly, the publications have a character of journal articles, exceptions are three textbooks, one monograph and one popularizing work. L.S. Rieger's scientific activities were devoted to three main mathematical areas: algebra (group and lattice theory), mathematical logic, and axiomatic set theory. The aim of this section is to outline the evolution of Rieger's professional interests. It must be pointed out that Rieger's isolated position in world mathematics and limited access to literature was a problem. Several times, he obtained results established previously by other authors without his knowledge. The more remarkable and inspiring his mathematical efforts are.

Group theory (1941–1952):

As early as 1941, L.S. Rieger started working in the theory of ordered and cyclically ordered groups. During the difficult wartime period, he discussed related problems by means of correspondence with his teacher Vladimír Kořínek. Under V. Kořínek's leadership, Rieger submitted the dissertation *On ordered groups* (later published as [R1], [R2], and [R3]). Excellence of the dissertation was rewarded by an award by the *Royal Bohemian Society of Sciences*.

There are many references to the papers [R1], [R2], [R3], including citations in significant monographs on ordered algebraic structures. Let us underline that L.S. Rieger was the first to study cyclically ordered groups and is considered the founder of this discipline. His results in this field were followed mainly by the eminent Slovak mathematician Ján Jakubík (*1923).

Lattice theory and mathematical logic (1949–1957):

In the subsequent period (1949–1951), L.S. Rieger turned his attention to problems of lattice theory, especially Boolean algebras (papers [R4]–[R7]). His position in research in lattice theory was quite unique in the Czechoslovak mathematics. For the majority of work from this area application to logic is typical. In [R5], a lattice-theoretical interpretation of the Heyting calculus is given and in [R7] Rieger found a new proof of Gödel's completeness theorem of the first-order predicate calculus.

In 1950, Rieger spent six months in Warsaw, Poland. This stay, particularly cooperation with the prominent Polish logician Andrzej Mostowski, was of a great importance for his scientific work. Here, L.S. Rieger probably reached the decision to pursue a continual research in mathematical logic. His algebraic approach to predicate calculus, which he presented at Mostowski's seminar, was then continued by some Polish mathematicians (e.g. H. Rasiowa (1917–1994) and R. Sikorski (1920–1983)).

It is hard to draw a line between algebra and logic in Rieger's work. Papers [R8], [R10], [R11] from 1951–1955 and [R12] are on the boundary between abstract algebra and mathematical logic, their common and essential feature is algebraization of mathematical logic. [R8] and [R10] are also dedicated to Gödel's completeness theorem, another of its proofs is given in [R21].

Let us note that publications from this area belong to Rieger's most cited works (including references in substantial monographs on lattice theory and mathematical logic). [R5], [R7], and [R8] are regarded the most significant papers of Rieger's research related to logic.

Axiomatic set theory (1954–1963):

Around 1954, L.S. Rieger focused on specific problems of axiomatic set theory. He worked with Gödel's axiomatic system following Gödel's investigation in [Göd40][1] which was his fundamental mathematical resource. Paper [R13] is of the greatest importance in Rieger's work in set theory. Later, he became primarily concerned with Gödel's theory of finite sets (in relation to special arithmetics) to which the papers [R15] and [R18] are devoted. [R13] and [R15] formed Rieger's doctoral dissertation *A contribution to Gödel's axiomatic set theory, I, II.* The publications [R16] and [R17] are based on results from [R15] and [R18]. In [R20], L.S. Rieger gave a new proof of the relative consistency of the axiom of choice and the generalized continuum hypothesis.

Other scientific treatises (1958–1963):

In a few last years of his life, L.S. Rieger also worked in the areas on the margin of mathematical logic. Connected to his research in the Mathematical Institute, he devoted himself to topics related to development of the first computers (papers [R14] and [R19]).

Presumably, the most valuable of Rieger's publications is the monograph [R22] where he summarized relevant results from algebraic logic achieved in that period. It comprises eight chapters, the last two contain his own scientific results. Rieger also made use of his results summed up in the manuscript [R21]. However, he was not able to finish his book and [R22] as well as [R21] were published after his death.

[1] Gödel, K., *The consistency of the axiom of choice and of the generalized continuum hypothesis with the axioms of set theory*, Annals of Mathematical Studies, No. 3, Princeton, 1940.

8.3 Group theory

8.3.1 *On ordered and cyclically ordered groups*

A group is called *ordered* if the set of its elements is linearly ordered in such a way that it is possible to multiply the ordering relation from both the left and the right side. Similarly, if a trinomial relation of cyclical ordering is given in the set of elements of a group such that it can be multiplied from both the sides, the group is called *cyclically ordered*[2].

The aim of the three papers *On ordered and cyclically ordered groups, I, II, III* ([R1], [R2], [R3]) was to investigate how the fact that a group can be (cyclically) ordered determines the algebraic structure of the group, and in what manner the structure is determined if the group is (cyclically) ordered in a specific way. The first paper is devoted exlusively to ordered groups, in the second one, cyclically ordered groups are introduced. In the third work, these groups are studied by a full use of topological means.

Let us remark that before Rieger, only commutative ordered groups were systematically studied, namely by H. Hahn (1879–1934) in [Hah07][3].

One of the principal results given in the work [R1] is a (purely algebraic) *necessary and sufficient condition for a group to be able to be ordered.* For that purpose, L.S. Rieger introduced the notion of *magnitude subgroup.* This result was later stated by other mathematicians by means of convex subgroup. The notion of magnitude subgroups is one of the central terms in the work and Rieger addressed some other problems and features related to it.

Further, L.S. Rieger stated *a necessary and sufficient condition for an ordered group to be commutative.* Thereby, he gave a connection between general and commutative groups, and integrated Hahn's results (especially those concerning a lexicographical product of ordered groups) into general theory of ordered groups.

In the paper [R2], cyclically ordered groups are studied. The relation of cyclical ordering was first introduced by Eduard Čech (1893–1960) in [Čech37][4]. As mentioned above, L.S. Rieger was the first who was intensively working in the field of cyclically ordered groups and who obtained substantial results.

Ordered groups are (in a certain sense) a special case of cyclically ordered groups. It is easy to realize that an arbitrary ordered group can form a cyclically

---

[2] A cyclically ordered group $\mathcal{G}$ is defined by the following axioms:
For every $x, y, z \in \mathcal{G}, x \neq y, x \neq z, y \neq z$, and every $v \in \mathcal{G}$

1. $< x, y, z < \Rightarrow < y, z, x <$,
2. one of the following possibilities is satisfied

$$< x, y, z < \text{ or } < y, x, z <,$$

3. $(< x, y, z < \ \& \ < x, z, v <) \Rightarrow < x, y, v <$,
4. $< x, y, z < \Rightarrow (< vx, vy, vz < \ \& \ < xv, yv, zv <)$.

[3] Hahn, H., *Über nichtarchimedische Grössensysteme*, Sitzungsber. d. Ak. d. Wiss. Wien, Abt. IIa **116** (1907), 601–655.

[4] Čech, E., *Topologické prostory*, Čas. pro pěst. mat. a fys. **66** (1937), D225–D264.

ordered one, simply by putting (for $x \neq y, x \neq z, y \neq z$) $< x, y, z <$ if either $x < y < z$ or $y < z < x$ or $z < x < y$.

Further, L.S. Rieger proved the "inverse" statement that *every cyclically ordered group can be represented as a factor group of a certain ordered group.*

The last treatise [R3] contains applications partly of results derived in the previous two works, and partly from general theory of topological groups. Ordered and cyclically ordered groups belong among well known *topological groups*[5]. First, Rieger introduced topologies to ordered and cyclically ordered groups and called them "natural".

Further, he stated several algebraic consequences derived from properties of natural topologies of both types of groups. The main theorem can be formulated as follows:

Every infinite cyclically ordered group which is compact in its natural topology is isomorphic with a multiplicative group of complex numbers with an absolute value equal to one.

8.3.2 *On groups and lattices* [R34]

Let us now mention Rieger's book *On groups and lattices* [R34] published in 1952. It has quite an unusual position in his works for its popularizing character. This book was meant for common readers, especially for secondary school students, and its aim was to present fundamental notions of group and lattice theory to wider public. The first part (*On lattices*) was slightly altered and republished in 1974.

As far as the situation in algebraic textbooks is concerned, only one textbook on groups had been in use in Czechoslovakia before Rieger's work [R34] was published. It was *Introduction to group theory* [Bor44][6] by Otakar Borůvka (1899–1995). This book was intended for university students and, in comparison with [R34], it was far more difficult for understanding. As for lattice theory, with respect to a short period of existence of this discipline (less than 20 years), it is natural that no other textbook had been published in Czechoslovakia before.

8.4 Lattice theory and mathematical logic

8.4.1 Lattice theory and theory of Boolean algebras

The first to influence L.S. Rieger's research in lattice theory was M.H. Stone (1903–1989). The work *A note on topological representations of distributive lattices* [R4] is a continuation of Stone's core investigation in topological representation of distributive lattices ([Sto37])[7].

[5] I.e. groups whose set of elements has such topology that multiplication as well as assignment of an inverse element are continuous mappings.

[6] Borůvka, O., *Úvod do theorie grup*, KČSN, Praha, 1944.

[7] Stone, M.H., *Topological representations of distributive lattices and Brouwerian logic*, Čas. pro pěst. mat. a fys. **67** (1937), 1–25.

A distributive lattice $\mathcal{L}$ is said to be *topologically represented* in a topological T_0-space $S(\mathcal{L})$ if there exists an isomorphism of $\mathcal{L}$ onto a set-ring $\mathcal{R}$ of certain open subsets of $S(\mathcal{L})$ such that $\mathcal{R}$ forms an open basis of $S(\mathcal{L})$.

M.H. Stone described a "universal" space $\bar{S}(\mathcal{L})$ which contains every representation space $S(\mathcal{L})$ as a dense subset. $\bar{S}(\mathcal{L})$ is the space of all prime filters of $\mathcal{L}$. L.S. Rieger presented another characterization of $\bar{S}(\mathcal{L})$ for distributive lattices with zero and unit. As a consequence, he obtained the assertion that *any distributive lattice with zero in which all prime filters are maximal is a generalized Boolean algebra and any distributive lattice with zero and unit in which all prime filters are maximal is a Boolean algebra.*

The main subject of the work *Some remarks on automorphisms of Boolean algebras* [R6] is construction of a Boolean algebra admitting no proper homomorphic mapping onto itself. L.S. Rieger found the solution by transferring the problem onto the topological problem of *existence of a zero dimensional compact space without proper homeomorphic transformations onto any of its subspaces.* Thus, he gave the (negative) answer to Birkhoff's Problem 74, one of the problems stated in the monograph *Lattice theory* [Bir48][8]. Rieger concluded the paper with some remarks on Birkhoff's Problem 75.

***On the lattice theory of Brouwerian propositional logic* [R5]**

The purpose of this paper was to show that by means of the notion of Heyting algebra[9], lattice theory can work as an efficient mathematical tool for both syntax and semantics of a language using Brouwerian logic[10]. First who studied lattice-theoretical interpretation of the Heyting calculus was G. Birkhoff (1911–1996) in [Bir40].[11]

However, timing of Rieger's work [R5] was rather unfortunate because a year before the paper by J.C.C. McKinsey and A. Tarski (1901–1983) [MT48][12] was published which deals with similar problems. L.S. Rieger did not have access to this treatise and proved several result independently.

One of Rieger's primary results is *characterization of the Heyting propositional calculus as a free Heyting algebra with countably infinite number of generators*[13], and thus its semantical interpretation by means of countably infinite number of "logical values". He showed that the same Heyting algebra can be constructed in various ways within four areas.

By simple algebraic considerations, L.S. Rieger obtained some theorems

[8] Birkhoff, G., *Lattice theory*, AMS, New York, 1948.

[9] Called by Rieger a **s**pecial **d**istributed **r**esiduated lattice with **u**nit and **z**ero, shortly a *sdruz-lattice.*

[10] *Intuitionistic (Brouwerian) logic* differs from the classical logic by replacing the "nonconstructive" postulate $A \vee nonA$ or the law of double negation $non(nonA) \Rightarrow A$ by the law of contradiction $(A \ \& \ nonA) \Rightarrow B$. Its formal system is called the *Heyting calculus.*

[11] Birkhoff, G., *Lattice theory*, AMS, New York, 1940.

[12] McKinsey, J.C.C. and Tarski, A., *Some theorems about the sentential calculi of Lewis and Heyting*, Journal of Symbolic Logic **13** (1948), 1–15.

[13] free 1-generated Heyting algebra is now called *Rieger-Nishimura lattice* and was first introduced in [R5].

by K. Gödel (1906–1948) ([Göd33])[14] and others. He also gave a relatively simple and elementary computational method for a decision problem of the Heyting calculus which can be applied to general topology and abstract algebra. Furthermore, Rieger presented the algebraic essence of relations between the classical and Heyting propositional calculus.

On free $\aleph_\xi$-complete Boolean algebras [R7]

A Boolean algebra $\mathcal{A}$ is called *$\aleph_\xi$-complete* if any of its subsets whose power does not exceed $\aleph_\xi$ has a supremum and an infimum in $\mathcal{A}$.

In the first and main part of the paper [R7], L.S. Rieger investigated general properties of free $\aleph_\xi$-complete Boolean algebras with a special attention paid to $\aleph_0$-complete Boolean algebras (shortly σ-algebras) due to their numerous applications. The second part comprises applications of obtained results to the domain of mathematical logic.

Initially, L.S. Rieger provided a constructive proof of *existence of a free $\aleph_\xi$-complete Boolean algebra* and addressed its uniquess and "universality". Further, he arrived at the conclusion that *a free σ-algebra can be σ-isomorphically represented by a σ-field of subsets of a set of its σ-prime filters.* He also proved that this assertion does not hold for free $\aleph_\xi$-complete Boolean algebras in general.

As a consequence, Rieger obtained the following statement:

A free σ-algebra with m generators can be σ-isomorphically represented by the minimal σ-field of Borel subsets of the generalized Cantor discontinuum C_m.

By achieved results, L.S. Rieger found solutions of Problems 78, 79, and 80 from the monograph [Bir48].

In the second part, as mentioned above, L.S. Rieger applied attained results to mathematical logic, namely to the *Tarski-Lindenbaum algebra*[15] of the first-order predicate calculus (further *TL-algebra*). Thus, he obtained an algebraic proof of the famous Gödel's completeness theorem[16] (translated into the language of theory of Boolean algebras).

The method of Rieger's proof comprises three steps. First, he showed that the TL-algebra is a certain extension of a free σ-algebra. Second, Rieger proved that the TL-algebra (as a specific type of a σ-algebra) can be σ-isomorphically represented by a certain σ-field of sets. Third, he defined a σ-homomorphism of the TL-algebra on the algebra $\{0, 1\}$ which corresponds to algebraic formulation of Gödel's statement.

[14]Gödel, K., *Über den intuitionistischen Aussagenkalkül*, Erg. Koll. Wien **4** (1933), 35–40.

[15]I.e. a set of classes of formulas when two formulas belong to the same class if they are logically equivalent.

[16]This theorem was first published in Gödel, K., *Die Vollständigkeit der Axiome des logischen Funktionenkalküls*, Monatsh. Math. Phys. **37** (1930), 349–360.

8.4.2 Works on the boundary between algebra and logic

In the paper *On countable generalized σ-algebras, with a new proof of Gödel's completeness theorem* [R8], L.S. Rieger introduced a generalization of the notion of σ-algebra in the sense that countably infinite joins and meets can be performed only on members of certain multiple sequencies (called *marked*). Further, he proved that *if the set of marked sequencies is countable then a countable generalized σ-algebra can be isomorphically represented by a set field.*

The main aim of the paper is the same as in the second part of [R7]: application of the results to the TL-algebra, leading to a new proof of Gödel's completeness theorem. L.S. Rieger showed that the TL-algebra is a countable generalized σ-algebra with a countable family of marked sequencies and then followed the second and third step of the general method employed in [R7].

The publication *On one fundamental theorem of mathematical logic* [R10] is also devoted to the similar problems. Here, L.S. Rieger obtained Gödel's completeness theorem as a consequence of Lindenbaum's thorem: Every consistent first-order theory can be extended to a complete consistent theory.[17] In this work, Lindenbaum's theorem is proved by use of the theory of generalized σ-algebras. At the end of [R10], various proofs of Gödel's completeness theorem known by that time are compared.

To close this section, let us make a remark on the paper *On Suslin algebras and their representations* [R11]. Here, the notion of *Suslin algebra* is introduced and used for the description of predicate variables of the second-order logic.

8.5 Axiomatic set theory

In 1954, L.S. Rieger became interested in specific areas within axiomatic set theory. Several of his papers are very extensive and deal with problems of considerable difficulty. In this domain of Rieger's research, the important role of K. Gödel and his investigation is also apparent. The work [Göd40] was a fundamental mathematical resource in set theory in Czechoslovakia at that time. Consequently, L.S. Rieger was working via the frame of Gödel's axiomatic system. Gödel's achievement in both mathematical logic and set theory strongly influenced the research of Czechoslovak mathematicians even after Rieger's death.

8.5.1 *A contribution to Gödel's axiomatic set theory*

The paper *A contribution to Gödel's axiomatic set theory, I* [R13] is considered Rieger's most valuable work in set theory. It is devoted to the study of models and of relations of specific axioms. L.S. Rieger gave a generalization of Gödel's model of constructible sets and classes from [Göd40]. The generalization is due to the omission of the axiom of foundation (from both the interpreting

[17] As an analogous consequence, Rieger also obtained the Löwenheim-Skolem theorem: If a countable theory has a model then it has a countable model.

and interpreted theory) which was needed in Gödel's definition of the axiom of constructibility. Rieger generalized the notion of constructibility using Robinson's definition of an ordinal number from [Rob37][18]. He thus obtained that *the axiom of foundation is a consequence of Gödel's axioms A, B, C combined with the (generalized) axiom of constructibility.*

The main result of the paper is the following statement:

Let T be a set theory given by Gödel's axioms A, B, C, the axiom of choice, and the generalized continuum hypothesis. The axiom of foundation and the (generalized) axiom of constructibility are independent of the axioms of T. Moreover, the axiom of predicative sets "$\exists x(x \in x)$" is relatively consistent with T.

The second and third publication of the sequel *A contribution to Gödel's axiomatic set theory* (papers [R15] and [R18]) pertain to Gödel's axiomatic theory of finite sets. This theory is given by Gödel's axioms A, B, C, the axiom of foundation, and axiom of choice. The axiom of infinity (C_1) however is replaced by its negation – the axiom of finity.

In [R15], L.S. Rieger dealt with a new type of arithmetically constructed models which he called *dyadic models*. His first model was constructed using the domain of integrity of Hensel's dyadic integral numbers. Further, construction of this simple model was generalized using the notion of *dyadic set-theoretical ring* (shortly an *s-t-ring*). The purpose of this paper was to construct two nonstandard and uncountable models of Gödel's theory of finite sets, which was achieved by the extension of countable *s*-*t*-rings to uncountable. In both models, the set of "finite ordinal numbers" has cardinality $\aleph_1$. This opened a new outlook on the (axiomatically defined) concept of finity, natural number, arithmetic etc.

[R18] is a continuation of [R15] in which particular examples (i.e. certain extensions of *s*-*t*-rings) of the axiomatic set theory describing the system of Hensel's dyadic integral numbers were constructed. In [R18], this theory called *dyadic arithmetic* is defined by means of 31 axioms with the following primitive notions: addition, multiplication and exponentiation of two.

The main theme of [R18] is to show that *dyadic arithmetic (A) and Gödel's theory of finite sets (B) are interpretable in each other*. This equivalence is established in the form of two statements – the equivalence theorem and the reproduction theorem. The equivalence theorem enables for each of the theories A and B construction of a model of one theory in the other one. Moreover, the reproduction theorem states that iteration of these two interpretations (A in B and B in A or conversely) results in the identical interpretation (A in A or B in B). Thus, considering Gödel's theory of finite sets, the inclusion relation is of an arithmetical character.

[18] Robinson, R.M., *The theory of classes. A modification of von Neumann's system*, Journal of Symbolic Logic **2** (1937) 29–36.

8.5.2 Other works

Results from [R15] and [R18] are exploited in two publications with similar subject, *On the problems of natural numbers* [R16] and *The problem of the so-called absolutely undecidable sentences of number theory* [R17]. In [R16], L.S. Rieger argued whether arithmetics at that time was in a similar position to geometry at the beginning of the 19th century (i.e. before knowing non-euclidean geometries). [R17] follows his considerations from [R16] and is devoted to the problem of finding sentences of number theory which hold in one model and do not hold in another one. This problem was first opened by K. Gödel in [Göd31][19] and further discussed by A. Mostowski in [Mos54]. [20]. L.S. Rieger suggested a new way of solving this issue by its reduction to the problem of solvability of certain exponential equations over special dyadic rings (defined in [R15]).

The last Rieger's publication on set theory, *On the consistency of the generalized continuum hypothesis* [R20], is devoted to the famous Gödel's theorem on the relative consistency of the axiom of choice and the generalized continuum hypothesis from [Göd40]. Rieger simplified Gödel's original method, particularly by simplifying Gödel's axiom of constructibility. Rieger followed Gödel's main lines in most of his proof. The principal simplification is in the proof of the relative consistency of the continuum hypothesis.

8.6 Other scientific works

8.6.1 Works on the margin of mathematical logic

The papers [R14] and [R19] form a separate group of publications on the margin of mathematical logic. They treat questions in areas of machine learning and algorithmic and numerical methods associated with development of the first computers.

On the theory of the neural nets [R14] is a survey article discussing some fundamental problems of the mathematical and logical theory of neural networks (neural nets). Rieger refers primarily to the collection of papers [SM56][21]. Two classical questions of the theory of neural nets – analysis and synthesis – are presented. Analysis lies in construction of a (Boolean) characteristic function describing activity of a given net.

The following problem of synthesis is considered: *For a Boolean function f, given in a table form, find a neural net having f as a characteristic function.* L.S. Rieger gave an algorithmic solution to this problem. However, his method is rather theoretical and not suitable for a practically conducted synthesis.

[19] Gödel, K., *Über formal unentscheidbare Sätze der Principia Mathematica und verwandter Systeme I*, Monatsh. Math. Phys. **38** (1931), 173–198.

[20] Mostowski, A., *Der gegenwärtige Stand der Grundlagenforschung in der Mathematik*, Die Hauptreferate des 8. Polnischen Mathematikerkongresses, Warschau, September 1953, Deutscher Verlag der Wissenschaften, Berlin, 1954, pp. 11–44.

[21] Shannon, C.E. and McCarthy, J. (ed.), *Automata studies*, Princeton University Press, Princeton, 1956.

Nevertheless, due to this result, the issue of characterization of events representable by neural nets can be reduced to the abstract theory of automata of J.Tu. Medvedev (see [SM56]).

The publication *On Kleene's normal form for computable functions* [R19] deals with functions whose values can be computed by the Turing machine (also called *Turing computable functions*). L.S. Rieger proved the well-known Kleene's theorem on uniform expression of Turing computable functions of a given number of arguments. He derived the following statement:

Every Turing computable function can be obtained as a composition of two primitive recursive functions f and g using a minimum operation, independently of the particular Turing machine.

Moreover, Rieger found explicit and relatively simple formulas for the functions f and g.

8.6.2 *Algebraic methods of mathematical logic* [R22]

We conclude this summary with an account of Rieger's presumably top work, the monograph [R22]. Its compilation took him as many as twelve years and was not completed until four years after his death. The book [R22] is comprised of eight chapters; the last two contain Rieger's own results, especially those summed up in the manuscript *On structures of the classical predicate calculus.* This treatise was found in Rieger's inheritance together with an incomplete version of [R22] and was published as [R21] in 1964.

In [R21], L.S. Rieger introduced the notion of *substituted indexed algebra*, which enables a precise and relatively simple description of the TL-algebra. Cylindric algebras (by A. Tarski and F. Thompson) and P. Halmos's polyadic algebras are in a certain sense a special type of substituted indexed algebras. This notion is further used for algebraic proof of Gödel's completeness theorem of the first-order predicate calculus. Rieger also gave an algebraic formulation of a sufficient and necessary condition for a theory to have a (semantic) model and for interpretation of a language to be interpretation of a theory.

The aim of the monograph [R22] was to summarize relevant achievements in algebraic logic obtained in that period. Its main result is algebraic characteristics of the TL-algebras. The fundamental notion is a relation of *logical* (*formal*) *consequence* ($\rightarrow$) between formulas which is defined recursively. This notion is further modified by adding a set of axioms of a certain mathematical theory. Thus, one obtains a *relative consequence* $\rightarrow_A$ where A denotes a given set of axioms. In this context, Rieger considered – apart from the classical TL-algebra L – an algebra L_A of the theory given by axioms of A which is given by the relation $\rightarrow_A$. By means of algebraic theory, L.S. Rieger characterized both syntax and semantics of the first-order predicate logic. For a more thorough analysis of [R22] see [Szc73][22].

As stated above, [R22] was completed and published in 1967 with the help of Rieger's former assistant P. Hájek and friend M. Katětov. It is one of the

[22]Szczech, W., *L. Rieger's logical achievement*, Studies in the history of mathematical logic (Surma, S.J., ed.), Wydawnictwo Polskiej Akademii Nauk, Wroclaw, 1973, pp. 261–265.

most significant post-war Czechoslovak publications on logic.

8.7 Conclusion

Czech mathematical logic and set theory has reached world significance in our days. However, at the beginning, many obstacles had to be overcome. Ladislav Svante Rieger was an outstanding personality of Czechoslovak mathematics who achieved to get over these obstacles and to get closer to the foundation of research in these disciplines. He gets the credit for that.