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ROLE OF THE HARNACK EXTENSION PRINCIPLE IN THE KURZWEIL-STIELTJES INTEGRAL

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Cordially dedicated to the memory of Jaroslav Kurzweil

Abstract. In the theories of integration and of ordinary differential and integral equations, convergence theorems provide one of the most widely used tools. Since the values of the Kurzweil-Stieltjes integrals over various kinds of bounded intervals having the same infimum and supremum need not coincide, the Harnack extension principle in the Kurzweil-Henstock integral, which is a key step to supply convergence theorems, cannot be easily extended to the Kurzweil-type Stieltjes integrals with discontinuous integrators. Moreover, in general, the existence of integral over an elementary set E does not always imply the existence of integral over every subset T of E. The goal of this paper is to construct the Harnack extension principle for the Kurzweil-Stieltjes integral with values in Banach spaces and then to demonstrate its role in guaranteeing the integrability over arbitrary subsets of elementary sets. New concepts of equiintegrability and equiregulatedness involving elementary sets are pivotal to the notion of the Harnack extension principle for the Kurzweil-Stieltjes integration.

Keywords: Kurzweil-Stieltjes integral; integral over arbitrary bounded sets; equiintegrability; equiregulatedness; convergence theorem; Harnack extension principle

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1. Introduction

One of the meaningful discussions on the topic of Kurzweil-Henstock integration concerns the Harnack extension principle and Cauchy property (see, e.g., [26], Corollaries 7.10–7.11, and [28], Theorems 1.4.6, 1.4.8 and 4.4.4). The Cauchy property was first used for the Riemann integral to integrate functions unbounded in the neighbor-

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hood of a finite number of points (see, e.g., [15], Theorems 2.12–2.16). A similar idea has also been applied to integrate in a Lebesgue sense functions not summable in the neighborhood of some points. Based on the Cauchy property, Harnack suggested a method to calculate the integrals of functions defined on an open set. The Cauchy property in the integral theory presents a sufficient condition for the integrable functions on every $[c,d] \subset (a,b)$ to be integrable on $[a,b]$ (see, e.g., [19], [26]–[28]). In the setting of the Kurzweil-Henstock integral for real-valued functions, the Harnack extension principle reads as follows (see, e.g., [13], Theorem 9.22, [26], Corollary 7.11, and [28], Theorem 4.4.4):

Theorem 1.1. *Let $T \subset [a,b]$ be a closed set and let $\{[a_i,b_i]\colon i \in \mathbb{N}\}$ be a collection of pairwise disjoint intervals such that $(a,b) \setminus T = \bigcup\limits_{i=1}^{\infty} (a_i,b_i)$. Then, if g is a real-valued function and the Kurzweil-Henstock integrals $\int_a^b g\chi_T \,\mathrm{d}t$ and $\int_{a_i}^{b_i} g \,\mathrm{d}t$ exist for all $i \in \mathbb{N}$ and the series*

$$\sum_{i=1}^{\infty} \sup\biggl\{\biggl|\int_r^t g \,\mathrm{d}t\biggr|\colon a_i \leqslant r \leqslant t \leqslant b_i\biggr\}$$

converges, then the Kurzweil-Henstock integral $\int_a^b g \,\mathrm{d}t$ exists and

$$\int_a^b g \,\mathrm{d}t = \int_a^b g\chi_T \,\mathrm{d}t + \sum_{i=1}^{\infty} \int_{a_i}^{b_i} g \,\mathrm{d}t. \tag{1.1}$$

Note that an analogous result to Theorem 1.1 can be obtained also for the Kurzweil-Henstock integrable real-valued functions defined on measure spaces endowed with locally compact metric topologies, see, e.g., [36], Theorem 5.1.

The definition of Kurzweil-Henstock (or Henstock-Kurzweil) integral based on Riemannian-type sums and refinements controlled by gauges (see, e.g., [10], [20], [23], [40]) leads to a non-absolutely convergent integral that is more powerful than the Lebesgue integral, and also contains a special case, i.e., Stieltjes-type integrals. Throughout this paper, we work with the Kurzweil-Stieltjes integrals for Banach space-valued functions. The simplest integral of this type is the Riemann-Stieltjes integral of the form $\int_a^b [\mathrm{d}f]\, g$, in which a function $g\colon [a,b] \to \mathbb{R}$ called the integrand is integrated with respect to another function $f\colon [a,b] \to \mathbb{R}$ referred to as the integrator (see, e.g., [13], [33], [37], [49]). This integral appeared for the first time in a famous treatise [49] by Stieltjes. Up to now, many authors have considered various kinds of Stieltjes integrals using the gauge integration (see, e.g., [14], [17], [33], [34], [37], [42], [51]), which have become highly popular in the fields of differential and integral equations and other applications such as in the finite element method, approximation of the Fourier transform having implications in digital image processing, economic estimates, and acoustic phonetics, integral equations to game theory, financial market modeling, etc. (see, e.g., [1], [5], [8], [10], [18], [29], [39], [43]). In the literature, these integrals are known under several different names (e.g., Henstock-Stieltjes, Perron-Stieltjes, and generalized Riemann-Stieltjes). All of these integrals are special cases of the Kurzweil integral referred from [21] or [24]. Therefore, we prefer to call this integral the *Kurzweil-Stieltjes integral*.

The Kurzweil-Henstock integral has been generalized in various ways. For instance, Cao (see [6]) noticed that Kurzweil's definition can be easily extended to functions with values in Banach spaces and investigated some of properties of the abstract Kurzweil-Henstock integral. This abstract Kurzweil-Henstock integral received further attention, such as the monograph by Schwabik and Ye (see [47]) that discusses these types of integrals, i.e., the McShane, Bochner, Dunford, and Pettis integrals for Banach space-valued functions, and compares the relationship between these various integrals. Moreover, the fundamental results concerning the Kurzweil-Stieltjes integral for Banach space-valued functions were given by Schwabik in [42] and [44], where he called it the *abstract Perron-Stieltjes integral*. The results obtained by Schwabik have been extended by Monteiro and Tvrdý, cf. [34] and [35], in such a way that they were applicable in proofs of some new results on the continuous dependence of solutions to generalized linear differential equations in a Banach space.

Convergence theorems, that concern the possibility of interchanging the limit and the integral (see, e.g., [2], [12], [14], [16], [27], [37], [45]), provide one of the most widely used tools in theories of integration and of ordinary differential and integral equations. In the theory of the Denjoy-Perron integral, a key step to prove convergence theorems by means of the category argument is the use of Harnack extension principle (see, e.g., [26], page 47, [38], page 253). The further extension of the Kurzweil-Stieltjes integral to the integration over elementary sets, i.e., sets that are finite unions of bounded intervals, was presented by Monteiro, Hanung, and Tvrdý in [32], Section 5, where it was a useful ingredient for proving the bounded convergence theorem for the abstract Kurzweil-Stieltjes integral which was further applied to the linear Stieltjes differential and integral equations, dynamic equations, etc. (see, e.g., [7], [30], [48]). However, based on [32], Theorems 5.8–5.10, Remark 5.12, Theorem 5.13, the Harnack extension principle for the Kurzweil-Henstock integral, see, e.g., Theorem 1.1, cannot be valid any longer for the Kurzweil-Stieltjes integral as, whenever the integrator F is not continuous on $[a,b]$, for a subinterval $J \subset [a,b]$ having an infimum and a supremum c and d, respectively, the integrals

$$(1.2)\qquad \int_c^d [\mathrm{d}F]\,g, \quad \int_{[c,d]} [\mathrm{d}F]\,g, \quad \int_{[c,d)} [\mathrm{d}F]\,g, \quad \int_{(c,d]} [\mathrm{d}F]\,g, \quad \int_{(c,d)} [\mathrm{d}F]\,g$$

need not have the same values, even if they all exist (see [32], Remark 5.12). As a consequence, the Harnack extension principle for the Kurzweil-Henstock integral cannot be easily extended to the setting of Stieltjes-type integrals.

From the study of convergence theorems for gauge-type integrals, the notion of equiintegrability appeared, whose idea is that there exists a single gauge δ that works for all the functions in a sequence (see, e.g., [4], [12], [25], [31], [33], [46], [47]). Aside from extending the definition of the Kurzweil-Stieltjes integral over arbitrary bounded sets equipped with all results formulated for the setting of this manuscript with a general bilinear triple (see Section 3), to deal with the Harnack extension principle for the Kurzweil-Stieltjes integral based on [32], it is necessary to develop the notion of equiintegrability touching the sequences of integrands and integrators for Banach space-valued functions, which is a new convergence theorem for the Kurzweil-Stieltjes integral, and investigate its fundamental properties including some notable results regarding equiregulatedness involving elementary sets, as presented in Section 4. Furthermore, the theory in Sections 3 and 4 leads to a new Harnack extension principle for the Kurzweil-Stieltjes integral, which significantly improves the results from [32]. Meanwhile, in general, the existence of the integral $\int_E [\mathrm{d}F]\, g$ does not (even in the case of the identity integrator $F(x) := x$ and $E = [a, b]$) always imply the existence of the integral $\int_T [\mathrm{d}F]\, g$ for every subset T of E. The goal of this paper is to provide sufficient conditions vouching the Harnack extension principle for the Kurzweil-Stieltjes integral with values in Banach spaces and then to show that it plays an important role in guaranteeing the existence of integration over arbitrary subsets of an elementary set, as shown in Section 5.

2. Preliminaries

In this section we recall some terminologies and notations commonly used in the literature.

Let X, Y, and Z be Banach spaces. The symbols $\|\cdot\|_X$, $\|\cdot\|_Y$, and $\|\cdot\|_Z$ stand for the norm in X, Y, and Z, respectively. If there are bilinear mapping $B\colon X \times Y \to Z$ and $\beta \in [0, \infty)$ such that

$$\|B(x, y)\|_Z \leqslant \beta \|x\|_X \|y\|_Y \quad \text{for } x \in X,\ y \in Y,$$

then the triple (X, Y, Z) is a *bilinear triple* with respect to B. In such a case, we write $\mathcal{B} = (X, Y, Z)$ and use the abbreviation xy for $B(x, y)$. Besides a classical situation with $X = Y = Z = \mathbb{R}$, a typical nontrivial example is, e.g., $\mathcal{B} = (\mathcal{L}(X, Z), X, Z)$, where $\mathcal{L}(X, Z)$ is the space of all linear bounded operators $L\colon X \to Z$, whereas $B(L, x) = Lx \in Z$ for $x \in X$ and $L \in \mathcal{L}(X, Z)$. Clearly, without any loss of generality, we may assume that $\beta = 1$.

Two intervals in $\mathbb{R}$ are said to be *disjoint* if their intersection is empty, whereas they are said to be *non-overlapping* if their intersection contains at most one point. In this study, by an *elementary set*, we understand a finite union of mutually disjoint bounded intervals. Note that bounded intervals are themselves elementary sets.

A finite set $\boldsymbol{\alpha} = \{\alpha_0, \alpha_1, \ldots, \alpha_m\} \subset [a,b]$ with $m \in \mathbb{N}$ is said to be a *division* of the interval $[a,b]$ if

$$a = \alpha_0 < \alpha_1 < \ldots < \alpha_m = b.$$

The set of all divisions of $[a,b]$ is denoted by $\mathcal{D}[a,b]$. The symbol $\nu(\boldsymbol{\alpha})$ is kept for the number of subintervals $[\alpha_{j-1}, \alpha_j]$ generated by the division $\boldsymbol{\alpha}$, i.e., $\nu(\boldsymbol{\alpha}) = m$ in the above case.

Let $f\colon [a,b] \to X$ be a function with values in a Banach space X. As in the case of the real-valued functions, the *variation* of f on $[a,b]$ is defined by

$$\operatorname*{var}_a^b f = \sup_{\boldsymbol{\alpha} \in \mathcal{D}[a,b]} \sum_{j=1}^{\nu(\boldsymbol{\alpha})} \|f(\alpha_j) - f(\alpha_{j-1})\|_X.$$

If $\operatorname{var}_a^b f < \infty$, then f has a bounded variation on $[a,b]$. $\mathrm{BV}([a,b],X)$ is the set of all functions $f\colon [a,b] \to X$ of a bounded variation on $[a,b]$.

Let $\mathcal{B} = (X,Y,Z)$ be a bilinear triple. For a division $\boldsymbol{\alpha} = \{\alpha_0, \alpha_1, \ldots, \nu(\boldsymbol{\alpha})\}$ of $[a,b]$ and a function $f\colon [a,b] \to X$ we put

$$(\mathcal{B})V_a^b(f,\boldsymbol{\alpha}) := \sup\Biggl\{\Biggl\| \sum_{j=1}^{\nu(\boldsymbol{\alpha})} [f(\alpha_j) - f(\alpha_{j-1})]y_j \Biggr\|_Z : \\ y_j \in Y,\ \|y_j\|_Y \leqslant 1,\ j \in \{1\ldots,\nu(\boldsymbol{\alpha})\}\Biggr\}$$

and

$$(\mathcal{B}) \operatorname*{var}_a^b f = \sup\{(\mathcal{B})V_a^b(f,\boldsymbol{\alpha})\colon\ \boldsymbol{\alpha} \in \mathcal{D}[a,b]\}.$$

A function $f\colon [a,b] \to X$ with $(\mathcal{B})\operatorname{var}_a^b(f) < \infty$ is said to have a *bounded $\mathcal{B}$-variation* on $[a,b]$ or a *bounded semi-variation*. The set of all functions $f\colon [a,b] \to X$ with bounded $\mathcal{B}$-variation on $[a,b]$ is denoted by $(\mathcal{B})\mathrm{BV}([a,b],X)$; $G([a,b],X)$ denotes the set of all X-valued functions which are regulated on $[a,b]$. Recall that $f\colon [a,b] \to X$ is *regulated* on $[a,b]$ if for any $t \in [a,b)$ there is a $f(t^+) \in X$ such that

$$\lim_{s \to t^+} \|f(s) - f(t^+)\|_X = 0,$$

and for any $t \in (a,b]$, there is a $f(t^-) \in X$ such that

$$\lim_{s \to t^-} \|f(s) - f(t^-)\|_X = 0.$$

For $f \in G([a,b],X)$ and $t \in [a,b]$, we put $\Delta^+ f(t) = f(t^+) - f(t)$, $\Delta^- f(t) = f(t) - f(t^-)$ and $\Delta f(t) = f(t^+) - f(t^-)$ (where by convention $\Delta^- f(a) = \Delta^+ f(b) = 0$).

Let $\mathcal{B} = (X,Y,Z)$ be a bilinear triple. A function $f\colon [a,b] \to X$ is called *$\mathcal{B}$-regulated* on $[a,b]$ (or *simply-regulated* on $[a,b]$) if the function $fy\colon t \in [a,b] \to f(t)y \in Z$ is regulated for all $y \in Y$. The set of all simply-regulated functions $f\colon [a,b] \to X$ is denoted by $(\mathcal{B})G([a,b],X)$. Clearly, $G([a,b],X) \subset (\mathcal{B})G([a,b],X)$. Moreover,

$$\mathrm{BV}([a,b],X) \subset G([a,b],X) \quad \text{and} \quad \mathrm{BV}([a,b],X) \subset (\mathcal{B})\mathrm{BV}([a,b],X).$$

A finite set of points in $[a,b]$

$$P = \{\alpha_0, \xi_1, \alpha_1, \xi_2, \ldots, \alpha_{m-1}, \xi_m, \alpha_m\}$$

where $\{\alpha_0, \alpha_1, \ldots, \alpha_m\} \in \mathcal{D}[a,b]$ and $\xi_j \in [\alpha_{j-1}, \alpha_j]$ for $j = 1, 2, \ldots, \nu(P)$ is called a *tagged partition* of $[a,b]$. The point ξ_j is called the *tag* of the subinterval $[\alpha_{j-1}, \alpha_j]$ for every $j = 1, 2, \ldots, \nu(P)$. We then write

$$P = \{([\alpha_{j-1}, \alpha_j], \xi_j)\} \quad \text{or } P = (\boldsymbol{\alpha}, \boldsymbol{\xi})$$

with $\boldsymbol{\alpha} = \{\alpha_0, \alpha_1, \ldots, \alpha_m\}$, $\boldsymbol{\xi} = \{\xi_1, \xi_1, \ldots, \xi_m\}$ and $\nu(P) = \nu(\boldsymbol{\alpha})$.

Positive functions $\delta\colon [a,b] \to (0,\infty)$ are called *gauges* on $[a,b]$. For a given gauge δ on $[a,b]$, a tagged partition $P = \{([\alpha_{j-1}, \alpha_j], \xi_j)\}$ of $[a,b]$ is called δ-fine if

$$[\alpha_{j-1}, \alpha_j] \subset (\xi_j - \delta(\xi_j), \xi_j + \delta(\xi_j)) \quad \text{for } j = 1, 2, \ldots, \nu(P).$$

If $\mathcal{B} = (X,Y,Z)$ is a bilinear triple, then for functions $f\colon [a,b] \to X$, $g\colon [a,b] \to Y$ and a tagged partition $P = \{([\alpha_{j-1}, \alpha_j], \xi_j)\}$ of $[a,b]$, we set

$$S(\mathrm{d}f, g, P) = \sum_{j=1}^{\nu(P)} [f(\alpha_j) - f(\alpha_{j-1})] g(\xi_j)$$

and

$$S(f, \mathrm{d}g, P) = \sum_{j=1}^{\nu(P)} f(\xi_j) [g(\alpha_j) - g(\alpha_{j-1})].$$

Now, we can present the definition of the abstract Kurzweil-Stieltjes integral as introduced by Schwabik in [42], Definition 5.

Definition 2.1. Let $\mathcal{B} = (X, Y, Z)$ be a bilinear triple and let $f\colon [a,b] \to X$ and $g\colon [a,b] \to Y$ be given. We say that the Kurzweil-Stieltjes integral (shortly KS-integral) $\int_a^b [\mathrm{d}f]\, g$ exists if there is $I \in Z$ such that for every $\varepsilon > 0$ there is a gauge δ on $[a,b]$ such that

$$\|S(\mathrm{d}f, g, P) - I\|_Z < \varepsilon \tag{2.1}$$

holds for every δ-fine tagged partition $P = (\boldsymbol{\alpha}, \boldsymbol{\xi})$ of $[a,b]$. In such a case, we put

$$\int_a^b [\mathrm{d}f]\, g = I.$$

Furthermore, we put

$$\int_a^a [\mathrm{d}f]\, g = 0 \quad \text{and} \quad \int_b^a [\mathrm{d}f]\, g = -\int_a^b [\mathrm{d}f]\, g \quad \text{if } a < b.$$

Similarly, if $f\colon [a,b] \to X$ and $g\colon [a,b] \to Y$, then $\int_a^b f\, [\mathrm{d}g] = I \in Z$ if and only if for every $\varepsilon > 0$ there is a gauge δ on $[a,b]$ such that

$$\|S(f, \mathrm{d}g, P) - I\|_Z < \varepsilon \tag{2.2}$$

holds for every δ-fine tagged partition $P = (\boldsymbol{\alpha}, \boldsymbol{\xi})$ of $[a,b]$.

Clearly, Definition 2.1 can be reasonable only if for any gauge δ on $[a,b]$ the set of δ-fine partitions of $[a,b]$ is nonempty. This crucial question is answered by the following lemma which is known as the Cousin lemma (see, e.g., [9], [13], Lemma 9.2, [26], Theorem 2.3.1, [28], Theorem 1.1.5).

Lemma 2.2 (Cousin). *Given an arbitrary gauge δ on $[a,b]$, there is a δ-fine tagged partition of $[a,b]$.*

Remark 2.3. Evidently, the Kurzweil-Stieltjes integral reduces to the Kurzweil-Henstock integral whenever the integrator f in (2.1) and the integrator g in (2.2) are the identity functions.

Throughout the paper, we assume that $\mathcal{B} = (X, Y, Z)$ is a bilinear triple. Furthermore, $[a,b]$ is a fixed bounded and closed interval in $\mathbb{R}$. All functions f are supposed to be defined on the entire interval $[a,b]$ and extended outside the interval $[a,b]$ in such a way that $f(t) = f(a)$ and $f(s) = f(b)$ for $t < a$ and $s > b$.

3. Integration over arbitrary bounded sets

In [32], Section 5, the Kurzweil-Stieltjes integral of operator-valued functions over elementary subsets of $[a, b]$ was introduced, and its basic properties were described. This definition can be easily extended to arbitrary subsets of $[a, b]$ and to setting in a general bilinear triple $\mathcal{B} = (X, Y, Z)$.

Definition 3.1. Let $f\colon [a, b] \to X$, $g\colon [a, b] \to Y$ and let S be an arbitrary subset of $[a, b]$. Then, the Kurzweil-Stieltjes integral (shortly KS-integral or integral) of g with respect to f over the set S, denoted by $\int_S [\mathrm{d}f]\, g$, is defined by

$$\int_S [\mathrm{d}f]\, g := \int_a^b [\mathrm{d}f]\, (g\chi_S)$$

whenever the integral on the right-hand side exists.

Similarly, if $f\colon [a, b] \to X$, $g\colon [a, b] \to Y$, then the integral $\int_S f\, [\mathrm{d}g]$ is defined by

$$\int_S f\, [\mathrm{d}g] := \int_a^b (f\chi_S)\, [\mathrm{d}g]$$

whenever the integral on the right-hand side exists.

Remark 3.2. By Definitions 2.1 and 3.1, the existence of the integral $\int_S [\mathrm{d}f]\, g$ means that there exists $I \in Z$ with the following property: for every $\varepsilon > 0$ there exists a gauge δ on $[a, b]$ such that

$$\|S(\mathrm{d}f, g\chi_S, P) - I\|_Z < \varepsilon$$

whenever $P = (\boldsymbol{\alpha}, \boldsymbol{\xi})$ is a δ-fine tagged partition of $[a, b]$.

Definition 5.1 from [32] is a special case of Definition 3.1. However, all the results presented in [32] for the special case $\mathcal{B} = (\mathcal{L}(X, Z), X, Z)$ can be reformulated for the setting of this paper with a general bilinear triple $\mathcal{B} = (X, Y, Z)$. In particular, Propositions 3.3 and 3.5 are valid.

Proposition 3.3. *Let S be an arbitrary subset of $[a, b]$. Then, the following assertions are true:*

(i) *Let $f\colon [a, b] \to X$ and $g_i\colon [a, b] \to Y$, $i = 1, 2$, such that the integrals $\int_S [\mathrm{d}f]\, g_i$ for $i = 1, 2$ exist. Then, the integral $\int_S [\mathrm{d}f](c_1 g_1 + c_2 g_2)$ also exists, and*

$$\int_S [\mathrm{d}f](c_1 g_1 + c_2 g_2) = c_1 \int_S [\mathrm{d}f]\, g_1 + c_2 \int_S [\mathrm{d}f]\, g_2 \quad \forall\, c_1, c_2 \in \mathbb{R}.$$

(ii) *Let $f_i\colon [a, b] \to X$, $i = 1, 2$, and $g\colon [a, b] \to Y$, such that the integrals $\int_S [\mathrm{d}f_i]\, g$ for $i = 1, 2$ exist. Then, the integral $\int_S [\mathrm{d}(c_1 f_1 + c_2 f_2)]\, g$ also exists, and*

$$\int_S [\mathrm{d}(c_1 f_1 + c_2 f_2)]\, g = c_1 \int_S [\mathrm{d}f_1]\, g + c_2 \int_S [\mathrm{d}f_2]\, g \quad \forall\, c_1, c_2 \in \mathbb{R}.$$

R e m a r k 3.4. Let $f\colon [a,b]\to X$ be $\mathcal{B}$-regulated on $[a,b]$. As $g\chi_{[a,b]}=g$ on $[a,b]$, the integral $\int_a^b[\mathrm{d}f]\,g$ exists if and only if the integral $\int_{[a,b]}[\mathrm{d}f]\,g$ exists. In such a case, these integrals have the same value, i.e.,

$$\int_{[a,b]}[\mathrm{d}f]\,g=\int_a^b[\mathrm{d}f]\,g. \tag{3.1}$$

Meanwhile,

$$g(t)\chi_{(a,b)}(t)-g(t)=\begin{cases}-g(a) & \text{if } t=a,\\ 0 & \text{if } t\in(a,b),\\ -g(b) & \text{if } t=b,\end{cases}$$

and hence, by [42], Lemma 12, we get for an arbitrary $d\in(a,b)$

$$\begin{aligned}\int_a^b[\mathrm{d}f](g\chi_{(a,b)}-g)&=\int_a^d[\mathrm{d}f](g\chi_{(a,b)}-g)+\int_d^b[\mathrm{d}f](g\chi_{(a,b)}-g)\\&=-\Bigl(\lim_{r\to a^+}[f(r)g(a)]-f(a)g(a)\Bigr)-\Bigl(f(b)g(b)-\lim_{r\to b^-}[f(r)g(b)]\Bigr),\end{aligned}$$

i.e., the integral $\int_a^b[\mathrm{d}f]\,g$ exists if and only if the integral $\int_{(a,b)}[\mathrm{d}f]\,g$ exists, and in such a case,

$$\begin{aligned}\int_{(a,b)}[\mathrm{d}f]\,g=\int_a^b[\mathrm{d}f]\,g+f(a)g(a)-f(b)g(b)\\+\lim_{r\to b^-}[f(r)g(b)]-\lim_{r\to a^+}[f(r)g(a)].\end{aligned} \tag{3.2}$$

The next proposition summarizes the properties of the KS-integral over all possible kinds of subintervals of $[a,b]$. The proofs of its assertions are easy modifications of those of [32], Theorems 5.8, 5.10, and 5.11. The above observations concerning the cases $c=a$ and/or $d=b$ will be included, considering the convention that the functions f and g are to be considered extended outside of the interval $[a,b]$ as constant functions on $(-\infty,a]\cup[b,\infty)$.

Proposition 3.5. *Let $f\in(\mathcal{B})G([a,b];X)$, $g\colon [a,b]\to Y$, and $a\leqslant c<d\leqslant b$. Then, the following assertions are true*:

(i) *The integral $\int_{(c,d)}[\mathrm{d}f]\,g$ exists if and only if the integral $\int_c^d[\mathrm{d}f]\,g$ exists. In such a case,*

$$\int_{(c,d)}[\mathrm{d}f]\,g=f(c)g(c)-\lim_{r\to c^+}[f(r)g(c)]+\int_c^d[\mathrm{d}f]\,g-f(d)g(d)+\lim_{r\to d^-}[f(r)g(d)].$$

(ii) *The integral* $\int_{[c,d)}[\mathrm{d}f]\,g$ *exists if and only if the integral* $\int_c^d[\mathrm{d}f]\,g$ *exists. In such a case,*

$$\int_{[c,d)}[\mathrm{d}f]\,g = f(c)g(c) - \lim_{r\to c^-}[f(r)g(c)] + \int_c^d[\mathrm{d}f]\,g - f(d)g(d) - \lim_{r\to d^-}[f(r)g(d)].$$

(iii) *The integral* $\int_{(c,d]}[\mathrm{d}f]\,g$ *exists if and only if the integral* $\int_c^d[\mathrm{d}f]\,g$ *exists. In such a case,*

$$\int_{(c,d]}[\mathrm{d}f]\,g = f(c)g(c) - \lim_{r\to c^+}[f(r)g(c)] + \int_c^d[\mathrm{d}f]\,g + \lim_{r\to d^+}[f(r)g(d)] - f(d)g(d).$$

(iv) *The integral* $\int_{[c,d]}[\mathrm{d}f]\,g$ *exists if and only if the integral* $\int_c^d[\mathrm{d}f]\,g$ *exists. In such a case,*

$$\int_{[c,d]}[\mathrm{d}f]\,g = f(c)g(c) - \lim_{r\to c^-}[f(r)g(c)] + \int_c^d[\mathrm{d}f]\,g + \lim_{r\to d^+}[f(r)g(d)] - f(d)g(d).$$

Remark 3.6. If $a \leqslant c < d \leqslant b$, $f \in (\mathcal{B})G([a,b];X)$, and $g\colon [a,b] \to Y$, then Proposition 3.5 implies that if any one of the integrals

$$\int_{(c,d)}[\mathrm{d}f]\,g,\quad \int_{[c,d)}[\mathrm{d}f]\,g,\quad \int_{(c,d]}[\mathrm{d}f]\,g,\quad \int_{[c,d]}[\mathrm{d}f]\,g,\quad \int_c^d[\mathrm{d}f]\,g \tag{3.3}$$

exists, then all the others exist as well. Of course, their values can differ, in general. If, in addition, f is continuous on $[a,b]$, then all the equalities

$$\int_{(c,d)}[\mathrm{d}f]\,g = \int_{[c,d)}[\mathrm{d}f]\,g = \int_{(c,d]}[\mathrm{d}f]\,g = \int_{[c,d]}[\mathrm{d}f]\,g = \int_c^d[\mathrm{d}f]\,g \tag{3.4}$$

are true.

Remark 3.7. The existence of the integral $\int_a^b[\mathrm{d}f]\,g$ need not (even in the case of the identity integrator $f(x) := x$) always imply the existence of the integral $\int_T[\mathrm{d}f]\,g$ for every subset T of $[a,b]$. This is demonstrated by the following example (cf. [22] or [26]): Let $g\colon [0,1] \to \mathbb{R}$ be defined by

$$g(t) = \begin{cases} 0 & \text{if } t = 0,\\ 2t\cos\dfrac{\pi}{t^2} + \dfrac{2\pi}{t}\sin\dfrac{\pi}{t^2} & \text{if } 0 < t \leqslant 1. \end{cases}$$

Then, g is Kurzweil-Henstock integrable on $[0,1]$. However, if $T{:=}\{t \in [0,1]\colon g(t) \geqslant 0\}$, then $\int_T g\,\mathrm{d}t$ does not exist.

As shown by the next assertion, this cannot happen when we restrict ourselves to elementary subsets of $[a,b]$.

The next definition is taken from [32], Definition 4.9.

Definition 3.8. A set $\{J_k\colon\ k=1,\dots,p\}$ of bounded intervals is said to be a minimal decomposition of the elementary set $E\subset[a,b]$ if $\bigcup\limits_{k=1}^{p} J_k=E$, while the union $J_k\cup J_l$ is not an interval whenever $k\neq l$. We will always assume that the ordering of the minimal decomposition is such that $\sup J_k\leqslant\inf J_l$ for $k\leqslant l$.

Remark 3.9. It is easy to see that if $\{J_k\colon\ k=1,\dots,p\}$ is a minimal decomposition of some elementary sets, then $\overline{J_k}\cap\overline{J_l}\neq\emptyset$ only if $l=k+1$ and $\sup J_k=\inf J_l\notin J_k\cap J_l$.

Proposition 3.10. *The following assertions are true for all* $f\in(\mathcal{B})G([a,b];X)$ *and* $g\colon\ [a,b]\to Y$.

(i) *Let* E *be an elementary subset of* $[a,b]$ *such that the integral* $\int_E[\mathrm{d}f]\,g$ *exists. Then, the integral* $\int_T[\mathrm{d}f]\,g$ *exists for every elementary subset* T *of* E.

(ii) *Let* $E=\bigcup\limits_{k=1}^{p} J_k$, *where* $\{J_k\colon\ k=1,2,\dots,p\}$ *are mutually disjoint subintervals of* $[a,b]$, *and let the integral* $\int_E[\mathrm{d}f]\,g$ *exist. Then, all the integrals*

$$\int_{J_k}[\mathrm{d}f]\,g,\quad k=1,2,\dots,p,$$

exist as well and

$$\int_E[\mathrm{d}f]\,g=\sum_{k=1}^{p}\int_{J_k}[\mathrm{d}f]\,g. \tag{3.5}$$

Proof. (i) See [32], Corollary 5.15.

(ii) By [32], Theorem 5.13, this assertion is true if the set $\{J_k\colon\ k=1,2,\dots,p\}$ is a minimal decomposition of E. In a general case, we may assume that the intervals $\{J_k\}$ are ordered in such a way that $x\leqslant y$ holds whenever $x\in J_k$, $y\in J_l$ and $k<l$. Then, if $\{J_k\colon\ k=1,2,\dots,p\}$ is not a minimal decomposition, there must exist $k\in\{1,2,\dots,p-1\}$ such that $J=J_k\cup J_{k+1}$ is an interval. Then, as $J_k\cap J_{k+1}=\emptyset$, we get

$$\int_{J_k}[\mathrm{d}f]\,g+\int_{J_{k+1}}[\mathrm{d}f]\,g=\int_a^b[\mathrm{d}f]\,g(\chi_{J_k}+\chi_{J_{k+1}})=\int_a^b[\mathrm{d}f](g\chi_J)=\int_J[\mathrm{d}f]\,g.$$

Hence, when we replace all such couples in the sum on the right-hand side of (3.5) by their unions, we get the sum over a minimal decomposition of E, while the sum itself does not change. This completes the proof. □

R e m a r k 3.11. Let $E = \bigcup_{k=1}^{p} J_k$ be an elementary subset of $[a,b]$.

(i) Let $\{J_k^*\colon\ k = 1,2,\ldots,p^*\}$ be the minimal decomposition of E. Then, Proposition 3.10 (ii) implies that $p^* \leqslant p$ and

$$\int_E [\mathrm{d}f]\, g = \sum_{k=1}^{p} \int_{J_k} [\mathrm{d}f]\, g = \sum_{k=1}^{p^*} \int_{J_k^*} [\mathrm{d}f]\, g.$$

(ii) Let c_k and d_k be the infimum and the supremum of J_k, respectively, for every $k = 1,2,\ldots,p$. Then, from Remark 3.6 with Proposition 3.10 (ii), the equality

$$\int_E [\mathrm{d}f]\, g = \sum_{k=1}^{p} \int_{J_k} [\mathrm{d}f]\, g = \sum_{k=1}^{p} \int_{c_k}^{d_k} [\mathrm{d}f]\, g \tag{3.6}$$

holds for all continuous integrators f, especially for the Kurzweil-Henstock integral. However, (3.6) is no longer valid for the KS-integral, in general.

R e m a r k 3.12. If a function $g\colon\ [a,b] \to Y$ and a subset S of $[a,b]$ are such that $g = 0$ on S, then $g\chi_S = 0$ on $[a,b]$, and hence $\int_S [\mathrm{d}f]\, g = \int_a^b [\mathrm{d}f]\,(g\chi_S) = 0$ and $\int_T [\mathrm{d}f]\, g = 0$ as well for every subset T of S and every $f\colon\ [a,b] \to X$. In particular, if the integral $\int_S [\mathrm{d}f]\, g$ exists and $h\colon\ [a,b] \to Y$ coincides with g on S, then $\int_S [\mathrm{d}f]\, h = \int_S [\mathrm{d}f]\, g$.

The next assertion discloses the additivity properties of the KS-integral over arbitrary subsets of $[a,b]$.

Proposition 3.13. *Let $f\colon\ [a,b] \to X$, $g\colon\ [a,b] \to Y$ and subsets S_1, S_2 in $[a,b]$ be given. Then, whenever three of the integrals*

$$\int_{S_1} [\mathrm{d}f]\, g, \quad \int_{S_2} [\mathrm{d}f]\, g, \quad \int_{S_1\cup S_2} [\mathrm{d}f]\, g, \quad \int_{S_1\cap S_2} [\mathrm{d}f]\, g$$

exist, then there also exists the remaining one and

$$\int_{S_1} [\mathrm{d}f]\, g + \int_{S_2} [\mathrm{d}f]\, g = \int_{S_1\cup S_2} [\mathrm{d}f]\, g + \int_{S_1\cap S_2} [\mathrm{d}f]\, g.$$

P r o o f. It follows directly from the identity $\chi_{S_1} + \chi_{S_2} = \chi_{S_1\cup S_2} + \chi_{S_1\cap S_2}$. $\square$

Proposition 3.14. *Let $f\colon [a,b] \to X$, $g\colon [a,b] \to Y$, and subsets $S_1, \dots, S_n$ in $[a,b]$ be such that $S_j \cap S_k = \emptyset$ for $j \neq k$ and the integral $\int_{S_j} [\mathrm{d}f]\, g$ exists for each $j \in \{1, \dots, n\}$. Put $S = \bigcup\limits_{j=1}^{n} S_j$. Then, the integral $\int_S [\mathrm{d}f]\, g$ exists and*

$$\int_S [\mathrm{d}f]\, g = \sum_{j=1}^{n} \int_{S_j} [\mathrm{d}f]\, g.$$

P r o o f. Follows directly from the identity $\chi_S = \sum\limits_{j=1}^{n} \chi_{S_j}$. $\square$

Proposition 3.15. *Let $f\colon [a,b] \to X$ and $g\colon [a,b] \to Y$ be given and let $S_1, \dots, S_p$ be subsets of $[a,b]$ and $p \geqslant 2$. Put*

$$S = \bigcup_{j=1}^{p} S_j \quad \text{and} \quad T_i = \Big(\bigcup_{j=1}^{i-1} S_j\Big) \cap S_i \quad \text{for } i = 2, 3, \dots, p$$

and assume that all the integrals

$$\int_{S_i} [\mathrm{d}f]\, g, \quad \int_{T_i} [\mathrm{d}f]\, g, \quad i = 1, 2, \dots, p,$$

exist. Then, the integral

$$\int_S [\mathrm{d}f]\, g$$

exists as well and

$$\int_S [\mathrm{d}f]\, g = \sum_{i=1}^{p} \int_{S_i} [\mathrm{d}f]\, g - \sum_{i=2}^{p} \int_{T_i} [\mathrm{d}f]\, g. \tag{3.7}$$

P r o o f. Observe that

$$S = S_1 \cup (S_2 \setminus S_1) \cup \bigcup_{i=3}^{p} \Big(S_i \setminus S_i \cap \bigcup_{j=1}^{i-1} S_j\Big) = \bigcup_{i=1}^{p} S_i \setminus \bigcup_{i=2}^{p} T_i$$

and, hence,

$$\chi_S = \sum_{i=1}^{p} \chi_{S_i} - \sum_{i=2}^{p} \chi_{T_i}$$

wherefrom our assertion follows. $\square$

Convergence theorems belong to the most important topics discussed in the frames of integration theory (see, e.g., [2], [14], [32], [42]). For the abstract KS-integral, the uniform convergence theorem given by Schwabik in [42], Theorem 11, is the simplest one. It states that if the sequence of integrands $\{g_n\}$ tends uniformly to g on $[a,b]$ and if all the integrals $\int_a^b [\mathrm{d}f]\, g_n$, $n \in \mathbb{N}$, exist, then the integral $\int_a^b [\mathrm{d}f]\, g$ exists as well and

$$\int_a^b [\mathrm{d}f]\, g = \lim_{n\to\infty} \int_a^b [\mathrm{d}f]\, g_n.$$

When the uniform convergence of the sequence of integrands $\{g_n\}$ to g is replaced by a just pointwise convergence on $[a,b]$, the situation is more difficult. One possible way is indicated by Monteiro, Hanung, and Tvrdý by means of the bounded convergence theorem for the abstract KS-integral in [32], Theorem 6.3, which requires the uniform boundedness of the sequence $\{g_n\}$ on $[a,b]$. Meanwhile, some convergence theorems for the abstract Stieltjes type integrals of Young, Dushnik and Kurzweil, which require a uniform convergence of the sequence of integrators with an integrand of bounded variation or a bounded variation convergence of the sequence of integrators with a bounded integrand, are presented by Hanung and Tvrdý in [14], Theorems 3.5–3.7. The next theorem deals with the case that the sequences of integrands and integrators $\{g_n\}$ and $\{f_n\}$ converge pointwise to g and f, respectively, and need neither boundedness nor bounded variation.

Theorem 4.1 (Equiintegrability convergence theorem). *Let $f_n\colon [a,b] \to X$ and $g_n\colon [a,b] \to Y$ for $n \in \mathbb{N}$ be such that the integral $\int_a^b [\mathrm{d}f_n] g_n$ exists for each $n \in \mathbb{N}$. Furthermore let the functions $f\colon [a,b] \to X$ and $g\colon [a,b] \to Y$ be such that the sequences $\{f_n\}$ and $\{g_n\}$ converge pointwise on $[a,b]$ to f and g, respectively. Finally suppose that*

(C_1) *for every $\eta > 0$ there is a gauge δ on $[a,b]$ such that $\big\|S(\mathrm{d}f_n, g_n, P) - \int_a^b [\mathrm{d}f_n] g_n\big\|_Z < \eta$ for every δ-fine tagged partition $P = (\boldsymbol{\alpha}, \boldsymbol{\xi})$ of $[a,b]$ and every $n \in \mathbb{N}$.*

Then, the integral $\int_a^b [\mathrm{d}f]\, g$ and the limit $\lim\limits_{n\to\infty} \int_a^b [\mathrm{d}f_n]\, g_n$ exist and

$$\int_a^b [\mathrm{d}f]\, g = \lim_{n\to\infty} \int_a^b [\mathrm{d}f_n]\, g_n. \tag{4.1}$$

P r o o f. It is quite analogous to that known for the real-valued case $X = Y = \mathbb{R}$ (cf., e.g., [33], Thorem 6.8.2) and can be omitted. $\square$

Definition 4.2. Let $f_n\colon [a,b]\to X$ and $g_n\colon [a,b]\to Y$ for $n\in\mathbb{N}$.

(i) The sequence $\{g_n\}$ is said to be *equiintegrable with respect to* $\{f_n\}$ *on* $[a,b]$ if the integrals $\int_a^b [\mathrm{d}f_n]\, g_n$ exist for all $n\in\mathbb{N}$ and the condition (C_1) in Theorem 4.1 is satisfied.

(ii) Similarly, if S is an arbitrary subset of $[a,b]$, then the sequence $\{g_n\}$ is *equiintegrable with respect to* $\{f_n\}$ *on* S if $\{g_n\chi_S\}$ is equiintegrable with respect to $\{f_n\}$ on $[a,b]$.

In view of Definition 4.2, Theorem 4.1 may be reformulated as follows:

Theorem 4.1′. *Let $f, f_n\colon [a,b]\to X$ and $g, g_n\colon [a,b]\to Y$, $n\in\mathbb{N}$, be such that $\lim\limits_{n\to\infty} f_n(t)=f(t)$ and $\lim\limits_{n\to\infty} g_n(t)=g(t)$ on $[a,b]$ and suppose that the sequence $\{g_n\}$ is equiintegrable with respect to $\{f_n\}$ on $[a,b]$. Then the integral $\int_a^b[\mathrm{d}f]\,g$ and the limit $\lim\limits_{n\to\infty}\int_a^b[\mathrm{d}f_n]\,g_n$ exist and (4.1) holds.*

Remark 4.3.

(i) Equiintegrability is often met in the literature dealing with the theory of Kurzweil-Henstock integrals (see, e.g., Bartle [3], Chapter 8; Gordon [13], Chapter 13 and [12]; Kurzweil [22], Chapter 5; Kurzweil and Jarník [25]; Schwabik [41], Chapter 1; Schwabik and Vrkoč [46]; Schwabik and Ye [47], Chapter 3). Nonetheless, little is known about the conditions that ensure the equiintegrability for Stieltjes-type integrals for real-valued functions (see [4], [33], Chapter 6, [31]).

(ii) Referring to, e.g., Gordon [13], Definition 13.15 or Schwabik and Vrkoč [46], Remark 6, a sequence $\{g_n\}$ equiintegrable with respect to $\{f_n\}$ on $[a,b]$ can be also called *uniformly integrable* with respect to $\{f_n\}$ on $[a,b]$.

(iii) If $f_n=f$ for all $n\in\mathbb{N}$, then $\{g_n\}$ is called equiintegrable with respect to f on $[a,b]$.

In general, it is rather difficult to verify that the condition (C_1) is satisfied. The following statement at least enables us to decide whether a given sequence $\{g_n\}$ is equiintegrable with respect to $\{f_n\}$ on $[a,b]$ without calculating the values of all the integrals $\int_a^b[\mathrm{d}f_n]g_n$, $n\in\mathbb{N}$.

Theorem 4.4 (Cauchy equiintegrability criterion)**.** *Let $f_n\colon [a,b]\to X$ and $g_n\colon [a,b]\to Y$ for $n\in\mathbb{N}$. Then, the sequence $\{g_n\}$ is equiintegrable with respect to $\{f_n\}$ on $[a,b]$ if and only if*

(C_2) *for every $\varepsilon>0$ there is a gauge δ such that $\|S(\mathrm{d}f_n,g_n,P)-S(\mathrm{d}f_n,g_n,Q)\|_Z<\varepsilon$ holds for all $n\in\mathbb{N}$ and all δ-fine tagged partitions $P=(\boldsymbol{\alpha},\boldsymbol{\xi})$, and $Q=(\boldsymbol{\beta},\boldsymbol{\eta})$ of $[a,b]$.*

P r o o f. (a) Assume that the sequence $\{g_n\}$ is equiintegrable with respect to $\{f_n\}$ on $[a,b]$. Let $\varepsilon > 0$ be given and let δ be an arbitrary gauge corresponding by (C_1) to $\eta = \frac{1}{2}\varepsilon$. Then, for any couple $P = (\boldsymbol{\alpha}, \boldsymbol{\xi})$, $Q = (\boldsymbol{\beta}, \boldsymbol{\eta})$ of the δ-fine tagged partitions of $[a,b]$ and any $n \in \mathbb{N}$, we obtain

$$\begin{aligned}&\|S(\mathrm{d}f_n, g_n, P) - S(\mathrm{d}f_n, g_n, Q)\|_Z \\ &\qquad \leqslant \left\| S(\mathrm{d}f_n, g_n, P) - \int_a^b [\mathrm{d}f_n]\, g_n \right\|_Z + \left\| S(\mathrm{d}f_n, g_n, Q) - \int_a^b [\mathrm{d}f_n]\, g_n \right\|_Z < \varepsilon.\end{aligned}$$

(b) Assume that the condition (C_2) is satisfied. Then, by the Cauchy-Bolzano criterion (see [42], Proposition 7) for the existence of the KS-integral, the integral $\int_a^b [\mathrm{d}f_n]\, g_n$ exists for every $n \in \mathbb{N}$. For a given $n \in \mathbb{N}$, gauge δ and $\varepsilon > 0$, put

$$\mathcal{I}_n(\varepsilon, \delta) = \{S(\mathrm{d}f_n, g_n, P)\colon\ P = (\boldsymbol{\alpha}, \boldsymbol{\xi}) \text{ is a } \delta\text{-fine tagged partition of } [a,b]\}.$$

Due to (C_2), we have

(4.2)
$$\begin{aligned}\operatorname{diam}(\mathcal{I}_n(\varepsilon, \delta)) = \sup\{&\|S(\mathrm{d}f_n, g_n, P) - S(\mathrm{d}f_n, g_n, Q)\|_Z\colon \\ &P = (\boldsymbol{\alpha}, \boldsymbol{\xi}),\ Q = (\boldsymbol{\beta}, \boldsymbol{\eta}) \text{ are } \delta\text{-fine tagged partitions of } [a,b]\} \leqslant \varepsilon.\end{aligned}$$

By Cousin lemma (cf. Lemma 2.2), any $\mathcal{I}_n(\varepsilon, \delta)$ is nonempty and, furthermore,

$$0 < \varepsilon_1 < \varepsilon_2 \Rightarrow \mathcal{I}_n(\varepsilon_1, \delta) \subset \mathcal{I}_n(\varepsilon_2, \delta) \quad \text{for every } n \in \mathbb{N} \text{ and gauge } \delta.$$

Thus, using the Cantor intersection theorem for complete metric spaces (see, e.g., [50], Theorem 5.1.17), we conclude that, for every $n \in \mathbb{N}$, the intersection $\bigcap_{\varepsilon>0} \overline{\mathcal{I}_n(\varepsilon, \delta)}$ is a one-point set $\{I_n\}$ with

$$I_n = \int_a^b [\mathrm{d}f_n]\, g_n \in Z.$$

As a consequence, if an arbitrary $\eta > 0$ and a gauge δ_ε are given, such that (C_2) is true with $\varepsilon = \frac{1}{2}\eta$, then $I_n \in \overline{\mathcal{I}_n(\varepsilon, \delta_\varepsilon)}$ for every $n \in \mathbb{N}$. In particular, considering (4.2), we have

$$\left\| S(\mathrm{d}f_n, g_n, P) - \int_a^b [\mathrm{d}f_n]\, g_n \right\|_Z = \|S(\mathrm{d}f_n, g_n, P) - I_n\|_Z \leqslant \varepsilon < \eta$$

for every δ_ε-fine tagged partition $P = (\boldsymbol{\alpha}, \boldsymbol{\xi})$ of $[a,b]$ and every $n \in \mathbb{N}$. In other words, the sequence $\{g_n\}$ is equiintegrable with respect to $\{f_n\}$ on $[a,b]$. This completes the proof. $\square$

In the special case when S in Definition 4.2 (ii) is a subinterval J of $[a,b]$ we have the following assertion.

Lemma 4.5. *Let J be a subinterval of $[a,b]$, $c=\inf J<d=\sup J$, let $\{f_n\}$ be a sequence in $G([a,b],X)$ and let $\{g_n\}$ be a sequence of functions mapping $[a,b]$ into Y. Furthermore assume that the following assertions are satisfied:*

(A_1) *there is $K\in[0,\infty)$ such that $\|g_n(c)\|_Y\leqslant K$ and $\|g_n(d)\|_Y\leqslant K$ for all $n\in\mathbb{N}$ whenever $c>a$ or $d<b$,*

(A_2) *if $c>a$ and $c\in J$, then for any $\varepsilon>0$ there is a $\Delta_c^->0$ such that $\|f_n(c^-)-f_n(t)\|_X<\varepsilon$ for all $t\in(c-\Delta_c^-,c)$ and $n\in\mathbb{N}$,*

(A_3) *if $c>a$ and $c\notin J$, then for any $\varepsilon>0$ there is a $\Delta_c^+>0$ such that $\|f_n(t)-f_n(c^+)\|_X<\varepsilon$ for all $t\in(c,c+\Delta_c^+)$ and $n\in\mathbb{N}$,*

(A_4) *if $d<b$ and $d\in J$, then for any $\varepsilon>0$ there is a $\Delta_d^+>0$ such that $\|f_n(t)-f_n(d^+)\|_X<\varepsilon$ for all $t\in(d,d+\Delta_d^+)$ and $n\in\mathbb{N}$,*

(A_5) *if $d<b$ and $d\notin J$, then for any $\varepsilon>0$ there is a $\Delta_d^->0$ such that $\|f_n(d^-)-f_n(t)\|_X<\varepsilon$ for all $t\in(d-\Delta_d^-,d)$ and $n\in\mathbb{N}$.*

Then, $\{g_n\}$ is equiintegrable with respect to $\{f_n\}$ on J if and only if

($*$) *the integrals $\int_c^d[\mathrm{d}f_n]\,g_n$ exist for all $n\in\mathbb{N}$ and for every $\varepsilon>0$ there is a gauge $\tilde\delta$ on $[c,d]$ such that $\big\|S(\mathrm{d}f_n,g_n,\widetilde P)-\int_c^d[\mathrm{d}f_n]\,g_n\big\|_Z<\varepsilon$ holds for every $\tilde\delta$-fine tagged partition $\widetilde P$ of $[c,d]$ and every $n\in\mathbb{N}$.*

Proof. (a) Assume that $a<c<d<b$ and $J=[c,d]$. Notice, that by Proposition 3.5 (iv), for a given $n\in\mathbb{N}$ the integral $\int_J[\mathrm{d}f_n]\,g_n$ exists if and only the integral $\int_c^d[\mathrm{d}f_n]\,g_n$ exists. Furthermore,

$$\int_a^b[\mathrm{d}f_n](g_n\chi_{[c,d]})=\int_a^c[\mathrm{d}f_n](g_n\chi_{\{c\}})+\int_c^d[\mathrm{d}f_n]\,g_n+\int_d^b[\mathrm{d}f_n](g_n\chi_{\{d\}}),\tag{4.3}$$

where

$$\int_a^c[\mathrm{d}f_n](g_n\chi_{\{c\}})=\Delta^-f_n(c)g_n(c),\qquad\int_d^b[\mathrm{d}f_n](g_n\chi_{\{d\}})=\Delta^+f_n(d)g_n(d).\tag{4.4}$$

Now, let ($*$) be true and let an arbitrary $\varepsilon>0$ be given. Put

$$\delta(t)=\begin{cases}\min\{\frac14(c-t),1\} & \text{if } t\in[a,c),\\ \min\{\Delta_c^-,\tilde\delta(c)\} & \text{if } t=c,\\ \min\{\frac14(t-c),\frac14(d-t),\tilde\delta(t)\} & \text{if } t\in(c,d),\\ \min\{\Delta_d^+,\tilde\delta(d)\} & \text{if } t=d,\\ \min\{\frac14(t-d),1\} & \text{if } t\in(d,b],\end{cases}$$

where $\tilde{\delta}$ is the gauge on $[c,d]$ from $(*)$, and Δ_c^- and Δ_d^+ are given by the conditions (A_2) and (A_4), respectively. Then (cf. [33], Lemma 6.2.11 and its proof) necessarily $\{c,d\} \subset \{\xi_j\} \cap \{\alpha_j\}$ for any δ-fine partition $P = \{([\alpha_{j-1},\alpha_j],\xi_j)\}$ of $[a,b]$. In particular, there are indices $k,l \in \mathbb{N}$ such that $2 \leqslant k < l \leqslant \nu(P) - 2$, $a < c - \Delta_c^- < \alpha_{k-1} < \xi_k = c = \alpha_k$ and $\xi_l = d = \alpha_l < \alpha_{l+1} < d + \Delta_d^+ < b$. Hence, for any $n \in \mathbb{N}$, we obtain

$$\begin{aligned} S(\mathrm{d}f_n, g_n\chi_{[c,d]}, P) &= (f_n(c) - f_n(\alpha_{k-1}))g_n(c) + [f_n(\alpha_{l+1}) - f_n(d)]g_n(d) \\ &\quad + \sum_{j=k+1}^{l} [f_n(\alpha_j) - f_n(\alpha_{j-1})]g_n(\xi_j) \\ &= \Delta^- f_n(c)g_n(c) + [f_n(c^-) - f_n(\alpha_{k-1})]g_n(c) + \Delta^+ f_n(d)g_n(d) \\ &\quad + [f_n(\alpha_{l+1}) - f_n(d^+)]g_n(d) + \sum_{j=k+1}^{l} [f_n(\alpha_j) - f_n(\alpha_{j-1})]g_n(\xi_j). \end{aligned}$$

Thus, using (A_1), (A_2), (A_4) and $(*)$, we get finally

$$\begin{aligned} &\Big\| \int_a^b [\mathrm{d}f_n](g_n\chi_{[c,d]}) - S(\mathrm{d}f_n, g_n\chi_{[c,d]}, P)\Big\|_Z \\ &\quad \leqslant \|f_n(c^-) - f_n(\alpha_{k-1})\|_X \|g_n(c)\|_Y + \|f_n(\alpha_{l+1}) - f_n(d^+)\|_X \|g_n(d)\|_Y \\ &\qquad + \Big\| \int_c^d [\mathrm{d}f_n]\, g_n - \sum_{j=k+1}^{l} [f_n(\alpha_j) - f_n(\alpha_{j-1})]g_n(\xi_j)\Big\|_Z \\ &\quad \leqslant \varepsilon(2K+1) \quad \forall\, n \in \mathbb{N}. \end{aligned}$$

It follows immediately that the sequence $\{g_n\}$ is equiintegrable with respect to $\{f_n\}$ on $[c,d]$. If $\{g_n\}$ is equiintegrable with respect to $\{f_n\}$ on $[c,d]$, then there is a gauge δ on $[a,b]$ such that

$$\Big\| \int_a^b [\mathrm{d}f_n](g_n\chi_{[c,d]}) - S(\mathrm{d}f_n, g_n\chi_{[c,d]}, P)\Big\|_Z < \varepsilon \quad \forall\, n \in \mathbb{N}.$$

By [33], Lemma 6.2.11 we can choose this gauge in such a way that $\{c,d\} \subset \{\xi_j\} \cap \{\alpha_j\}$ for any δ-fine partition $P = \{([\alpha_{j-1},\alpha_j],\xi_j)\}$ of $[a,b]$. Let again

$$a < c - \Delta_c^- < \alpha_{k-1} < \xi_k = c = \alpha_k \quad \text{and} \quad \xi_l = d = \alpha_l < \alpha_{l+1} < d + \Delta_d^+ < b.$$

Modifying a little bit the previous part of the proof it is now easy to prove that the restriction $\tilde{\delta}$ of δ to $[c,d]$ is the proper gauge to ensure that $(*)$ is true.

(b) Let $a < c < d < b$ and $J = (c, d)$. Then, according to Proposition 3.5 (i) we have

$$\int_c^d [\mathrm{d}f_n]\, g_n = \Delta^+ f_n(c) g_n(c) + \int_{(c,d)} [\mathrm{d}f_n]\, g_n + \Delta^- f_n(d) g_n(d)$$
$$= \Delta^+ f_n(c) g_n(c) + \int_a^b [\mathrm{d}f_n](g_n \chi_{(c,d)}) + \Delta^- f_n(d) g_n(d) \quad \forall\, n \in \mathbb{N}.$$

The proof can be completed analogously as in the part (a), only instead of (A_2) and (A_4) we have to make use of (A_3) and (A_5).

(c) All the remaining cases can be treated analogously. □

Now, let us recall the notion of equiregulatedness due to Fraňková (see [11]) which will be useful in what follows.

Definition 4.6. A subset M of $G([a, b]; X)$ is called *equiregulated* if the following conditions hold.

(i) For any $\varepsilon > 0$ and $\tau \in (a, b]$ there is a $\delta_1(\tau) \in (0, \tau - a)$ such that

$$\|f(\tau^-) - f(t)\|_X < \varepsilon \quad \forall\, t \in (\tau - \delta_1(\tau), \tau) \text{ and } f \in M.$$

(ii) For any $\varepsilon > 0$ and $\tau \in [a, b)$ there is a $\delta_2(\tau) \in (0, b - \tau)$ such that

$$\|f(\tau^+) - f(t)\|_X < \varepsilon \quad \forall\, t \in (\tau, \tau + \delta_2(\tau)) \text{ and } f \in M.$$

Since any equiregulated sequence $\{f_n\}$ in $G([a, b]; X)$ automatically satisfies the assumption (A_2)–(A_5) in Lemma 4.5, we can state the following assertion.

Corollary 4.7. *Let J be an arbitrary subinterval of $[a, b]$, $c = \inf J$ and $d = \sup J$. Let the sequence $\{f_n\} \subset G([a, b]; X)$ be equiregulated and let $\{g_n\}$ be a sequence of functions mapping $[a, b]$ into Y such that $\{g_n(t)\}$ is bounded in Y whenever $t = c > a$ or $t = d < b$. Then, $\{g_n\}$ is equiintegrable with respect to $\{f_n\}$ on J if and only if $(*)$ is true.*

Corollary 4.7 together with the Cauchy equiintegrability criterion, cf. Theorem 4.4, enable us to prove the following assertion.

Corollary 4.8. *Let J be an arbitrary subinterval of $[a, b]$, $c = \inf J$ and $d = \sup J$. Let the sequence $\{f_n\} \subset G([a, b]; X)$ be equiregulated and let $\{g_n\}$ be a sequence of functions mapping $[a, b]$ into Y such that $\{g_n(t)\}$ is bounded in Y whenever $t = c > a$ or $t = d < b$. Then:*

(i) *If the sequence $\{g_n\}$ is equiintegrable with respect to $\{f_n\}$ on $[a, b]$, then it is equiintegrable with respect to $\{f_n\}$ on any subinterval J of $[a, b]$.*

(ii) *For every $c \in (a,b)$, the sequence $\{g_n\}$ is equiintegrable with respect to $\{f_n\}$ on $[a,b]$ if and only if it is equiintegrable with respect to $\{f_n\}$ on both the intervals $[a,c]$ and $[c,b]$.*

P r o o f. (i) Let the sequence $\{g_n\}$ be equiintegrable with respect to $\{f_n\}$ on $[a,b]$. Obviously, cf., e.g., [42], Proposition 8, $\int_c^d [\mathrm{d} f_n]\, g_n$ exists in Z for each $n \in \mathbb{N}$. Furthermore, by the Cauchy equiintegrability criterion, the hypothesis (C_2) is satisfied. It can be shown in a rather routine way that then (C_2) holds also on $[c,d]$. Therefore, by Theorem 4.4, $\{g_n\}$ is equiintegrable with respect to $\{f_n\}$ when $[a,b]$ is replaced by $[c,d]$. Finally, due to Corollary 4.7, we can conclude that $\{g_n\}$ is equiintegrable with respect to $\{f_n\}$ on J.

(ii) It remains to show just the sufficiency part of the implication. Let $\eta > 0$ and $c \in (a,b)$ be given. Let δ_a and δ_b be the gauges satisfying (C_1) for $[a,b]$ replaced by $[a,c]$ or $[c,b]$, respectively. Then, for a given $\eta > 0$, the condition (C_1) is satisfied if we put

$$\delta(t) = \begin{cases} \delta_a(t) & \text{if } \ t \in [a,c), \\ \min\{\delta_a(c), \delta_b(c)\} & \text{if } \ t = c, \\ \delta_b(t) & \text{if } \ t \in (c,b]. \end{cases}$$

This completes the proof. □

Next assertion is a direct consequence of Lemma 4.5 and Corollary 4.8.

Corollary 4.9. *Let the sequence $\{f_n\} \subset G([a,b];X)$ be equiregulated and let the sequence $\{g_n\}$ of mappings of $[a,b]$ into Y be pointwise bounded, i.e., $\{g_n(t)\}$ is bounded in Y for any $t \in [a,b]$. Then, $\{g_n\}$ is equiintegrable with respect to $\{f_n\}$ on $[a,b]$ if and only if the sequence $\{g_n\}$ is equiintegrable with respect to $\{f_n\}$ on arbitrary elementary subsets E of $[a,b]$.*

The Saks-Henstock lemma (see, e.g., [42], Lemma 16) states that the Riemannian sums not only approximate the integrals in the "gauge topology" over the entire interval, but also over suitably chosen systems of subintervals. Next, we show that the equiintegrability implies a uniform Saks-Henstock property. However, first, let us introduce the notion of a δ-fine system, cf., e.g., [42], Lemma 16.

Definition 4.10. The set $W = \{([\beta_j, \gamma_j], \xi_j),\ j = 1, 2, \ldots, m\}$ is a δ-fine system in $[a,b]$ if

$$a \leqslant \beta_1 \leqslant \xi_1 \leqslant \gamma_1 \leqslant \beta_2 \leqslant \xi_2 \leqslant \gamma_2 \leqslant \ldots \leqslant \beta_m \leqslant \xi_m \leqslant \gamma_m \leqslant b$$

and

$$[\beta_j, \gamma_j] \subset (\xi_j - \delta(\xi_j), \xi_j + \delta(\xi_j)) \quad \text{for } j = 1, 2, \ldots, m.$$

Similarly, for divisions and tagged partitions of the interval $[a,b]$, we denote by $\nu(W)$ the number of the intervals contained in W, i.e., $\nu(W)=m$ in the above situation.

Let $T\subset[a,b]$ and let $W=\{([\beta_j,\gamma_j],\xi_j),\ j=1,2,\dots,m\}$ be a δ-fine system in $[a,b]$. Then, W is called a *δ-fine T-tagged system* in $[a,b]$ if $\xi_j\in T$ for every $j=1,2,\dots,m$.

Proposition 4.11. *Let $\{f_n\}\subset G([a,b],X)$ be equiregulated and let the sequence $\{g_n\}$ of functions mapping $[a,b]$ into Y be equiintegrable with respect to $\{f_n\}$ on $[a,b]$. Furthermore, let $\varepsilon>0$ be given arbitrarily and let δ be a gauge on $[a,b]$ such that*

$$\Big\|S(\mathrm{d}f_n,g_n,P)-\int_a^b[\mathrm{d}f_n]\,g_n\Big\|_Z<\varepsilon$$

for every δ-fine tagged partition $P=(\boldsymbol{\alpha},\boldsymbol{\xi})$ of $[a,b]$ and every $n\in\mathbb{N}$. Then,

$$\Big\|\sum_{j=1}^{\nu(S)}\Big([f_n(\gamma_j)-f_n(\beta_j)]g_n(\xi_j)-\int_{\beta_j}^{\gamma_j}[\mathrm{d}f_n]\,g_n\Big)\Big\|_Z\leqslant\varepsilon$$

holds for every δ-fine system $W=\{([\beta_j,\gamma_j],\xi_j)\}$ in $[a,b]$ and every $n\in\mathbb{N}$.

Proof. Assume that the system $\{([\beta_j,\gamma_j],\xi_j),\ j=1,2,\dots,m\}$ in $[a,b]$ satisfies the conditions

$$a\leqslant\beta_1\leqslant\xi_1\leqslant\gamma_1\leqslant\beta_2\leqslant\xi_2\leqslant\gamma_2\leqslant\ldots\leqslant\beta_m\leqslant\xi_m\leqslant\gamma_m\leqslant b,$$
$$[\beta_j,\gamma_j]\subset(\xi_j-\delta(\xi_j),\xi_j+\delta(\xi_j))\quad\text{for } j=1,2,\dots,m,$$

and put $\gamma_0=a$ and $\beta_{m+1}=b$. Then, by Corollary 4.9, the sequence $\{g_n\}$ is equiintegrable with respect to $\{f_n\}$ on any subinterval $[\gamma_j,\beta_{j+1}]$ for $j=0,1,\dots,m$. Hence, we can apply the method of the proof of the Saks-Henstock lemma in [33], Lemma 6.5.1 to complete the proof of this proposition. □

5. Harnack extension principle and its applications

In this section we extend the Harnack extension principle, cf. Theorem 1.1, to KS-integrals. As we already mentioned in the introduction, unlike the case with identity integrators, the integrals (1.2) (or see (3.3)) need not have, in general, the same values. This indicates that it cannot be done in a straightforward way.

The following definition is used in this section.

Definition 5.1. Let E be an elementary set in $\mathbb{R}$ and let S be a subset of E. Then a sequence $\{E_i\}$ of mutually disjoint elementary sets in $\mathbb{R}$ is called a *proper cover* of S if $S=\bigcup_{i\in\mathbb{N}}E_i$.

Having in mind Theorem 4.1 and Definition 5.1, we are now ready to state and prove the Harnack extension principle for the KS-integral.

Theorem 5.2 (Harnack extension principle)**.** *Let* $f \in (\mathcal{B})G([a,b],X)$ *and* $g\colon [a,b] \to Y$. *Let* T *be a closed subset of* $[a,b]$ *such that the integral* $\int_T [\mathrm{d}f]\, g$ *exists. Furthermore, let* $\{E_i\}$ *be a proper cover of* $[a,b] \setminus T$. *Put* $S_n = \bigcup_{i=1}^{n} E_i$ *for* $n \in \mathbb{N}$ *and assume that the sequence* $\{g\chi_{S_n}\}$ *is equiintegrable with respect to* f *on* $[a,b]$. *Then, the integrals* $\int_{[a,b]\setminus T} [\mathrm{d}f]\, g$, $\int_{[a,b]} [\mathrm{d}f]\, g$, *and* $\int_{E_i} [\mathrm{d}f]\, g$ *exist for all* $i \in \mathbb{N}$ *and*

$$\int_{[a,b]} [\mathrm{d}f]\, g = \int_T [\mathrm{d}f]\, g + \int_{[a,b]\setminus T} [\mathrm{d}f]\, g, \tag{5.1}$$

where

$$\int_{[a,b]\setminus T} [\mathrm{d}f]\, g = \sum_{i=1}^{\infty} \int_{E_i} [\mathrm{d}f]\, g. \tag{5.2}$$

P r o o f. Obviously, S_n is an elementary set in $[a,b]$ for every $n \in \mathbb{N}$ and $E_i \subset S_n$ for every $i = 1, \ldots, n$. Since the integral $\int_{S_n} [\mathrm{d}f]\, g = \int_a^b [\mathrm{d}f](g\chi_{S_n})$ exists for every $n \in \mathbb{N}$, the sets E_i, $i = 1, \ldots, n$, are mutually disjoint, and $f \in (\mathcal{B})G([a,b],X)$ by Propositions 3.10 (i) and 3.14, the integrals $\int_{E_i} [\mathrm{d}f]\, g$ exist for all $i = 1, \ldots, n$, and

$$\int_{S_n} [\mathrm{d}f]\, g = \sum_{i=1}^{n} \int_{E_i} [\mathrm{d}f]\, g \quad \text{for any } n \in \mathbb{N}. \tag{5.3}$$

Furthermore, as

$$(g\chi_{[a,b]\setminus T})(t) = \lim_{n\to\infty} (g\chi_{S_n})(t) \quad \forall\, t \in [a,b], \tag{5.4}$$

making use of Theorem 4.1 and (5.3), we obtain

$$\int_{[a,b]\setminus T} [\mathrm{d}f]\, g = \lim_{n\to\infty} \int_{S_n} [\mathrm{d}f]\, g = \sum_{i=1}^{\infty} \int_{E_i} [\mathrm{d}f]\, g,$$

i.e., (5.2) is true. Finally, (5.2) together with Proposition 3.14 imply (5.1). This completes the proof. □

R e m a r k 5.3. Recall, cf. Remark 3.4, that under the assumptions of Theorem 5.2, the integrals $\int_{[a,b]} [\mathrm{d}f]\, g$ and $\int_a^b [\mathrm{d}f]\, g$ coincide.

If T is an arbitrary subset of the elementary set $E \subseteq [a,b]$, then, as noticed in Remark 3.7 when $E = [a,b]$, the existence of the integral $\int_E [\mathrm{d}f]\, g$ does not always imply the existence of the integral $\int_T [\mathrm{d}f]\, g$ (except when T is an elementary subset of E; see Proposition 3.10 (i)). This means that the assumptions on existence of the integral $\int_T [\mathrm{d}f]\, g$ in Proposition 3.10 cannot be omitted, in general. Next theorem provides a certain affirmative result for the case $E = [a,b]$.

Theorem 5.4. *Let $f \in (\mathcal{B})G([a,b],X)$ and $g\colon [a,b] \to Y$. Let T be a closed subset of $[a,b]$ and let $\{E_i\}$ be a proper cover of $[a,b] \setminus T$. Put $S_n = \bigcup_{i=1}^{n} E_i$ for $n \in \mathbb{N}$ and assume that the sequence $\{g\chi_{S_n}\}$ is equiintegrable with respect to f on $[a,b]$. Furthermore, assume that the integral $\int_{[a,b]} [\mathrm{d}f]\, g$ exists. Then,*

(i) *All integrals $\int_{E_i} [\mathrm{d}f]\, g$, $\int_{[a,b]\setminus T} [\mathrm{d}f]\, g$, and $\int_T [\mathrm{d}f]\, g$ exist for all $i \in \mathbb{N}$ and the equalities*

$$\int_{[a,b]\setminus T} [\mathrm{d}f]\, g = \sum_{i=1}^{\infty} \int_{E_i} [\mathrm{d}f]\, g \quad \text{and} \quad \int_{[a,b]} [\mathrm{d}f]\, g = \int_T [\mathrm{d}f]\, g + \int_{[a,b]\setminus T} [\mathrm{d}f]\, g$$

are true.

(ii) *For every $\varepsilon > 0$, there is a gauge δ on T, such that*

$$\Big\| \sum_{j=1}^{\nu(Q)} \Big(\int_{\alpha_j}^{\beta_j} [\mathrm{d}f]\, g\chi_T - \int_{\alpha_j}^{\beta_j} [\mathrm{d}f]\, g \Big) \Big\|_Z < \varepsilon$$

for every δ-fine T-tagged system $W = \{([\alpha_j,\beta_j],\xi_j)\}$ in $[a,b]$.

P r o o f. (i) Similarly like in the proof of Theorem 5.2, we can show that all the integrals $\int_{E_i} [\mathrm{d}f]\, g$, $i \in \mathbb{N}$, exist and

$$\int_{[a,b]\setminus T} [\mathrm{d}f]\, g = \sum_{i=1}^{\infty} \int_{E_i} [\mathrm{d}f]\, g.$$

Evidently, $([a,b]\setminus T) \cup T = [a,b]$ and $([a,b]\setminus T) \cap T = \emptyset$. Therefore, by Proposition 3.14 we obtain that the integral $\int_T [\mathrm{d}f]\, g$ exists and

$$\int_{[a,b]} [\mathrm{d}f]\, g = \int_{[a,b]\setminus T} [\mathrm{d}f]\, g + \int_T [\mathrm{d}f]\, g.$$

(ii) Let $\varepsilon > 0$ be given, and let δ be a gauge on $[a,b]$ such that

$$\Big\| S(\mathrm{d}f, g, P) - \int_a^b [\mathrm{d}f]\, g \Big\|_Z < \frac{\varepsilon}{3} \quad \text{and} \quad \Big\| S(\mathrm{d}f, g\chi_T, P) - \int_a^b [\mathrm{d}f](g\chi_T) \Big\|_Z < \frac{\varepsilon}{3}$$

whenever $P = (\boldsymbol{\alpha}, \boldsymbol{\xi})$ is a δ-fine tagged partition of $[a,b]$.

Impose now that $W = \{([\alpha_j, \beta_j], \xi_j)\}$ is a δ-fine T-tagged system in $[a, b]$. Then, using the Saks-Henstock lemma (see [42], Lemma 16), we deduce

$$\begin{aligned}
&\Big\| \sum_{j=1}^{\nu(W)} \Big(\int_{\alpha_j}^{\beta_j} [\mathrm{d}f](g\chi_T) - \int_{\alpha_j}^{\beta_j} [\mathrm{d}f]\, g \Big) \Big\|_Z \\
&\qquad \leqslant \Big\| \sum_{j=1}^{\nu(W)} \Big(\int_{\alpha_j}^{\beta_j} [\mathrm{d}f](g\chi_T) - [f(\beta_j) - f(\alpha_j)](g\chi_T)(\xi_j) \Big) \Big\|_Z \\
&\qquad\quad + \Big\| \sum_{j=1}^{\nu(W)} \Big([f(\beta_j) - f(\alpha_j)] g(\xi_j) - \int_{\alpha_j}^{\beta_j} [\mathrm{d}f]\, g \Big) \Big\|_Z \\
&\qquad < \varepsilon.
\end{aligned}$$

This completes the proof. □

If E could be an arbitrary elementary subset of $[a, b]$, then to obtain an assertion similar to that of Theorem 5.4, we have to require that the integrator f is (strongly) regulated, instead of only belonging to $(\mathcal{B})G([a, b], X)$.

Theorem 5.5. *Let $f \in G([a, b], X)$ and $g\colon [a, b] \to Y$. Let E be an elementary set in $[a, b]$ such that the integral $\int_E [\mathrm{d}f]\, g$ exists. Let T be a closed subset of E and $\{E_i\}$ be a proper cover of $E \setminus T$. Furthermore, put $S_n = \bigcup_{i=1}^{n} E_i$ for $n \in \mathbb{N}$ and assume that the sequence $\{g\chi_{S_n}\}$ is equiintegrable with respect to f on $[a, b]$. Then, both the integrals $\int_T [\mathrm{d}f]\, g$ and $\int_{E \setminus T} [\mathrm{d}f]\, g$ exist and the equality*

$$\int_E [\mathrm{d}f]\, g = \int_T [\mathrm{d}f]\, g + \int_{E \setminus T} [\mathrm{d}f]\, g \tag{5.5}$$

is true.

Proof. Without loss of generality we may assume that $E = \bigcup_{k=1}^{m} J_k$ where $\{J_k\}_{k=1}^m$ is a minimal decomposition of E, cf. Definition 3.8. By Proposition 3.10 with Remark 3.6, the integral $\int_{\overline{J}_k} [\mathrm{d}f]\, g$, where $\overline{J}_k$ denotes, as usual, the closure of J_k, exists for every $k = 1, 2, \ldots, m$. Let us put

$$T_k = T \cap \overline{J}_k \quad \text{for } k = 1, 2, \ldots, m.$$

Then all T_k, $k = 1, 2, \ldots, m$, are closed and, cf. Remark 3.9, $T_k \cap T_l \subset \overline{J}_k \cap \overline{J}_l$ for $k, l = 1, 2, \ldots, m$.

Now, let us fix an arbitrary $k = 1, 2, \ldots, m$, and put

$$E_{i,k} := E_i \cap \overline{J}_k \text{ for } i \in \mathbb{N} \quad \text{and} \quad S_{n,k} := \bigcup_{i=1}^{n} E_{i,k} \text{ for } n \in \mathbb{N}.$$

Notice that $S_{n,k} = S_n \cap \overline{J}_k$ for $k = 1, 2, \ldots, m$, and hence $S_{n,k} \subseteq S_n$ for $n \in \mathbb{N}$. The collection $\{E_i\}$ is supposed to be a proper cover of $E \setminus T$, i.e., (cf. Definition 5.1), E_i are mutually disjoint elementary sets and $\bigcup_{i\in\mathbb{N}} E_i = E \setminus T$. The sets $E_{i,k}$ are clearly elementary sets and since $E_{i,k} \subset E_i$ for each $i \in \mathbb{N}$, they are mutually disjoint as well. Finally,

$$\bigcup_{i\in\mathbb{N}} E_{i,k} = \bigcup_{i\in\mathbb{N}} (E_i \cap \overline{J}_k) = \Big(\bigcup_{i\in\mathbb{N}} E_i\Big) \cap \overline{J}_k = (E \setminus T) \cap \overline{J}_k = \overline{J}_k \setminus T_k.$$

To summarize, the collection $\{E_{i,k} \colon i \in \mathbb{N}\}$ is a proper cover of $\overline{J}_k \setminus T_k$.

Moreover, from (5.4) in the proof of Theorem 5.2, the sequence $\{g\chi_{S_n}\}$ converges pointwise on $[a, b]$, which further implies the boundedness of the sequence $\{(g\chi_{S_n})(t)\}$ in Y for any $t \in [a, b]$. Accordingly, we obtain that the sequence $\{(g\chi_{S_{n,k}})(t)\}$ is bounded in Y for any $t \in [a, b]$ and $k = 1, 2, \ldots, m$. Hence, by Corollary 4.8 (i), the sequence $\{g\chi_{S_{n,k}} \colon n \in \mathbb{N}\}$ is equiintegrable with respect to f on $\overline{J}_k$. Consequently, Theorem 5.4 gives the existence of the integrals $\int_{T_k} [\mathrm{d}f]\, g$ and $\int_{\overline{J}_k \setminus T_k} [\mathrm{d}f]\, g$ and the equality

$$\int_{\overline{J}_k} [\mathrm{d}f]\, g = \int_{T_k} [\mathrm{d}f]\, g + \int_{\overline{J}_k \setminus T_k} [\mathrm{d}f]\, g$$

for all $k = 1, 2, \ldots, m$.

Finally, by Proposition 3.15, where we insert $\{T_k\}_{k=1}^m$ instead of $\{S_j\}_{j=1}^p$, the integral $\int_T [\mathrm{d}f]\, g$ exists. The existence of the integral $\int_{E\setminus T} [\mathrm{d}f]\, g$ and the equality (5.5) then follow directly from Proposition 3.13. $\square$

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