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# GEOMETRY OF UNIVERSAL EMBEDDING SPACES FOR ALMOST COMPLEX MANIFOLDS

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Abstract. We investigate the geometry of universal embedding spaces for compact almost-complex manifolds of a given dimension, and related constructions that allow for an extrinsic study of the integrability of almost-complex structures. These embedding spaces were introduced by J-P. Demailly and H. Gaussier, and are complex algebraic analogues of twistor spaces. Their goal was to study a conjecture made by F. Bogomolov asserting the "transverse embeddability" of arbitrary compact complex manifolds into foliated algebraic varieties. In this work, we introduce a more general category of universal embedding spaces, and elucidate the geometric structure of related bundles, such as the integrability locus characterizing integrable almost-complex structures. Our approach could potentially lead to finding new obstructions to the existence of a complex structure, which may be useful for tackling Yau's Challenge.

## 1. Introduction

In the article [9], J-P. Demailly and H. Gaussier settled almost-complex, and weakened versions of a conjecture made by F. Bogomolov [3] about the transverse embeddability of compact complex manifolds into complex projective manifolds, equipped with an algebraic foliation (cf. Basic Question 1.1, [9]). Their Theorem 1.2 addresses the almost-complex Bogomolov conjecture, while Theorem 1.6 does so for the weakened version. Both theorems are stated in a combined way below (Theorem 1).

The theme of our paper is the application of these results to the study of almost-complex geometry from an extrinsic point of view. This perspective gives rise to new ways of approaching what may be deemed the principal open question in the field [12].

**Yau's Challenge (YC).** Determine whether there exists a compact almost-complex manifold of real dimension at least 6 that cannot be given a complex structure.

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In order to state the Demailly-Gaussier main results of interest to us, let us first recall some basic concepts in (almost-)complex geometry.

1.1. **Background on the Demailly-Gaussier paper.** Recall that an almost--complex structure $J$ is said to be integrable (or a complex structure) iff its Nijenhuis tensor, which is given by the formula

$$N_J(\zeta,\eta) = [J(\zeta), J(\eta)] - J([J(\zeta),\eta] + [\zeta, J(\eta)]) - [\zeta,\eta]\,,$$

vanishes identically [11].

Let $W$ be a complex manifold. A real structure, $s$, on $W$ is an anti-holomorphic involution on $W$. The set of fixed points of $s$, which we denote by $W^{\mathbb{R}}$ and call the real part of $W$ (w.r.t. $s$), is either empty or it is a real manifold with $\dim_{\mathbb{R}} W^{\mathbb{R}} = \dim_{\mathbb{C}} W$ (see, for example, [2, 5]). Real structures can be defined outside of the smooth category as well, but here we focus on manifolds as this is enough for our purposes. Let $X$ be a real manifold. An embedding $f\colon X \hookrightarrow W$ is called totally real if there is a real structure on $W$ such that the image, $f(X)$, is contained in the real part of $W$; i.e. $f(X) \subset W^{\mathbb{R}}$. Moreover, if $\mathcal{D} \subset T_W$ is a holomorphic distribution, we say that the embedding $f$ is transverse to $\mathcal{D}$ if for all $x \in X$, $T_{W,f(x)} \simeq f_*(T_{X,x}) \oplus \mathcal{D}_{f(x)}$.

**Theorem 1** (Theorem 1.2, Theorem 1.6, [9])**.** *Let $n \geq 1$ and $k \geq 4n$ be integers. Every compact almost-complex manifold $(X, J_X)$ of real dimension $2n$ admits an embedding $F\colon X \hookrightarrow Z_{n,k}$, where $Z_{n,k}$ is a complex affine algebraic manifold, that is*

(1) *totally real,*

(2) *transverse to a holomorphic complex algebraic distribution $\mathcal{D}_{n,k} \subset T_{Z_{n,k}}$, and*

(3) *$J_X$-inducing (i.e. $J_X$ is a pullback by $F$ of an almost-complex structure on $F(X)$, coming from an induced complex structure on $T_{Z_{n,k}}/\mathcal{D}_{n,k}$).*

(4) *Moreover, if $J_X$ is integrable, then $Im(\bar{\partial}_{J_X} F)$ is contained in a subvariety $\mathcal{I}_{n,k}$ of the Grassmannian bundle $Gr^{\mathbb{C}}(\mathcal{D}_{n,k}, n) \to Z_{n,k}$ of $n$-planes in $\mathcal{D}_{n,k}$ that is given as the isotropic locus of the torsion operator $\theta\colon \mathcal{D}_{n,k} \times \mathcal{D}_{n,k} \to T_{Z_{n,k}}/\mathcal{D}_{n,k}$, $\theta(\zeta,\eta) = [\zeta,\eta] \mod \mathcal{D}_{n,k}$.*

Deamailly and Gaussier called these embeddings, and embedding spaces *universal* – the name stems from the fact that they provide universal solutions to the almost-complex, and weakened Bogomolov conjectures. The definition of the universal embedding space, $(Z_{n,k}, \mathcal{D}_{n,k})$, and the construction of the universal embedding, $F$, will be provided later, in section 2. Since part of our work involves generalizing all of the above, it seems more correct to recover Demailly's and Gaussier's original notions from our own.

1.2. **Yau's Challenge via universal embeddings.** In July of 2018, Demailly gave a talk, where he proposed using Theorem 1 to tackle the long-standing open problem of deciding whether the 6-dimensional sphere $S^6$ is a complex manifold or not [7].

Despite the level of difficulty of the problem, $S^6$ appears to be the most accessible candidate solution to the YC. There seems to be more evidence in support of $S^6$ not admitting a complex structure. Let us point out that here, we deal merely with the preparation of steps that comprise what we call Demailly's strategy to solve the $S^6$ problem. However, in order to test Demailly's idea, one would need to carry out a number of involved computations that are beyond the scope of this paper. All we claim to do in this article is to enounce Demailly's strategy to study the $S^6$ problem, each of which steps is discussed more thoroughly in Section 3.2.1. In the final section, we briefly discuss what would need to be done in order to complete each step.

Observe that once we fix an orientation, we find that the space of almost-complex structures on $S^6$ is connected. Put another way, the standard, octonion, non-integrable almost-complex structure on (the oriented) $S^6$, $J_{\mathbb{O}}$, is homotopically unique [4]. Let us briefly describe this almost-complex structure, and the octonion embedding from which it arises. If $(e_j)_{j=0}^7$ are the unit octonions, where the $e_j$ are purely imaginary for all $j > 0$ and they span $\Im(\mathbb{O}) \simeq \mathbb{R}^7$, then the octonion embedding $f_{\mathbb{O}}\colon S^6 \hookrightarrow \Im(\mathbb{O}) \subset \mathbb{O}$ is simply the inclusion $f_{\mathbb{O}}(u_1, \dots, u_7) = \sum_{j=1}^7 u_j e_j$. Any $u \in S^6$, determines a complex structure $J_{\mathbb{O}}(u) \in \mathrm{End}_{\mathbb{R}}(\mathbb{O})$ that is essentially right octonionic multiplication by $u$: $J_{\mathbb{O}}(u)\zeta = \zeta u$. The octonion almost-complex structure on $S^6$, $J_{\mathbb{O}}$, is given by the mapping $u \mapsto J_{\mathbb{O}}(u)$. The non-integrability of this structure is due to the non-associativity of the octonions. Indeed, the Nijenhuis tensor can be found to be

$$N_{J_{\mathbb{O}}}(u)(\zeta, \eta) = -\big(\zeta(\eta u) - (\zeta u)\eta\big) - \big(\eta(\zeta u) - (\eta\zeta)u\big) \qquad [10].$$

Demailly wanted to somehow use the inclusion $f_{\mathbb{O}}$ together with Theorem 1 to obtain a universal embedding $(S^6, J_{\mathbb{O}}) \hookrightarrow Z_{3,4}$ that would produce a contradiction from the assumption that $S^6$ supports a complex structure. He believed that the topology, and geometry of $\mathcal{I}_{3,4}$ would play an important role. However, when it comes to implementing Demailly's idea to prove the non-existence of a complex structure on $S^6$, Theorem 1 has some deficiencies. Firstly, (the proof of) Theorem 1 does not allow us to input an initial embedding of choice, such as $f_{\mathbb{O}}$. And also, the optimal dimension of the universal embedding space $(Z_{3,k}, \mathcal{D}_{3,k})$ cannot be reached (i.e. $k > 4$). To be precise, in real dimension 6, Theorem 1 makes use of a Whitney embedding $S^6 \hookrightarrow \mathbb{R}^{12}$ to build a universal embedding $(S^6, J_{\mathbb{O}}) \hookrightarrow (Z_{3,12}, \mathcal{D}_{3,12})$.

1.3. **Objectives and main results.** Our first task is to state, and prove a universal embedding result with which we can properly formulate Demailly's proposal for tackling the $S^6$ problem. This will, in turn, lead to a generalization of Theorem 1 to a functorially defined, and much larger category of embedding spaces. These spaces will be denoted by $(\mathcal{Z}_n(Y), \mathbf{D}_{n,k})$, where $\mathcal{Z}_n(Y)$ is an $N_{n,k}$ dimensional complex manifold depending on a $2k$-dimensional complex manifold $Y$, equipped with a complex corank $n$ distribution $\mathbf{D}_{n,k} \subset T_{\mathcal{Z}_n(Y)}$ to be defined in Section 2.

For instance, as we will explain in Section 2, $(\mathcal{Z}_n(\mathbb{C}^{2k}), \mathbf{D}_{n,k})$ coincides with the Demailly-Gaussier embedding space $(Z_{n,k}, \mathcal{D}_{n,k})$. Later on, we introduce bundles, arising from these generalized embedding spaces $(\mathcal{Z}_n(Y), \mathbf{D}_{n,k})$, for the extrinsic study of the (non-)integrability of almost-complex structures.

Let $W$ be a smooth real affine algebraic variety, and let $I(W)$ be its ideal. The complexification $W^{\mathbb{C}}$ of $W$ is the complex affine algebraic manifold that is defined by the complex solutions to the ideal $I(W)$; i.e.

$$W^{\mathbb{C}} = \{p \in \mathbb{C}^N \mid f(p) = 0 \quad \forall f \in I(W)\}.$$

If $\sigma$ denotes the standard conjugation on $\mathbb{C}^N$, $\sigma(p) = \overline{p}$, then the restriction $\sigma|_{W^{\mathbb{C}}}$ is a real structure on $W^{\mathbb{C}}$ (cf. Section 1.1). The real part of $W^{\mathbb{C}}$ w.r.t. $\sigma|_{W^{\mathbb{C}}}$ coincides with $W$.

The adjustment of Theorem 1 that is needed to state Demailly's strategy is contained in the following observation.

**Proposition 1.** *Let $(X, J_X)$ be a compact almost-complex manifold of real dimension $2n$. Assume that there is a $C^\infty$ embedding of $X$ into a real affine algebraic $2k$-dimensional manifold $Y^{\mathbb{R}}$. Assume further that the normal bundle $N_{X/Y^{\mathbb{R}}}$ admits a complex structure $J_N$. Let $Y$ be the complexification of $Y^{\mathbb{R}}$. Then, there is a totally real embedding $F\colon X \hookrightarrow \mathcal{Z}_n(Y)$ that is transverse to $\mathbf{D}_{n,k}$ and that induces the almost-complex structure $J_X$.*

We will often use the notation $W^{\mathbb{R}}$ to denote a given real affine algebraic manifold with complexification $W$.

Indeed, we will show that such an embedding always exists, so we are able to reach the (generalized) analogue of Theorem 1

**Theorem 2.** *Let $n \geq 1$, and $k \geq 4n$. Then, any compact almost-complex $2n$-dimensional manifold $(X, J_X)$ admits a totally real, $J_X$-inducing, transverse to $\mathbf{D}_{n,k}$ embedding $F\colon (X, J_X) \hookrightarrow \mathcal{Z}_n(Y)$, where $Y = (M \times M)^{\mathbb{C}}$, and $M$ is any real affine algebraic manifold of dimension $k$.*

Theorem 2 (and Proposition 1) follow the way paved by Demailly and Gaussier in [9]. However, we branch off into a new direction that is concerned with the geometry of universal bundle constructions for the study of almost-complex structures.

Let $\mathrm{Gr}^{\mathbb{C}}(\mathbf{D}_{n,k}, n)$ be the Grassmannian bundle of $n$-planes in the distribution $\mathbf{D}_{n,k}$, and $\mathbf{I}_{n,k}$ be the generalized integrability locus $\mathcal{I}_{n,k}$ from Theorem 1. In the results that follow, we consider Zariski open subsets $\mathbf{Gr}^{\circ}_{n,k} \subset \mathrm{Gr}^{\mathbb{C}}(\mathbf{D}_{n,k}, n)$ and $\mathbf{I}^{\circ}_{n,k} := \mathbf{I}_{n,k} \cap \mathbf{Gr}^{\circ}_{n,k}$ that are bundles of trains (so termed in the spirit of [1]), whose definitions are given on p. 48.

Our main contribution starts with the following result.

**Theorem 3.** *The spaces $\boldsymbol{Gr}^{\circ}_{n,k}$ and $\mathbf{I}^{\circ}_{n,k}$ have the structure of holomorphic affine linear bundles over the total space of a Grassmannian bundle, $\pi_{n,k}\colon Gr^{\mathbb{C}}(\boldsymbol{\Delta}_{n,k}, n) \to \mathcal{Z}_n(Y)$.*

Among our main motivations for considering these bundles is a strategy to tackle the

**Homotopy YC (HYC).** Classify all compact manifolds of real dimension at least 6, that do not support a complex structure that is homotopic to a non-integrable almost-complex structure.

The strategy will be discussed in Section 4.3. Notice how the $S^6$ problem fits the scheme of the HYC (cf. section 1.2).

Now, let $\gamma_{n,k} \to \mathrm{Gr}^{\mathbb{C}}(\boldsymbol{\Delta}_{n,k}, n)$ be the tautological bundle.

**Proposition 2.** *The quotient* $\boldsymbol{Gr}^{\circ}_{n,k}/\mathbf{I}^{\circ}_{n,k}$ *can be viewed as a holomorphic vector bundle on* $Gr^{\mathbb{C}}(\Delta_{n,k}, n)$, *and we have a vector bundle isomorphism*

$$\boldsymbol{Gr}^{\circ}_{n,k}/\mathbf{I}^{\circ}_{n,k} \simeq \Lambda^2\gamma^*_{n,k} \otimes \pi^*_{n,k}(T_{\mathcal{Z}_n(Y)}/\mathbf{D}_{n,k}) \,.$$

Let $F\colon (X, J_X) \hookrightarrow \mathcal{Z}_n(Y)$ be a universal embedding as in Theorem 2. Then, $F$ has a lift $\tilde{F}\colon (X, J_X) \to \mathbf{Gr}^{\circ}_{n,k}$. As a consequence of the above results and [9, Proposition 5.1], we obtain the following linearization formula for the Nijenhuis tensor. The map $\tilde{\Theta}$ below may be thought of as the quotient mapping $\mathbf{Gr}^{\circ}_{n,k} \to \mathbf{Gr}^{\circ}_{n,k}/\mathbf{I}^{\circ}_{n,k}$.

**Proposition 3.** $N_{J_X} = 4\tilde{\Theta} \circ \tilde{F}$.

1.4. **Organization.** In Section 2, we define our generalized embedding spaces $(\mathcal{Z}_n(Y), \mathbf{D}_{n,k})$, and point out some basic categorical aspects. We then prove Proposition 1, and Theorem 2. In section 3, we describe the (local) geometry of our generalized embedding spaces, and the (global) homogeneous nature of their simplest instance. Subsection 3.2.1 is dedicated to Demailly's proposal to tackle the $S^6$ problem. In Section 4, we prove Theorem 3, Proposition 2, and Proposition 3. Section 4.3 is about a strategy to study the HYC with our various bundle constructions. In the final Section 5, we mention possible future research directions, related to the content of this article.

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## 2. Universal embedding spaces associated to even-dimensional complex manifolds

A complex directed manifold is a pair $(X, \mathcal{D})$ of complex manifold $X$ and holomorphic distribution $\mathcal{D} \subset T_X$. Complex directed manifolds form a category whose morphisms are holomorphic maps $\Psi\colon X \to X'$ with $\Psi_*(\mathcal{D}) \subset \mathcal{D}'$ [8]. A morphism is étale if it is a local isomorphism. For example, an étale morphism of real analytic manifolds is a real analytic map that is locally a diffeomorphism, and an étale morphism of complex manifolds is a holomorphic map that is locally a biholomorphism. If $(X, J)$ and $(X', J')$ are almost-complex manifolds, a map $f\colon X \to X'$ is pseudo-holomorphic provided that it satisfies the corresponding Cauchy-Riemann equation with $\bar{\partial}_{J,J'} f := \frac{1}{2}(df + J' \circ df \circ J)$.

Let $k \geq n \geq 1$, and $Y$ be a complex manifold of complex dimension $2k$. For every $y \in Y$, consider the complex projective manifold of flags of signature $(k-n, k)$ in $T_{Y,y}$

$$\begin{aligned} F_{(k-n,k)}(T_{Y,y}) =\{(S', \Sigma') \big| S' \subset \Sigma' \subset T_{Y,y} \ \text{ is a sequence of linear subspaces,} \\ \dim_{\mathbb{C}}(S') = k-n \ \text{ and } \ \dim_{\mathbb{C}}(\Sigma') = k\}\,, \end{aligned}$$

and the product manifold

$$F^2_{(k-n,k)}(T_{Y,y}) = \{(S', S'', \Sigma', \Sigma'') \big| (S', \Sigma'), (S'', \Sigma'') \in F_{(k-n,k)}(T_{Y,y})\}\,.$$

Let

$$Q_y = \{(S', S'', \Sigma', \Sigma'') \in F^2_{(k-n,k)}(T_{Y,y}) \big| \Sigma' \oplus \Sigma'' = T_{Y,y}\}\,.$$

Define

$$\mathcal{Z}_n(Y) := \coprod_{y \in Y} Q_y\,.$$

The space $\mathcal{Z}_n(Y)$ is a complex manifold of complex dimension

$$N_{n,k} := 2k + 2(k^2 + n(k-n))\,.$$

It bears a resemblance to twistor bundles and Grassmannians [9]: twistor bundles parametrize almost-complex structures whereas $\mathcal{Z}_n(Y)$ includes all almost-complex structures $J$ on $Y$ via the pair of eigenspaces in the decomposition $T_Y = T_Y^{1,0} \oplus T_Y^{0,1}$ that each $J$ determines.

Let $\pi_Y : \mathcal{Z}_n(Y) \to Y$ be the projection map defined for any $y \in Y$ and $q_y \in Q_y$ by $\pi_Y(y, q_y) = y$. Define $\mathbf{\Delta}_{n,k}$ to be the sub-bundle of $\pi_Y^*(T_Y)$ such that for any $w = (y, S', S'', \Sigma', \Sigma'') \in \mathcal{Z}_n(Y)$, we have $\mathbf{\Delta}_{n,k,w} = S' \oplus \Sigma''$. Now, define a distribution $\mathbf{D}_{n,k} \subset T_{\mathcal{Z}_n(Y)}$ by $\mathbf{D}_{n,k} := d\pi_Y^{-1}(\mathbf{\Delta}_{n,k})$.

In the case $Y = \mathbb{C}^{2k}$, if we let

$$Q := \{(S', S'', \Sigma', \Sigma'') \in F^2_{(k-n,k)}(\mathbb{C}^{2k}) \mid \Sigma' \oplus \Sigma'' = \mathbb{C}^{2k}\},$$

then we simply get that $\mathcal{Z}_n(\mathbb{C}^{2k}) = \mathbb{C}^{2k} \times Q$. Therefore, the above construction recovers Demailly's and Gaussier's original embedding spaces, $(Z_{n,k}, \mathcal{D}_{n,k})$, which are complex directed manifolds with $Z_{n,k}$ being the complex, quasi-projective manifold of all 5-tuples

$$\{(z, S', S'', \Sigma', \Sigma'') \mid z \in \mathbb{C}^{2k}, (S', S'', \Sigma', \Sigma'') \in Q\}$$

and $\mathcal{D}_{n,k}$, the corank $n$ sub-bundle of $T_{Z_{n,k}}$ whose fiber at any $w \in Z_{n,k}$ is

$$\mathcal{D}_{n,k,w} = \{(\zeta, u', u'', v', v'') \in T_{Z_{n,k},w} \mid \zeta \in S' \oplus \Sigma''\}[9].$$

For $n$ fixed, the above defined complex directed manifold $(\mathcal{Z}_n(Y), \mathbf{D}_{n,k})$ will be called here the universal embedding space associated with $Y$. We will soon see that it has a universal property.

**Remark 1.** $\mathcal{Z}_n(Y) \xrightarrow{\pi_Y} Y$ is a (holomorphic) fiber bundle with typical fiber $Q$, and if $Y = \mathbb{C}^{2k}$, the bundle is trivial. The universal embedding space $\mathcal{Z}_n(Y)$ is locally diffeomorphic to $\mathcal{Z}_n(\mathbb{C}^{2k})$.

**Proof.** Given a holomorphic atlas $(U_\alpha, \psi_\alpha)$ for $Y$ and $p \in U_\alpha$, the isomorphism $T_{Y,p} \simeq \mathbb{C}^{2k}$ induces a biholomorphism $Q_p \simeq_{q_\alpha} Q$, and so $(U_\alpha, \mathbf{Id}_{U_\alpha} \times q_\alpha)$ is a local trivialization.

Now observe that $\mathcal{Z}_n(U_\alpha) = \pi_Y^{-1}(U_\alpha) \simeq U_\alpha \times Q \simeq \mathbb{C}^{2k} \times Q = \mathcal{Z}_n(\mathbb{C}^{2k})$, where $\mathcal{Z}_n(U_\alpha) = \coprod_{p \in U_\alpha} Q_p$, where the first identification comes from the local trivialization and the second one, from the map $\psi_\alpha \colon U_\alpha \to \mathbb{C}^{2k}$. $\square$

Depending on the context, we will say that the dimension of an almost-complex manifold is $2n$ (over $\mathbb{R}$) or $n$ (over $\mathbb{C}$).

2.1. **Universal embedding property.** Next, we construct embeddings of compact almost-complex manifolds. Our approach follows closely the proof of Theorem 1.2 [9]. In a way, we also fill in some of the details of Demailly's and Gaussier's original embedding construction. Throughout this section, $(X, J_X)$ is a compact almost-complex manifold. So let us begin with the proof of our first result.

**Proof of Proposition 1.** Define $\tilde{J} := J_X \oplus J_N$, which is a complex structure on $T_X \oplus N_{X/Y^{\mathbb{R}}}$. A choice of isomorphism $T_{Y^{\mathbb{R}}}|_X \simeq T_X \oplus N_{X/Y^{\mathbb{R}}}$ allows us to view $\tilde{J}$ as a complex structure on $T_{Y^{\mathbb{R}}}|_X$. Consider the $J_Y$-complexification of $\tilde{J}$, $\tilde{J}^{\mathbb{C}} = J_X^{\mathbb{C}} \oplus J_N^{\mathbb{C}} \colon T_Y|_X \to T_Y|_X$, and put $S := \{0\} \oplus N^{\mathbb{C}}_{X/Y^{\mathbb{R}}}$. The complexification of the above chosen isomorphism provides an identification $T_Y|_X \simeq T_X^{\mathbb{C}} \oplus N^{\mathbb{C}}_{X/Y^{\mathbb{R}}}$. So we may regard $S$ as a sub-bundle of $T_Y|_X$. For any $x \in X$ and $(0, \eta_x) \in S_x = \{0\} \oplus N^{\mathbb{C}}_{X/Y^{\mathbb{R}},x}$, $\tilde{J}^{\mathbb{C}}(x)(0, \eta_x) = (0, J_N^{\mathbb{C}}(\eta_x)) \in S_x$, implying that $S_x$ is $\tilde{J}^{\mathbb{C}}(x)$-stable so that $\tilde{J}^{\mathbb{C}}(x)|_{S_x} \in \operatorname{End}(S_x)$. Let $\Sigma'_x$ be the $+i$ eigenspace for $\tilde{J}^{\mathbb{C}}(x)$ and $\Sigma''_x$ be the $-i$ eigenspace for $\tilde{J}^{\mathbb{C}}(x)$. Then, the $+i$, respectively $-i$, eigenspaces for $\tilde{J}^{\mathbb{C}}(x)|_{S_x}$ are $S'_x := S_x \cap \Sigma'_x$ and $S''_x := S_x \cap \Sigma''_x$. More explicitly, these eigenspaces are $\Sigma'_x = T^{1,0}_{X,x} \oplus \operatorname{Eig}(J_N^{\mathbb{C}}(x), i)$, $\Sigma''_x = T^{0,1}_{X,x} \oplus \operatorname{Eig}(J_N^{\mathbb{C}}(x), -i)$, $S'_x = \{0\} \oplus \operatorname{Eig}(J_N^{\mathbb{C}}(x), i)$, and $S''_x = \{0\} \oplus \mathrm{Eig}(J_N^{\mathbb{C}}(x), -i)$. Note that $S'_x \subset \Sigma'_x$, $S''_x \subset \Sigma''_x$, $\Sigma'_x \oplus \Sigma''_x = T_Y|_{X,x}$, and $S_x = S'_x \oplus S''_x$, where $\dim_{\mathbb{C}}(\Sigma'_x) = \dim_{\mathbb{C}}(\Sigma''_x) = \frac{1}{2}\dim_{\mathbb{C}}(T_Y|_{X,x}) = k$ and $\dim_{\mathbb{C}}(S'_x) = \dim_{\mathbb{C}}(S''_x) = \frac{1}{2}\dim_{\mathbb{C}}(N^{\mathbb{C}}_{X/Y^{\mathbb{R}},x}) = \frac{1}{2}(2k-2n) = k-n$. Therefore, if $f\colon X \hookrightarrow Y^{\mathbb{R}}$ is the given $C^\infty$ embedding, for any $x \in X$ – which we simultaneously regard as a point in $f(X)$, adopting this slight abuse of notation for convenience – we have that $(S'_x, S''_x, \Sigma'_x, \Sigma''_x) \in Q_x = \{(S', S'', \Sigma', \Sigma'') \in F^2_{(k-n,k)}(T_{Y,x}) \mid \Sigma' \oplus \Sigma'' = T_{Y,x}\}$, and in this way, get an embedding $F\colon X \hookrightarrow \mathcal{Z}_n(Y)$, where $F(x) = (f(x), S'_x, S''_x, \Sigma'_x, \Sigma''_x)$.

Let $\sigma$ be the conjugation on $Y$ (cf. Section 1.3). For $(S', S'', \Sigma', \Sigma'') \in Q_y$, put $\overline{S'} := d\sigma|_y(S') \subset T_{Y,\overline{y}}$, $\overline{\Sigma'} := d\sigma|_y(\Sigma') \subset T_{Y,\overline{y}}$, and define $\overline{S''}$ and $\overline{\Sigma''}$ similarly. Then, $\sigma$ gives rise to the anti-holomorphic involution $\tilde{\sigma}\colon \mathcal{Z}_n(Y) \to \mathcal{Z}_n(Y)$, $(y, S', S'', \Sigma', \Sigma'') \mapsto (\overline{y}, \overline{S''}, \overline{S'}, \overline{\Sigma''}, \overline{\Sigma'})$. The real points $\mathcal{Z}_n(Y)^{\mathbb{R}}$ of $\mathcal{Z}_n(Y)$ are the fixed points of $\tilde{\sigma}$, so $\mathcal{Z}_n(Y)^{\mathbb{R}} = \{(y, S', S'', \Sigma', \Sigma'') \in \mathcal{Z}_n(Y) \mid y \in Y^{\mathbb{R}}, S'' = \overline{S'}, \Sigma'' = \overline{\Sigma'}\}$. The anti-holomorphic character of $\sigma$ implies that $d\sigma$ is type-reversing and point-wise conjugate linear, so $\overline{\Sigma'_x} = T^{0,1}_{X,x} \oplus \mathrm{Eig}(J_N^{\mathbb{C}}(x), -i) = \Sigma''_x$ and similarly, $\overline{S'_x} = S''_x$. Therefore, $F(X) \subset \mathcal{Z}_n(Y)^{\mathbb{R}}$.

Since $\dim_{\mathbb{R}}(dF|_x(T_{X,x})) + \dim_{\mathbb{R}}(\mathbf{D}_{n,k,F(x)}) = 2N_{n,k} = \dim_{\mathbb{R}}(T_{\mathcal{Z}_n(Y),F(x)})$, the embedding $F$ is transverse to $\mathbf{D}_{n,k}$ if $dF|_x(T_{X,x})$ and $\mathbf{D}_{n,k,F(x)}$ intersect trivially. But the latter follows from $d\pi_Y|_{F(x)}(dF|_x(T_{X,x}) \cap \mathbf{D}_{n,k,F(x)}) = df|_x(T_{X,x}) \cap S'_x \oplus \Sigma''_x = \{0\}$. Hence $T_{\mathcal{Z}_n(Y),F(x)} = dF|_x(T_{X,x}) \oplus \mathbf{D}_{n,k,F(x)}$.

Let $J_{\mathcal{Z}_n(Y)}$ be the given complex structure on $\mathcal{Z}_n(Y)$. The quotient $T_{\mathcal{Z}_n(Y)}/\mathbf{D}_{n,k}$ is a holomorphic vector bundle on $\mathcal{Z}_n(Y)$. Since $\mathbf{D}_{n,k} \subset T_{\mathcal{Z}_n(Y)}$ is a holomorphic distribution, $J_{\mathcal{Z}_n(Y)}(\mathbf{D}_{n,k}) = \mathbf{D}_{n,k}$. So $J_{\mathcal{Z}_n(Y)}$ descends to a complex structure on $T_{\mathcal{Z}_n(Y)}/\mathbf{D}_{n,k}$. The transversality of $F$ implies that at any $x \in X$, there is a real isomorphism $\rho\colon T_{F(X),F(x)} \to T_{\mathcal{Z}_n(Y),F(x)}/\mathbf{D}_{n,k,F(x)}$. Then, $J^{\mathcal{Z}_n(Y),\mathbf{D}_{n,k}}_{F(X)}(x) := \rho^{-1} \circ J_{\mathcal{Z}_n(Y)}(x) \circ \rho$ defines an almost-complex structure $J^{\mathcal{Z}_n(Y),\mathbf{D}_{n,k}}_{F(X)}$ on $F(X)$, and then since $F$ is an embedding, the pullback section $J_F := F^*(J^{\mathcal{Z}_n(Y),\mathbf{D}_{n,k}}_{F(X)})$ is an almost-complex structure on $X$. Note that $T_{\mathcal{Z}_n(Y),F(x)}/\mathbf{D}_{n,k,F(x)} \simeq \Sigma'_x/S'_x$, and so $T_{\mathcal{Z}_n(Y),F(x)}/\mathbf{D}_{n,k,F(x)}$ is isomorphic to the holomorphic tangent space $T^{1,0}_{X,x}$, which is $T_{X,x}$ endowed with the linear complex structure $J_X(x)$. Run the above procedure now with $T^{1,0}_{X,x}$ playing the role of $T_{\mathcal{Z}_n(Y),F(x)}/\mathbf{D}_{n,k,F(x)}$, to find that $J_F = J_X$. □

**Lemma 1.** *Let $M$ be a real manifold, $g\colon X \hookrightarrow M$ be a $C^\infty$ embedding, and $i_\Delta\colon M \hookrightarrow M \times M$ be the diagonal embedding $i_\Delta(x) = (x, x)$. Embed $X$ into $M \times M$ via $i_\Delta \circ g$. Then, the normal bundle $N_{X/M\times M}$ has a complex structure $J_N$.*

**Proof.** Choose isomorphisms $T_M|_X \simeq N_{M/M\times M}|_X$, $T_M|_X \simeq T_X \oplus N_{X/M}$, $T_{M\times M}|_M \simeq T_M \oplus T_M$, and $T_{M\times M}|_X \simeq T_X \oplus N_{X/M\times M}$. Then,

$$T_X \oplus N_{X/M} \oplus T_M|_X \simeq f^*(T_{M\times M}|_M) = T_{M\times M}|_X \simeq T_X \oplus N_{X/M\times M}.$$

This implies that

$$N_{X/M\times M} \simeq N_{X/M} \oplus N_{X/M} \oplus T_X\,.$$

Let $J_{N_{X/M}\oplus N_{X/M}}$ be the tautological complex structure that is given by $J_{N_{X/M}\oplus N_{X/M}}(\zeta, \eta) = (-\eta, \zeta)$. Put $J_N := J_{N_{X/M}\oplus N_{X/M}} \oplus (-J_X)$, which defines a complex structure on $N_{X/M} \oplus N_{X/M} \oplus T_X$. Thanks to the above identifications, we may regard $J_N$ as a complex structure on $N_{X/M\times M}$. Note that this complex structure is not canonically defined as it depends on the chosen isomorphisms. Indeed, different choices of isomorphisms will lead to different complex structures, but this will not matter for how the lemma is applied later on. □

If we take $Y^{\mathbb{R}} = M \times M$ as in the lemma, we then have that

$$\begin{aligned} \Sigma'_x &= T^{1,0}_{X,x} \oplus \{(u,-iu) \mid u \in N^{\mathbb{C}}_{X/M,x}\} \oplus T^{0,1}_{X,x}\,, \\ \Sigma''_x &= T^{0,1}_{X,x} \oplus \{(u,iu) \mid u \in N^{\mathbb{C}}_{X/M,x}\} \oplus T^{1,0}_{X,x}\,, \\ S'_x &= \{0\} \oplus \{(u,-iu) \mid u \in N^{\mathbb{C}}_{X/M,x}\} \oplus T^{0,1}_{X,x}, \quad \text{and} \\ S''_x &= \{0\} \oplus \{(u,iu) \mid u \in N^{\mathbb{C}}_{X/M,x}\} \oplus T^{1,0}_{X,x}\,. \end{aligned}$$

This is why it is nicer to take $-J_X$ in the definition of $J_N$, as otherwise, $\Sigma'_x$ and $\Sigma''_x$ would have repeated direct sum factors of $T^{1,0}_{X,x}$ and $T^{0,1}_{X,x}$.

**Proof of Theorem 2.** First, note that there will always be an embedding $\psi\colon X^{2n} \hookrightarrow \mathbb{R}^k$ for some $k > 2n$. The strong Whitney embedding gives the optimal lower bound $k = 4n$, hence the assumption $k \geq 4n$. Let $M$ be a real manifold with $\dim_{\mathbb{R}} M = k$, and put $Y^{\mathbb{R}} = M \times M$. Let $\phi\colon \mathbb{R}^k \to B$ be a diffeomorphism of $\mathbb{R}^k$ with some open ball $B \subset M$. A $C^\infty$ embedding of $X$ into $Y^{\mathbb{R}}$ satisfying the hypothesis of Proposition 1 ($f$) is given by taking $g = \phi \circ \psi$ in Lemma 1 so that $f = i_\Delta \circ g$, with $J_N$ being the associated complex structure on the normal bundle. Apply Proposition 1 to $f$ in order to obtain the universal embedding $F\colon (X, J_X) \hookrightarrow (\mathcal{Z}_n(Y), \mathbf{D}_{n,k})$. □

**Example 1.** Let $S^6 \subset \mathbb{R}^7$ be the unit 6-dimensional sphere. Recall the standard octonion embedding of $S^6$ ($f_{\mathbb{O}}$), and the induced almost-complex structure ($J_{\mathbb{O}}$) that were defined in Subsection 1.2 of the introduction. Indeed, the complexification of $\mathbb{O} \simeq \mathbb{R}^8$ can be viewed as $\mathbb{C}^8 = \mathbb{O} \oplus i\mathbb{O}$. So $f_{\mathbb{O}}(S^6) \subset \Im(\mathbb{O})$ may be assumed to lie in the first factor, $\mathbb{O}$. We illustrate how to manufacture a universal embedding out of this initial embedding data, using Proposition 1.

Observe that for any $u \in S^6$,

$$\mathbb{O} \simeq T_{\mathbb{O}}|_{S^6,u} \simeq T_{S^6,u} \oplus N_{S^6/\mathbb{O},u} \simeq u^\perp \oplus \big(\mathbb{R}(1) \oplus \mathbb{R}(u)\big)\,,$$

where $T_{S^6,u} = u^\perp = \{v \in \Im(\mathbb{O}) \mid \langle v, u\rangle := \Re(v\bar{u}) = -\sum_{i=1}^7 v_i u_i = 0\}$.

Since $N_{S^6/\mathbb{O},u} \simeq \mathbb{R}(1) \oplus \mathbb{R}(u)$, $J_{\mathbb{O}}(u)\big(N_{S^6/\mathbb{O},u}\big) = N_{S^6/\mathbb{O},u}$, and so $J_{\mathbb{O}}|_{N_{S^6/\mathbb{O}}}$ is a complex structure on $N_{S^6/\mathbb{O}}$.

The complexification $J^{\mathbb{C}}_{\mathbb{O}}$ is such that for any $u \in S^6$, $J^{\mathbb{C}}_{\mathbb{O}}(u)(\zeta \otimes z) = \zeta u \otimes z$. The method in the proof of Proposition 1 outputs the universal embedding

$$F_{J_{\mathbb{O}}} : (S^6, J_{\mathbb{O}}) \hookrightarrow (\mathcal{Z}_3(\mathbb{C}^8), \mathbf{D}_{3,4}) = (Z_{3,4}, \mathcal{D}_{3,4})\,,$$

where

$$F_{J_{\mathbb{O}}}(u) = \big(f_{\mathbb{O}}(u), N^{1,0}_{S^6/\mathbb{O},u}, N^{0,1}_{S^6/\mathbb{O},u}, T^{1,0}_{\mathbb{O}}|_{S^6,u}, T^{0,1}_{\mathbb{O}}|_{S^6,u}\big)\,.$$

2.2. **Functorial property with respect to étale morphisms.** For each $n \in \mathbb{N}$, let $\mathbb{C}$-$\mathbf{Man}^n_{Et}$ be the category of complex manifolds of dimension $n$ whose morphisms are étale morphisms. For each $k \geq n \geq 1$, we have functors $\mathbf{Z}_n \colon \mathbb{C}\text{-}\mathbf{Man}^{2k}_{Et} \to \mathbb{C}\text{-}\mathbf{Man}^{N_{n,k}}_{Et}$, given by $\mathbf{Z}_n(Y) = \mathcal{Z}_n(Y)$, and for any étale morphism $f\colon Y \to Y'$, $\mathbf{Z}_n(f)\colon \mathcal{Z}_n(Y) \to \mathcal{Z}_n(Y')$ is the map that is defined at each $w = (y, S', S'', \Sigma', \Sigma'') \in \mathcal{Z}_n(Y)$ by

$$\mathbf{Z}_n(f)(w) = \left(f(y), df|_y(S'), df|_y(S''), df|_y(\Sigma'), df|_y(\Sigma'')\right).$$

Throughout, we will write $\mathcal{Z}_n(f)$ instead of $\mathbf{Z}_n(f)$.

For a fixed $n$, the universal embedding space associated with $Y$ is a complex directed manifold $(\mathcal{Z}_n(Y), \mathbf{D}_{n,k})$ such that every compact almost-complex $n$-dimensional manifold $(X, J_X)$ admits a totally real, transverse to $\mathbf{D}_{n,k}$, $J_X$-inducing embedding $F\colon X \hookrightarrow \mathcal{Z}_n(Y)$, and it satisfies the following universal property. For any $n$-dimensional compact almost-complex manifold $(X', J_{X'})$, any $C^\infty$ embeddings $g\colon X \hookrightarrow Y^{\mathbb{R}}$ and $g'\colon X' \hookrightarrow Y'^{\mathbb{R}}$, any pseudo-holomorphic étale map $\psi_X\colon (X, J_X) \to (X', J_{X'})$, and any étale morphism $\psi_Y\colon Y^{\mathbb{R}} \to Y'^{\mathbb{R}}$ that fit into a commutative diagram

$$\begin{array}{ccc} (X, J_X) & \xhookrightarrow{\;g\;} & Y^{\mathbb{R}} \\ \big\downarrow \psi_X & & \big\downarrow \psi_Y \\ (X', J_{X'}) & \xhookrightarrow{\;g'\;} & Y'^{\mathbb{R}}, \end{array}$$

there is a corresponding functorially defined morphism $\mathcal{Z}_n(\psi_Y)\colon (\mathcal{Z}_n(Y), \mathbf{D}_{n,k}) \to (\mathcal{Z}_n(Y'), \mathbf{D}'_{n,k})$ of complex directed manifolds making the diagram

$$\begin{array}{ccc} (X, J_X) & \xhookrightarrow{\;F\;} & \mathcal{Z}_n(Y) \\ \big\downarrow \psi_X & & \big\downarrow \exists \mathcal{Z}_n(\psi_Y) \\ (X', J_{X'}) & \xhookrightarrow{\;F'\;} & \mathcal{Z}_n(Y'), \end{array}$$

commute. By the Nash-Tognoli Theorem, we may assume that $X$ and $X'$ are smooth real algebraic varieties and that $g : X \hookrightarrow Y^{\mathbb{R}}$ and $g'\colon X' \hookrightarrow Y'^{\mathbb{R}}$ are algebraic. Then, our construction gives that $F$ and $F'$ are real algebraic as well.

## 3. The geometry of $\mathcal{Z}_n(Y)$ and related bundles

3.1. **Coordinates on $\mathcal{Z}_n(Y)$.** Let $U_y \simeq_\psi \mathbb{C}^{2k}$ be any holomorphic coordinate chart that is centered at a given point $y \in Y$. Let $p \in U_y$. We write $\psi(p) = (x_1, \dots, x_{2k})$ for the holomorphic coordinates of the point $p$. Let $S'_p = \mathrm{Span}_{\mathbb{C}}(\frac{\partial}{\partial x_j}\big|_p)_{j=n+1}^{k}$, $S''_p = \mathrm{Span}_{\mathbb{C}}(\frac{\partial}{\partial x_j}\big|_p)_{j=n+k+1}^{2k}$, $\Sigma'_p = \mathrm{Span}_{\mathbb{C}}(\frac{\partial}{\partial x_j}\big|_p)_{j=1}^{k}$, $\Sigma''_p = \mathrm{Span}_{\mathbb{C}}(\frac{\partial}{\partial x_j}\big|_p)_{j=k+1}^{2k}$, and put $f_p := (S'_p, S''_p, \Sigma'_p, \Sigma''_p)$. From this point on, we will omit reference to the point $p$ in the vector $\frac{\partial}{\partial x_j}\big|_p$, and denote it simply by $\frac{\partial}{\partial x_j}$. We develop a coordinate chart $U_y \times \mathcal{A}(f_y) \simeq \mathbb{C}^{2k} \times \mathbb{C}^{N_{n,k}-2k} = \mathbb{C}^{N_{n,k}}$ that is centered at $w_y := (y, f_y)$. To that end, let $E_{S'_p} = \mathrm{Span}_{\mathbb{C}}(\frac{\partial}{\partial x_j})_{j=1}^{n}$ and $E_{S''_p} = \mathrm{Span}_{\mathbb{C}}(\frac{\partial}{\partial x_j})_{j=k+1}^{n+k}$. In addition to $\Sigma'_p \oplus \Sigma''_p = \mathrm{T}_{Y,p}$, there are direct sum decompositions $S'_p \oplus E_{S'_p} = \Sigma'_p$ and $S''_p \oplus E_{S''_p} = \Sigma''_p$. Define

$$\mathcal{A}(f_y) := \Big\{(S', S'', \Sigma', \Sigma'') \in Q_p \mid p \in U_y, S' \cap E_{S'_p} = \{0\}, S'' \cap E_{S''_p} = \{0\}, \\ \Sigma' \cap \Sigma''_p = \{0\}, \Sigma'' \cap \Sigma'_p = \{0\}\Big\}.$$

For any $(S', S'', \Sigma', \Sigma'') \in \mathcal{A}(f_y)$, $S'$, $S''$, $\Sigma'$ and $\Sigma''$ correspond uniquely to maps $f_{S'} \in \mathrm{Hom}(S'_p, E_{S'_p}), f_{S''} \in \mathrm{Hom}(S''_p, E_{S''_p}), f_{\Sigma'} \in \mathrm{Hom}(\Sigma'_p, \Sigma''_p)$ and $f_{\Sigma''} \in \mathrm{Hom}(\Sigma''_p, \Sigma'_p)$, respectively, in the sense that $S' = \Gamma(f_{S'})$, $S'' = \Gamma(f_{S''})$, $\Sigma' = \Gamma(f_{\Sigma'})$, and $\Sigma'' = \Gamma(f_{\Sigma''})$, where $\Gamma(g)$ denotes the graph of the function $g$; e.g. $\Gamma(f_{S'}) = \{x + f_{S'}(x) | x \in S'_p\}$. Suppose that

$$f_{S'}(\frac{\partial}{\partial x_j}) = \sum_{i=1}^{n} z_{ij} \frac{\partial}{\partial x_i}, \text{ for } n+1 \le j \le k\,,$$

$$f_{S''}(\frac{\partial}{\partial x_j}) = \sum_{i=k+1}^{n+k} z_{ij} \frac{\partial}{\partial x_i}, \text{ for } n+k+1 \le j \le 2k\,,$$

$$f_{\Sigma'}(\frac{\partial}{\partial x_j}) = \sum_{i=k+1}^{2k} z_{ij} \frac{\partial}{\partial x_i} \text{ for } 1 \le j \le k\,, \quad \text{and}$$

$$f_{\Sigma''}(\frac{\partial}{\partial x_j}) = \sum_{i=1}^{k} z_{ij} \frac{\partial}{\partial x_i}, \text{for } k+1 \le j \le 2k\,.$$

Then,

$$S' = \mathrm{Span}_{\mathbb{C}}\Big(\frac{\partial}{\partial x_j} + \sum_{i=1}^{n} z_{ij} \frac{\partial}{\partial x_i}\Big)_{j=n+1}^{k}, S'' = \mathrm{Span}_{\mathbb{C}}\Big(\frac{\partial}{\partial x_j} + \sum_{i=k+1}^{n+k} z_{ij} \frac{\partial}{\partial x_i}\Big)_{j=n+k+1}^{2k},$$

$$\Sigma' = \mathrm{Span}_{\mathbb{C}}\Big(\frac{\partial}{\partial x_j} + \sum_{i=k+1}^{2k} z_{ij} \frac{\partial}{\partial x_i}\Big)_{j=1}^{k} \quad \text{and} \quad \Sigma'' = \mathrm{Span}_{\mathbb{C}}\Big(\frac{\partial}{\partial x_j} + \sum_{i=1}^{k} z_{ij} \frac{\partial}{\partial x_i}\Big)_{j=k+1}^{2k}.$$

This defines a coordinate map $q \colon \mathcal{A}(f_y) \to \mathbb{C}^{2(k^2+n(k-n))} = \mathbb{C}^{N_{n,k}-2k}$,

$$q(S', S'', \Sigma', \Sigma'') = Z := \left(\begin{array}{c|c} Z_{S'} & Z_{\Sigma''} \\ \hline Z_{\Sigma'} & Z_{S''} \end{array}\right),$$

where

$$Z_{S'} = \left(\begin{array}{c|c} \mathbf{0_{n\times n}} & z_{ij} \\ \hline \multicolumn{2}{c}{\mathbf{0_{(k-n)\times k}}} \end{array}\right)_{1 \le i \le n, n+1 \le j \le k},$$

$$Z_{S''} = \left(\begin{array}{c|c} \mathbf{0_{n\times n}} & z_{ij} \\ \hline \multicolumn{2}{c}{\mathbf{0_{(k-n)\times k}}} \end{array}\right)_{k+1 \le i \le n+k, n+k+1 \le j \le 2k},$$

$$Z_{\Sigma'} = (z_{ij})_{k+1 \le i \le 2k, 1 \le j \le k}\,,$$

and

$$Z_{\Sigma''} = (z_{ij})_{1\le i\le k, k+1\le j\le 2k}\ ,$$

and note that $q(f_y) = 0$.

Coordinates centered at $w_y$ are then given by the map $\phi := \psi \times q\colon U_y \times \mathcal{A}(f_y) \to \mathbb{C}^{N_{n,k}}$, $\phi(y, S', S'', \Sigma', \Sigma'') = (\psi(y), q(S', S'', \Sigma', \Sigma'')) = (\psi(y), Z)$, and note that $\phi(w_y) = 0$.

3.2. **Sub-bundles of the Grassmannian bundle.** The torsion operator is the section

$$\theta \in H^0\big(\mathcal{Z}_n(Y), \mathcal{O}(\Lambda^2(\mathbf{D}^*_{n,k}) \otimes T_{\mathcal{Z}_n(Y)}/\mathbf{D}_{n,k})\big)\,,$$

where

$$\theta(w)\colon \mathbf{D}_{n,k,w} \times \mathbf{D}_{n,k,w} \to T_{\mathcal{Z}_n(Y),w}/\mathbf{D}_{n,k,w}\,,$$

$\theta(w)(\zeta, \eta) = [\zeta, \eta] \mod \mathbf{D}_{n,k,w}$. At the central point, we have the coordinate form

**Lemma 2.**

$$\theta(w_y) = -2 \sum_{j=n+1}^{2k} \sum_{i=1}^{n} dx_j \wedge dz_{ij} \otimes \frac{\partial}{\partial x_i}\,.$$

**Proof.** Let $\mathcal{I} := \{1,\dots,n\}\times\{n+1,\dots,k\}\cup\{k+1,\dots,n+k\}\times\{n+k+1,\dots,2k\}\cup \{k+1,\dots,2k\}\times\{1,\dots,k\}\cup\{1,\dots,k\}\times\{k+1,\dots,2k\}$ and define

$$\begin{aligned}(\mathcal{I}) :=\big(&(1,n+1),\dots,(1,k),\dots,(n,n+1),\dots,(n,k),(k+1,n+k+1),\\ &\dots,(k+1,2k),\dots,(n+k,n+k+1),\dots,(n+k,2k),(k+1,1),\\ &\dots,(k+1,k),\dots,(2k,1),\dots,(2k,k),(1,k+1),\dots,(1,2k),\\ &\dots,(k,k+1),\dots,(k,2k)\big)\,.\end{aligned}$$

Here we work with the less compact, but equivalent coordinates $\phi(p, S', S'', \Sigma', \Sigma'') = (x, (z_{ij})_{(i,j)\in(\mathcal{I})})$. Let $w = (p, S', S'', \Sigma', \Sigma'') \in U_y \times \mathcal{A}(f_y)$ so that

$$\Big(\Big(\frac{\partial}{\partial x_l}\Big)_{l=1}^{2k}, \Big(\frac{\partial}{\partial z_{ij}}\Big)_{(i,j)\in(\mathcal{I})}\Big)$$

is a basis of the tangent space $T_{\mathcal{Z}_n(Y),w}$. Then,

$$\mathbf{D}_{n,k,w_y} = \Big\{(\zeta, u', u'', v', v'') \in T_{\mathcal{Z}_n(Y),w_y} \mid \zeta \in S'_y \oplus \Sigma''_y = \mathrm{Span}_{\mathbb{C}}\Big(\frac{\partial}{\partial x_l}\Big)_{l=n+1,\dots,2k}\Big\}$$

and

$$\mathbf{D}_{n,k,w} = \{(\zeta, u', u'', v', v'') \in T_{\mathcal{Z}_n(Y),w} \mid \zeta \in S' \oplus \Sigma''\}\,.$$

Let $a\colon U_y \times \mathcal{A}(f_y) \to \mathrm{M}_{n\times(N_{n,k}-n)}(\mathbb{C})$ be the function $a(w) = (\chi_{\mathcal{I}} z_{ij})$, where

$$\chi_{\mathcal{I}} = \begin{cases} 1 & \text{if } (i,j)\in\mathcal{I} \\ 0 & \text{otherwise.}\end{cases}$$

Then, since $\chi_{\mathcal{I}} = 0$ on $\{n+1, \ldots, k\} \times \{n+1, \ldots, k\}$,

$$S' \oplus \Sigma'' = \mathrm{Span}_{\mathbb{C}}\Big(\frac{\partial}{\partial x_j} + \sum_{i=1}^{k} a_{ij}(w)\frac{\partial}{\partial x_i}\Big)_{j=n+1,\ldots,2k}.$$

Therefore,

$$\mathbf{D}_{n,k,w_y} = \mathrm{Span}_{\mathbb{C}}\Big(\frac{\partial}{\partial x_l}\Big)_{l=n+1,\ldots,2k} \oplus \mathrm{Span}_{\mathbb{C}}\Big(\frac{\partial}{\partial z_{ij}}\Big)_{(i,j)\in(\mathcal{I})},$$
$$\mathbf{D}_{n,k,w} = \mathrm{Span}_{\mathbb{C}}\Big(\frac{\partial}{\partial x_j} + \sum_{i=1}^{k} a_{ij}(w)\frac{\partial}{\partial x_i}\Big)_{j=n+1,\ldots,2k} \oplus \mathrm{Span}_{\mathbb{C}}\Big(\frac{\partial}{\partial z_{ij}}\Big)_{(i,j)\in(\mathcal{I})},$$

and so $T_{\mathcal{Z}_n(Y),w_y}/\mathbf{D}_{n,k,w_y} \simeq \mathrm{Span}_{\mathbb{C}}\Big(\frac{\partial}{\partial x_l}\Big)_{l=1,\ldots,n}$.

Note that $a(w_y) = 0$. At $w_y$,

$$\Big[\frac{\partial}{\partial x_j} + \sum_{i=1}^{k} a_{ij}\frac{\partial}{\partial x_i}, \frac{\partial}{\partial x_l} + \sum_{i=1}^{k} a_{il}\frac{\partial}{\partial x_i}\Big] = \sum_{i=1}^{k}\Big(\frac{\partial a_{il}}{\partial x_j} - \frac{\partial a_{ij}}{\partial x_l}\Big)\frac{\partial}{\partial x_i}$$
$$= 0$$

because the function $a$ is independent of $x_i$ for all $i$, and

$$\theta(w_y)\Big(\frac{\partial}{\partial x_j} + \sum_{i=1}^{k} a_{ij}\frac{\partial}{\partial x_i}, \frac{\partial}{\partial z_{ml}}\Big) = \sum_{i=1}^{k}\theta(w_y)\Big(a_{ij}\frac{\partial}{\partial x_i}, \frac{\partial}{\partial z_{ml}}\Big)$$
$$= \sum_{i=1}^{k}\Big(-\frac{\partial a_{ij}}{\partial z_{ml}}\frac{\partial}{\partial x_i} \mod \mathbf{D}_{n,k,w_y}\Big)$$
$$= \sum_{i=1}^{n} -\frac{\partial a_{ij}}{\partial z_{ml}}\frac{\partial}{\partial x_i}.$$

Then since, $1 \le i \le n$, $n+1 \le j \le 2k$, and $(m,l) \in \mathcal{I}$, the index domains (i.e. the domains of $i$, $j$, $l$ and $m$) overlap exactly at $\{1, \ldots, n\} \times \{n+1, \ldots, 2k\} \cap \mathcal{I} = \{1, \ldots, n\} \times \{n+1, \ldots, 2k\}$. Therefore,

$$\theta(w_y) = \sum_{j,l=n+1}^{2k}\sum_{m=1}^{n}\theta(w_y)\Big(\frac{\partial}{\partial x_j} + \sum_{i=1}^{k} a_{ij}\frac{\partial}{\partial x_i}, \frac{\partial}{\partial z_{ml}}\Big)dx_j \wedge dz_{ml}$$
$$+ \sum_{j,l=n+1}^{2k}\sum_{m=1}^{n}\theta(w_y)\Big(\frac{\partial}{\partial z_{ml}}, \frac{\partial}{\partial x_j} + \sum_{i=1}^{k} a_{ij}\frac{\partial}{\partial x_i}\Big)dz_{ml} \wedge dx_j$$
$$= \sum_{j,l=n+1}^{2k}\sum_{m=1}^{n}\Big(\sum_{i=1}^{n} -\frac{\partial a_{ij}}{\partial z_{ml}}\frac{\partial}{\partial x_i}\Big)dx_j \wedge dz_{ml}$$

$$
\begin{aligned}
&+ \sum_{j,l=n+1}^{2k} \sum_{m=1}^{n} \Big( \sum_{i=1}^{n} \frac{\partial a_{ij}}{\partial z_{ml}} \frac{\partial}{\partial x_i} \Big) dz_{ml} \wedge dx_j \\
&= -2 \sum_{j,l=n+1}^{2k} \sum_{i,m=1}^{n} \frac{\partial a_{ij}}{\partial z_{ml}} dx_j \wedge dz_{ml} \otimes \frac{\partial}{\partial x_i} \\
&= -2 \sum_{j=n+1}^{2k} \sum_{i=1}^{n} dx_j \wedge dz_{ij} \otimes \frac{\partial}{\partial x_i} \,. \qquad \square
\end{aligned}
$$

**Remark 2.** For any $X = \sum_{l=n+1}^{2k} X_l \frac{\partial}{\partial x_l} + \sum_{(i,j)\in(\mathcal{I})} X_{ij} \frac{\partial}{\partial z_{ij}}$, $Y = \sum_{l=n+1}^{2k} Y_l \frac{\partial}{\partial x_l} + \sum_{(i,j)\in(\mathcal{I})} Y_{ij} \frac{\partial}{\partial z_{ij}} \in \mathbf{D}_{n,k,w_y}$,

$$
\theta(w_y)(X,Y) = -2 \sum_{i=1}^{n} \sum_{j=n+1}^{2k} (X_j Y_{ij} - Y_j X_{ij}) \frac{\partial}{\partial x_i} \,.
$$

Recall the projection mapping $\pi_Y \colon \mathcal{Z}_n(Y) \to Y$, $\pi_Y(y, q_y) = y$. Consider the Grassmannian bundle $\mathrm{Gr}^{\mathbb{C}}(\mathbf{D}_{n,k}, n)$ on $\mathcal{Z}_n(Y)$, whose fiber at $w \in \mathcal{Z}_n(Y)$ is the Grassmannian $\mathrm{Gr}^{\mathbb{C}}(\mathbf{D}_{n,k,w}, n)$. For any $w \in \mathcal{Z}_n(Y)$, define

$$
\mathbf{Gr}^{\circ}_{n,k,w} := \{ S \in \mathrm{Gr}^{\mathbb{C}}(\mathbf{D}_{n,k,w}, n) \mid d\pi_Y(w)|_S \quad \text{is injective} \}
$$

and

$$
\mathbf{I}_{n,k,w} := \{ S \in \mathrm{Gr}^{\mathbb{C}}(\mathbf{D}_{n,k,w}, n) \mid \theta(w)|_{S \times S} = 0 \} \,.
$$

Let $\mathbf{Gr}^{\circ}_{n,k}$ be the sub-bundle of the Grassmannian bundle with fiber $\mathbf{Gr}^{\circ}_{n,k,w}$ at $w$, and define $\mathbf{I}_{n,k}$ similarly. Now let $\mathbf{I}^{\circ}_{n,k}$ be the sub-bundle whose fiber over $w$ is $\mathbf{I}^{\circ}_{n,k,w} := \mathbf{Gr}^{\circ}_{n,k,w} \cap \mathbf{I}_{n,k,w}$. When $Y = \mathbb{C}^{2k}$, we denote these bundles by $Gr^{\circ}_{n,k}, \mathcal{I}_{n,k}$, and $\mathcal{I}^{\circ}_{n,k}$.

**Remark 3.** Let $F$ be a universal embedding as in Theorem 2. Then, $d\pi_Y(F(x)) \circ \bar{\partial}_{J_X} F(x)$ is injective for all $x \in X$. So, $\mathrm{Im}(\bar{\partial}_{J_X} F) \subset \mathbf{Gr}^{\circ}_{n,k}$, and if $J_X$ is integrable, then $\mathrm{Im}(\bar{\partial}_{J_X} F) \subset \mathbf{I}^{\circ}_{n,k}$.

3.2.1. *Demailly's strategy.* Let $J$ be a hypothetical complex structure on (the oriented) $S^6$. It must be homotopic to $J_{\mathbb{O}}$. The proof of Proposition 1 outputs isotopic universal embeddings $F_{J_{\mathbb{O}}} \colon (S^6, J_{\mathbb{O}}) \hookrightarrow \mathcal{Z}_3(\mathbb{C}^8)$ and $F_J \colon (S^6, J) \hookrightarrow \mathcal{Z}_3(\mathbb{C}^8)$, where the latter embedding is built up from $\widetilde{J} = J \oplus J_{\mathbb{O}}$ (cf. Example 1). The lifts $\mathrm{Im}(\overline{\partial}_{J_{\mathbb{O}}} F_{J_{\mathbb{O}}}) \colon (S^6, J_{\mathbb{O}}) \to \mathrm{Gr}^{\circ}_{3,4}$ and $\mathrm{Im}(\overline{\partial}_J F_J) \colon (S^6, J) \to \mathcal{I}^{\circ}_{3,4}$ are again homotopic (see Remark 3 above), so $[\mathrm{Im}(\overline{\partial}_{J_{\mathbb{O}}} F_{J_{\mathbb{O}}})] = [\mathrm{Im}(\overline{\partial}_J F_J)] \in \pi_6(\mathcal{I}^{\circ}_{3,4})$. Say that we could show that $[\mathrm{Im}(\overline{\partial}_{J_{\mathbb{O}}} F_{J_{\mathbb{O}}})]$ is a non-trivial homotopy class, and $\pi_6(\mathcal{I}^{\circ}_{3,4}) = 0$. This would be a contradiction, implying that $J$ cannot exist.

From the definition of homotopy groups, it should be evident that this method does not carry through to other almost-complex manifolds beyond $S^6$.

3.3. **A functorial group action.** This is an overview of basic facts that may be useful for understanding the geometry of $\mathcal{Z}_n(Y)$, and related bundles via group actions.

Let $(e_j)_{j=1}^{2k}$ be the standard basis of $\mathbb{C}^{2k}$, $S_0' = \mathrm{Span}_{\mathbb{C}}(e_j)_{j=n+1}^{k}$, $S_0'' = \mathrm{Span}_{\mathbb{C}}(e_j)_{j=n+k+1}^{2k}$, $\Sigma_0' = \mathrm{Span}_{\mathbb{C}}(e_j)_{j=1}^{k}$, and $\Sigma_0'' = \mathrm{Span}_{\mathbb{C}}(e_j)_{j=k+1}^{2k}$. Put $f_0 := (S_0', S_0'', \Sigma_0', \Sigma_0'') \in F^2_{(k-n,k)}(\mathbb{C}^{2k})$ and $w_0 := (0, f_0)$.

**Lemma 3.** *The group $GL_{2k}(\mathbb{C})$ acts transitively on $Q$ and the stabilizer $\Lambda$ of $f_0$ is the subgroup of $GL_{2k}(\mathbb{C})$ of all matrices of the form*

$$\left(\begin{array}{c|c} B_{k\times k} & 0_{k\times k} \\ \hline 0_{k\times k} & B'_{k\times k}\end{array}\right),$$

*where*

$$B_{k\times k} = \left(\begin{array}{c|c} \mathbf{B_{n\times n}} & \mathbf{0_{n\times(k-n)}} \\ \hline \multicolumn{2}{c}{\mathbf{B_{(k-n)\times k}}}\end{array}\right) \in GL_k(\mathbb{C})$$

*and*

$$B'_{k\times k} = \left(\begin{array}{c|c} \mathbf{B'_{n\times n}} & \mathbf{0_{n\times(k-n)}} \\ \hline \multicolumn{2}{c}{\mathbf{B'_{(k-n)\times k}}}\end{array}\right) \in GL_k(\mathbb{C})\,.$$

*As a result, $Q \simeq GL_{2k}(\mathbb{C})/\Lambda$.*

**Proof.** For any subspace $S = \mathrm{Span}_{\mathbb{C}}(s_j)_{j=1}^{r}$ of $\mathbb{C}^{2k}$, define $GS := \mathrm{Span}_{\mathbb{C}}(Gs_j)_{j=1}^{r}$, for any $G \in \mathrm{GL}_{2k}(\mathbb{C})$. The action of $\mathrm{GL}_{2k}(\mathbb{C})$ on $Q$, given by $(G, (S', S'', \Sigma', \Sigma'')) \mapsto G(S, S'', \Sigma', \Sigma'')$, where $G(S, S'', \Sigma', \Sigma'') = (GS', GS'', G\Sigma', G\Sigma'')$, is transitive. To compute the stabilizer of $f_0$, let $B = \big(B_{ij}\big)_{1\le i,j\le 2k} \in \mathrm{GL}_{2k}(\mathbb{C})$ and note that $Bf_0 = f_0$ iff

$$\begin{aligned}
&\text{for each}\quad n+1 \le j \le k,\ B_{ij} = 0 \quad \text{for all}\ \ 1 \le i \le n \quad \text{and}\quad k+1 \le i \le 2k\,,\\
&\text{for each}\quad n+k+1 \le j \le 2k,\ B_{ij} = 0 \quad \text{for all}\ \ k+1 \le i \le n+k\,,\\
&\text{for each}\quad 1 \le j \le k,\ B_{ij} = 0 \quad \text{for all}\ \ k+1 \le i \le 2k\,,
\end{aligned}$$

and

$$\text{for each}\quad k+1 \le j \le 2k,\ B_{ij} = 0 \quad \text{for all}\ \ 1 \le i \le k\,.$$

Therefore, $\big(B_{ij}\big)_{1\le i\le n, n+1\le j\le k} = 0_{n\times(k-n)}$, $\big(B_{ij}\big)_{k+1\le i\le n+k, n+k+1\le j\le 2k} = 0_{n\times(k-n)}$, $\big(B_{ij}\big)_{k+1\le i\le 2k, 1\le j\le k} = 0_{k\times k}$, and $\big(B_{ij}\big)_{1\le i\le k, k+1\le j\le 2k} = 0_{k\times k}$ so that $B$ is as claimed. □

The group $\mathrm{Aut}^{\mathrm{hol}}(Y)$ of biholomorphisms of $Y$ acts functorially on $\mathcal{Z}_n(Y)$:

$$\begin{aligned}
&\mathrm{Aut}^{\mathrm{hol}}(Y) \times \mathcal{Z}_n(Y) \to \mathcal{Z}_n(Y)\,,\\
&(f, (y, S', S'', \Sigma', \Sigma'')) \mapsto \mathcal{Z}_n(f)(y, S', S'', \Sigma', \Sigma'')\,,
\end{aligned}$$

where recall $\mathcal{Z}_n(f)(y, S', S'', \Sigma', \Sigma'') = (f(y), df|_y(S', S'', \Sigma', \Sigma''))$. Of course it can happen that $\mathrm{Aut}^{\mathrm{hol}}(Y) = \{\mathbf{Id}_Y\}$. Although this action is generally non-transitive, there are exceptions, for example when $Y = \mathbb{C}^{2k}$. Consider the subgroup $\mathrm{Aff}(\mathbb{C}^{2k})$ of $\mathrm{Aut}^{\mathrm{hol}}(\mathbb{C}^{2k})$. For any $f = Bz + c \in \mathrm{Aff}(\mathbb{C}^{2k})$, note that

$$\begin{aligned}\mathcal{Z}_n(f)(y, S', S'', \Sigma', \Sigma'') &= \big(By + c, df|_y(S', S'', \Sigma', \Sigma'')\big)\\ &= \big(By + c, B(S', S'', \Sigma', \Sigma'')\big)\,.\end{aligned}$$

**Lemma 4.** *The functorial action of $\mathit{Aff}(\mathbb{C}^{2k})$ on $\mathcal{Z}_n(\mathbb{C}^{2k})$ is transitive, the stabilizer of $w_0$ is $\mathcal{L} := \{f = Bz \mid B \in \Lambda\}$, and $\mathcal{Z}_n(\mathbb{C}^{2k}) \simeq \mathit{Aff}(\mathbb{C}^{2k})/\mathcal{L}$.*

**Proof.** Let $(y, S', S'', \Sigma', \Sigma'') \in \mathcal{Z}_n(\mathbb{C}^{2k}) = \mathbb{C}^{2k} \times Q$. We saw that $\mathrm{GL}_{2k}(\mathbb{C})$ acts transitively on $Q$. So there is a $B_0 \in \mathrm{GL}_{2k}(\mathbb{C})$ such that $(S', S'', \Sigma', \Sigma'') = B_0 f_0$. Put $c_0 = y$ so that $\mathcal{Z}_n(B_0 z + c_0)(w_0) = (y, S', S'', \Sigma', \Sigma'')$. □

## 4. Affine bundle structure

Let $\pi_{n,k}\colon \mathrm{Gr}^{\mathbb{C}}(\mathbf{\Delta}_{n,k}, n) \to \mathcal{Z}_n(Y)$ be the Grassmannian bundle. Let $\gamma_{n,k} \to \mathrm{Gr}^{\mathbb{C}}(\mathbf{\Delta}_{n,k}, n)$ be the tautological bundle with fiber $\gamma_{n,k,L} = L$ over any point $L \in \mathrm{Gr}^{\mathbb{C}}(\mathbf{\Delta}_{n,k}, n)$, viewed as a vector subspace of the corresponding fiber of $\mathbf{\Delta}_{n,k} \to \mathcal{Z}_n(Y)$. Since both $\gamma_{n,k}$ and $\pi^*_{n,k}(T_{\mathcal{Z}_n(Y)/Y})$ are vector bundles on $\mathrm{Gr}^{\mathbb{C}}(\mathbf{\Delta}_{n,k}, n)$, we can form the vector bundle

$$h\colon \mathrm{Hom}\big(\gamma_{n,k}, \pi^*_{n,k}(T_{\mathcal{Z}_n(Y)/Y})\big) \to \mathrm{Gr}^{\mathbb{C}}(\mathbf{\Delta}_{n,k}, n)\,.$$

The typical fiber of this bundle is $\mathrm{Hom}(\mathbb{C}^n, \mathbb{C}^{N_{n,k}-2k})$, coming from the fact that for any $(w, S_w) \in \mathrm{Gr}^{\mathbb{C}}(\mathbf{\Delta}_{n,k}, n)$,

$$h^{-1}(w, S_w) = \mathrm{Hom}\big(\gamma_{n,k,S_w}, \pi^*_{n,k}(T_{\mathcal{Z}_n(Y)/Y})\big)\,,$$

$\dim_{\mathbb{C}}(S_w) = n$, and $\dim_{\mathbb{C}}(T_{\mathcal{Z}_n(Y)/Y,w}) = N_{n,k} - 2k$.

Given any vector bundle homomorphism $F \in \mathrm{Hom}(\gamma_{n,k}, \pi^*_{n,k}(T_{\mathcal{Z}_n(Y)/Y}))$, if $\mathbf{Id}_{\gamma_{n,k}}$ is the identity morphism, we can produce another vector bundle morphism $\mathbf{Id}_{\gamma_{n,k}} \oplus F\colon \gamma_{n,k} \to \gamma_{n,k} \oplus \pi^*_{n,k}(T_{\mathcal{Z}_n(Y)/Y}) \subset \pi^*_{n,k}(\mathbf{D}_{n,k})$, and the graph of $F$ is then $\Gamma(F) := \mathrm{Im}(\mathbf{Id}_{\gamma_{n,k}} \oplus F)$. This can be regarded as a sub-bundle $\Gamma(F) \subset \gamma_{n,k} \oplus \pi^*_{n,k}(T_{\mathcal{Z}_n(Y)/Y}) \to \mathrm{Gr}^{\mathbb{C}}(\mathbf{\Delta}_{n,k}, n)$ and the fiber over $(w, V)$ is the usual graph of the linear map $F(w, V)$ in $\mathrm{Hom}(V, T_{\mathcal{Z}_n(Y)/Y,w})$, i.e. $\Gamma(F)_{(w,V)} = \mathrm{Im}(\mathbf{Id}_V + F(w, V)) = \Gamma(F(w, V))$. Let

$$\Theta\colon \mathrm{Hom}\big(\gamma_{n,k}, \pi^*_{n,k}(T_{\mathcal{Z}_n(Y)/Y})\big) \to \Lambda^2\gamma^*_{n,k} \otimes \pi^*_{n,k}(T_{\mathcal{Z}_n(Y)}/\mathbf{D}_{n,k})$$

be the bundle morphism $\Theta(F) = \theta|_{\Gamma(F)\times\Gamma(F)}$. On fibers, this becomes a map

$$\Theta(w, V)\colon \mathrm{Hom}\big(\gamma_{n,k,V}, T_{\mathcal{Z}_n(Y)/Y,w}\big) \to \Lambda^2\gamma^*_{n,k,V} \otimes T_{\mathcal{Z}_n(Y),w}/\mathbf{D}_{n,k,w}\,,$$

where $\Theta(w, V)(f) = \theta(w)|_{\Gamma(f)\times\Gamma(f)}$.

Let $\rho\colon \mathbf{Gr}^{\mathrm{o}}_{n,k} \to \mathrm{Gr}^{\mathbb{C}}(\mathbf{\Delta}_{n,k}, n)$ be the map with definition $\rho(w, V_w) = (w, d\pi(w)(V_w))$ for any $w \in \mathcal{Z}_n(Y)$ and $V_w \in \mathbf{Gr}^{\mathrm{o}}_{n,k,w}$.

**Lemma 5.** *If $w \in \mathcal{Z}_n(Y)$ and $V \in \mathit{Gr}^{\mathbb{C}}(\mathbf{\Delta}_{n,k,w}, n)$, then*

(1) $\Theta(w, V)$ *is linear, and*

(2) $\mathbf{I}^{\mathrm{o}}_{n,k,w} \cap \rho^{-1}(w, V) \simeq \ker\big(\Theta(w, V)\big)$.

**Proof.** To prove (1), let $\zeta$, $\eta \in V$. Remark 2 implies that $\theta(w)(\zeta,\eta) = \theta(w)\big(f(\zeta), f(\eta)\big) = 0$. From the bilinearity and anti-symmetry of $\theta(w)$, it then follows that $\theta(w)\big(\zeta + f(\zeta), \eta + f(\eta)\big) = \theta(w)\big(\zeta, f(\eta)\big) - \theta(w)\big(\eta, f(\zeta)\big)$. Apply this identity to $f + g$ to find that

$$\theta(w)(\zeta+(f+g)\big(\zeta), \eta+(f+g)(\eta)\big) = \theta(w)\big(\zeta+f(\zeta), \eta+f(\eta)\big)+\theta(w)\big(\zeta+g(\zeta), \eta+g(\eta)\big)\,.$$

Clearly, for any $\lambda \in \mathbb{C}$, $\theta(w)\big(\zeta + \lambda f(\zeta), \eta + \lambda f(\eta)\big) = \lambda\theta(w)\big(\zeta + f(\zeta), \eta + f(\eta)\big)$. The above also shows that $\Theta$ is a true vector bundle homomorphism.

The second claim will follow at once from the description of the fiber $\rho^{-1}(w, V)$ that we provide in the proof of the theorem below. $\square$

4.1. **The proof of Theorem 3.** Let $(U_\alpha)$ be a local trivialization of the holomorphic fiber bundle $\mathcal{Z}_n(Y) \xrightarrow{\pi_Y} Y$ so that $\mathcal{Z}_n(U_\alpha) = \pi_Y^{-1}(U_\alpha) \simeq U_\alpha \times Q$ (cf. Remark 1). The relative tangent bundle sequence need not be globally split. However, it is locally split with respect to a trivialization, meaning that, in particular, the short exact sequence

$$0 \to T_{\mathcal{Z}_n(Y)/Y}|_{\mathcal{Z}_n(U_\alpha)} \to T_{\mathcal{Z}_n(Y)}|_{\mathcal{Z}_n(U_\alpha)} \to \pi_Y^*(T_Y)|_{\mathcal{Z}_n(U_\alpha)} \to 0$$

is split and induces the split sequence

$$(*) \quad 0 \to T_{\mathcal{Z}_n(Y)/Y}|_{\mathcal{Z}_n(U_\alpha)} \to \mathbf{D}_{n,k}|_{\mathcal{Z}_n(U_\alpha)} \to \boldsymbol{\Delta}_{n,k}|_{\mathcal{Z}_n(U_\alpha)} \to 0\,.$$

Not all splittings of $(*)$ necessarily come from a trivialization of $\mathcal{Z}_n(Y) \xrightarrow{\pi_Y} Y$.

Define $G_\alpha := \mathrm{Gr}^{\mathbb{C}}(\boldsymbol{\Delta}_{n,k}, n)|_{\mathcal{Z}_n(U_\alpha)}$. Let $w_\alpha \in \mathcal{Z}_n(U_\alpha)$ and $S_{w_\alpha} \in \mathrm{Gr}^{\mathbb{C}}(\boldsymbol{\Delta}_{n,k,w_\alpha}, n)$. The direct sum of vector spaces $\gamma_{n,k}|_{G_\alpha, S_{w_\alpha}} \oplus \pi_{n,k}^*(T_{\mathcal{Z}_n(Y)/Y}|_{\mathcal{Z}_n(U_\alpha), w_\alpha})$ is defined thanks to the splitting of $(*)$, and here $\gamma_{n,k}|_{G_\alpha}$ is the restriction of the tautological bundle to $G_\alpha$. Notice that

$$\begin{aligned}
\rho^{-1}(w_\alpha, S_{w_\alpha}) &= \{V_{w_\alpha} \in \mathbf{Gr}^{\mathrm{o}}_{n,k,w_\alpha} \mid d\pi_Y(w_\alpha)(V_{w_\alpha}) = S_{w_\alpha}\} \\
&= \{V_{w_\alpha} \in \mathrm{Gr}^{\mathbb{C}}(\gamma_{n,k}|_{G_\alpha, S_{w_\alpha}} \oplus \pi_{n,k}^*(T_{\mathcal{Z}_n(Y)/Y}|_{\mathcal{Z}_n(U_\alpha), w_\alpha}), n) \mid \\
&\quad V_{w_\alpha} \cap \pi_{n,k}^*(T_{\mathcal{Z}_n(Y)/Y}|_{\mathcal{Z}_n(U_\alpha), w_\alpha}) = \{0\}\} \\
&= \{\Gamma(f) \mid f \in \mathrm{Hom}(\gamma_{n,k}|_{G_\alpha, S_{w_\alpha}}, \pi_{n,k}^*(T_{\mathcal{Z}_n(Y)/Y}|_{\mathcal{Z}_n(U_\alpha), w_\alpha}))\}\,.
\end{aligned}$$

Consider the map

$$\mathbf{t}\colon \mathrm{Hom}\big(\gamma_{n,k}, \pi_{n,k}^*(T_{\mathcal{Z}_n(Y)/Y})\big) \times_{\mathrm{Gr}^{\mathbb{C}}(\boldsymbol{\Delta}_{n,k}, n)} \mathbf{Gr}^{\mathrm{o}}_{n,k} \to \mathbf{Gr}^{\mathrm{o}}_{n,k}\,,$$

where the domain is the fiber product, and where the map $\mathbf{t}$ is defined on the fiber over $(w_\alpha, S_{w_\alpha})$ by

$$\mathrm{Hom}\big(\gamma_{n,k}|_{G_\alpha, S_{w_\alpha}}, \pi_{n,k}^*(T_{\mathcal{Z}_n(Y)/Y}|_{\mathcal{Z}_n(U_\alpha), w_\alpha})\big) \times \rho^{-1}(w_\alpha, S_{w_\alpha}) \to \rho^{-1}(w_\alpha, S_{w_\alpha})\,,$$

$$\big(f, \Gamma(g)\big) \mapsto \Gamma(f+g)\,.$$

Now, if $\Gamma(f'), \Gamma(f'') \in \rho^{-1}(w_\alpha, S_{w_\alpha})$, $\Gamma(f'') = \Gamma(f + f')$ iff $f = f'' - f'$, which is to say that the action of $\mathrm{Hom}\big(\gamma_{n,k}|_{G_\alpha, S_{w_\alpha}}, \pi_{n,k}^*(T_{\mathcal{Z}_n(Y)/Y}|_{\mathcal{Z}_n(U_\alpha), w_\alpha})\big)$ on $\rho^{-1}(w_\alpha, S_{w_\alpha})$ is free and transitive. The map $\mathbf{t}$ thus realizes the fiber $\rho^{-1}(w_\alpha, S_{w_\alpha})$ as an affine linear space modelled on the vector space

$$\mathrm{Hom}\big(\gamma_{n,k}|_{G_\alpha, S_{w_\alpha}}, \pi_{n,k}^*(T_{\mathcal{Z}_n(Y)/Y}|_{\mathcal{Z}_n(U_\alpha), w_\alpha})\big)\,.$$

There are a few subtleties. Since $\rho^{-1}(w_\alpha, S_{w_\alpha})$ is not genuinely a vector space, an isomorphism with $\mathrm{Hom}\big(\gamma_{n,k}|_{G_\alpha, S_{w_\alpha}}, \pi^*_{n,k}(T_{\mathcal{Z}_n(Y)/Y}|_{\mathcal{Z}_n(U_\alpha), w_\alpha})\big)$ cannot be defined. The biholomorphisms between fibers of the bundles $\rho : \mathbf{Gr}^{\mathrm{o}}_{n,k} \to \mathrm{Gr}^{\mathbb{C}}(\mathbf{\Delta}_{n,k}, n)$ and $h\colon \mathrm{Hom}\big(\gamma_{n,k}, \pi^*_{n,k}(T_{\mathcal{Z}_n(Y)/Y})\big) \to \mathrm{Gr}^{\mathbb{C}}(\mathbf{\Delta}_{n,k}, n)$ are not canonical. They depend on the local holomorphic splitting of the relative tangent bundle sequence. That $Y$ is a generic complex even dimensional manifold is the underlying reason for there being no natural splitting of the sequence, which in turn implies the non-naturality of the biholomorphisms.

Note that the sets $G_\alpha$, which are preimages of the open subsets $\mathcal{Z}_n(U_\alpha) \subset \mathcal{Z}_n(Y)$ under the continuous bundle projection $\pi_{n,k}$, form an open cover of the total space $\mathrm{Gr}^{\mathbb{C}}(\mathbf{\Delta}_{n,k}, n)$. Let $\mathbf{\Gamma} := \{\Gamma(f) \mid f \in \mathrm{Hom}(\mathbb{C}^n, \mathbb{C}^{N_{n,k}-2k})\}$, which is an affine linear space modelled after $\mathrm{Hom}(\mathbb{C}^n, \mathbb{C}^{N_{n,k}-2k})$. For all $(w, S_w) \in G_\alpha$, we have a biholomorphism $\rho^{-1}(w, S_w) \simeq \mathbf{\Gamma}$, inducing a biholomorphism $\rho^{-1}(G_\alpha) \simeq G_\alpha \times \mathbf{\Gamma}$. So the $G_\alpha$ are a trivialization of $\rho\colon \mathbf{Gr}^{\mathrm{o}}_{n,k} \to \mathrm{Gr}^{\mathbb{C}}(\mathbf{\Delta}_{n,k}, n)$ as a holomorphic fiber bundle with typical fiber $\mathbf{\Gamma}$, and so $\rho\colon \mathbf{Gr}^{\mathrm{o}}_{n,k} \to \mathrm{Gr}^{\mathbb{C}}(\mathbf{\Delta}_{n,k}, n)$ is a holomorphic affine linear bundle modelled on

$$h\colon \mathrm{Hom}\big(\gamma_{n,k}, \pi^*_{n,k}(T_{\mathcal{Z}_n(Y)/Y})\big) \to \mathrm{Gr}^{\mathbb{C}}(\mathbf{\Delta}_{n,k}, n)\,.$$

Part 2 of Lemma 5 follows from the biholomorphism

$$\rho^{-1}(w_\alpha, S_{w_\alpha}) \simeq \mathrm{Hom}(\gamma_{n,k}|_{G_\alpha, S_{w_\alpha}}, \pi^*_{n,k}(T_{\mathcal{Z}_n(Y)/Y}|_{\mathcal{Z}_n(U_\alpha), w_\alpha}))\,.$$

So the fiber over $(w_\alpha, S_{w_\alpha})$ of the sub-bundle $\rho|_{\mathbf{I}^{\mathrm{o}}_{n,k}} : \mathbf{I}^{\mathrm{o}}_{n,k} \to \mathrm{Gr}^{\mathbb{C}}(\mathbf{\Delta}_{n,k}, n)$ is an affine linear space modelled on the vector subspace $\ker\big(\Theta(w_\alpha, S_{w_\alpha})\big)$ of

$$\mathrm{Hom}\big(\gamma_{n,k,S_{w_\alpha}}, \pi^*_{n,k}(T_{\mathcal{Z}_n(Y)/Y, w_\alpha})\big)\,.$$

The bundle $\mathbf{I}^{\mathrm{o}}_{n,k}$ is thus an affine bundle modelled on the (sub-)vector bundle $h|_{\ker\Theta}\colon \ker\Theta \to \mathrm{Gr}^{\mathbb{C}}(\mathbf{\Delta}_{n,k}, n)$, where $\ker\Theta := \{F \in \mathrm{Hom}\big(\gamma_{n,k}, \pi^*_{n,k}(T_{Z_n(Y)/Y})\big) \mid \Theta(F) = 0\}$. □

An immediate consequence of the theorem is that the total spaces $\mathbf{Gr}^{\mathrm{o}}_{n,k}$ and $\mathbf{I}^{\mathrm{o}}_{n,k}$ are of the same homotopy type. In particular, $\pi_i(\mathbf{Gr}^{\mathrm{o}}_{n,k}, \mathbf{I}^{\mathrm{o}}_{n,k}) = 0$, for all $i \geq 0$. However, one could instead look at, perhaps, fibered homotopy groups [6]. This could be a topic worthwhile studying.

4.2. **The proof of Proposition 2 and Proposition 3.** Theorem 3 and its proof imply that the quotient $\mathbf{Gr}^{\mathrm{o}}_{n,k}/\mathbf{I}^{\mathrm{o}}_{n,k}$ is the vector bundle $\mathrm{Hom}\big(\gamma_{n,k}, \pi^*_{n,k}(T_{\mathcal{Z}_n(Y)/Y})\big)/\ker\Theta$. Proposition 2 then follows from the argument presented below.

**Proof of Proposition 2.** We exhibit an isomorphism of vector bundles on $\mathrm{Gr}^{\mathbb{C}}(\mathbf{\Delta}_{n,k}, n)$,

$$\mathrm{Hom}\big(\gamma_{n,k}, \pi^*_{n,k}(T_{\mathcal{Z}_n(Y)/Y})\big)/\ker\Theta \simeq \mathrm{Hom}\big(\Lambda^2\gamma_{n,k}, \pi^*_{n,k}(T_{\mathcal{Z}_n(Y)}/\mathbf{D}_{n,k})\big)\,.$$

Observe that there is an isomorphism

$$T_{\mathcal{Z}_n(Y)/Y,w_y} \simeq \mathrm{Hom}(S'_y, \Sigma'_y/S'_y) \oplus \mathrm{Hom}(S''_y, \Sigma''_y/S''_y) \\ \oplus \mathrm{Hom}(\Sigma'_y, \Sigma''_y) \oplus \mathrm{Hom}(\Sigma''_y, \Sigma'_y)\,,$$

which depends on choices of direct sum decompositions $\Sigma'_y = Q'_y \oplus S'_y$ and $\Sigma''_y = Q''_y \oplus S''_y$. Set

$$A_{w_y} := \mathrm{Hom}(S''_y, \Sigma''_y/S''_y) \oplus \mathrm{Hom}(\Sigma'_y, \Sigma''_y) \oplus \mathrm{Hom}(\Sigma''_y, S'_y)$$

so that $T_{\mathcal{Z}_n(Y)/Y,w_y} \simeq A_{w_y} \oplus \mathrm{Hom}(\mathbf{\Delta}_{n,k,w_y}, \Sigma'_y/S'_y)$, and

$$\mathrm{Hom}(V, T_{\mathcal{Z}_n(Y)/Y,w_y}) \simeq \mathrm{Hom}\big(V, \mathrm{Hom}\big(\mathbf{\Delta}_{n,k,w_y}, \Sigma'_y/S'_y\big)\big) \oplus \mathrm{Hom}(V, A_{w_y})\,,$$

for any $V \in \mathrm{Gr}^{\mathbb{C}}(\mathbf{\Delta}_{n,k,w_y}, n)$. We show that the linear in $\mathrm{Hom}(V, T_{\mathcal{Z}_n(Y)/Y,w_y})$ function $\Theta(w_y, V)$ vanishes on $\mathrm{Hom}(V, A_{w_y})$. This implies that

$$\ker\left(\Theta(w_y, V)\right) = \ker\left(\Theta(w_y, V)|_{\mathrm{Hom}(V,\mathrm{Hom}(\mathbf{\Delta}_{n,k,w_y},\Sigma'_y/S'_y))}\right) \oplus \mathrm{Hom}(V, A_{w_y})\,,$$

and so

$$\mathrm{Hom}(V, T_{\mathcal{Z}_n(Y)/Y,w_y})/\ker\left(\Theta(w_y, V)\right) \simeq \\ \mathrm{Hom}\big(V, \mathrm{Hom}(\mathbf{\Delta}_{n,k,w_y}, \Sigma'_y/S'_y)\big)/\ker\left(\Theta(w_y, V)|_{\mathrm{Hom}(V,\mathrm{Hom}(\mathbf{\Delta}_{n,k,w_y},\Sigma'_y/S'_y))}\right).$$

Observe that for any $f \in \mathrm{Hom}(V, A_{w_y})$, there exist unique $g \in \mathrm{Hom}\big(V, \mathrm{Hom}(S''_y, \Sigma''_y/S''_y)\big)$, $g' \in \mathrm{Hom}\big(V, \mathrm{Hom}(\Sigma'_y, \Sigma''_y)\big)$ and $g'' \in \mathrm{Hom}\big(V, \mathrm{Hom}(\Sigma''_y, S'_y)\big)$ such that $f = g + g' + g''$. If $(X^m)_{m=1}^n$ is a basis of $V$, then in terms of the basis $\left(\frac{\partial}{\partial z_{ij}}\right)_{(i,j)\in(\mathcal{I})}$ of $T_{\mathcal{Z}_n(Y)/Y,w_y}$,

$$\begin{aligned} f(X^m) &= g(X^m) + g'(X^m) + g''(X^m) \\ &= \sum_{k+1\le i\le k+n, n+k+1\le j\le 2k} g(X^m)_{ij}\frac{\partial}{\partial z_{ij}} + \sum_{k+1\le i\le 2k, 1\le j\le k} g'(X^m)_{ij}\frac{\partial}{\partial z_{ij}} \\ &\quad + \sum_{n+1\le i\le k, k+1\le j\le 2k} g''(X^m)_{ij}\frac{\partial}{\partial z_{ij}}\,, \end{aligned}$$

for any $1 \le m \le n$. In particular, we see that $f(X^m)_{ij} = 0$ for all $1 \le i \le n$ and $n+1 \le j \le 2k$, from which it follows (cf. Remark 2) that

$$\theta(w_y)(X^a + f(X^a), X^b + f(X^b)) = \\ -2\sum_{i=1}^{n}\sum_{j=n+1}^{2k}(X^a_j f(X^b)_{ij} - X^b_j f(X^a)_{ij})\frac{\partial}{\partial x_i} = 0\,.$$

Next we make use of the isomorphism $\Psi\colon \mathrm{Hom}(V, \mathrm{Hom}\big(\mathbf{\Delta}_{n,k,w_y}, \Sigma'_y/S'_y)\big) \to \mathrm{Hom}(V \otimes \mathbf{\Delta}_{n,k,w_y}, \Sigma'_y/S'_y)$, $f \mapsto \Psi(f)$, where for any $a \in V$ and $b \in \mathbf{\Delta}_{n,k,w_y}$, $\Psi(f(a)(b)) = f(a \otimes b)$, to re-express the quotient vector space of interest. If

$\zeta, \eta \in V$, then

$$\begin{aligned}\theta(w_y)\big(\zeta + f(\zeta), \eta + f(\eta)\big) &= -2 \sum_{i=1}^{n} \sum_{j=n+1}^{2k} (\zeta_j f(\eta)_{ij} - \eta_j f(\zeta)_{ij}) \frac{\partial}{\partial x_i} \\ &= -2\big(f(\eta)\zeta - f(\zeta)\eta\big) \\ &= -2 f(\eta \otimes \zeta - \zeta \otimes \eta)\,.\end{aligned}$$

Let $J = \langle a \otimes b - b \otimes a \mid a, b \in V\rangle \subset V \otimes V$ be the ideal such that $S^2(V) = V \otimes V/J$. Then, $\Theta(w_y, V)(f) = 0$ iff $\theta(w_y)\big(\zeta + f(\zeta), \eta + f(\eta)\big) = 0$ for all $\zeta, \eta \in V$ iff $f|_J = 0$. The isomorphism

$$\{f \in \mathrm{Hom}(V \otimes V, \Sigma'_y/S'_y) \mid f|_J = 0\} \simeq \mathrm{Hom}(S^2(V), \Sigma'_y/S'_y)$$

along with a choice of decomposition $V \oplus R \simeq \boldsymbol{\Delta}_{n,k,w_y}$ lead to

$$\mathrm{Hom}(V \otimes \boldsymbol{\Delta}_{n,k,w_y}, \Sigma'_y/S'_y) \simeq \mathrm{Hom}(V \otimes V, \Sigma'_y/S'_y) \oplus \mathrm{Hom}(V \otimes R, \Sigma'_y/S'_y)\,.$$

Since $\Theta(w_y, V)$ is independent of $\mathrm{Hom}(V \otimes R, \Sigma'_y/S'_y)$,

$$\begin{aligned}&\ker\big(\Theta(w_y, V)|_{\mathrm{Hom}(V, \mathrm{Hom}(\boldsymbol{\Delta}_{n,k,w_y}, \Sigma'_y/S'_y))}\big) \\ &\qquad\qquad \simeq \mathrm{Hom}(S^2(V), \Sigma'_y/S'_y) \oplus \mathrm{Hom}(V \otimes R, \Sigma'_y/S'_y)\,.\end{aligned}$$

Finally, note that since $V \otimes V = S^2(V) \oplus \Lambda^2(V)$, $\mathrm{Hom}(V \otimes V, \Sigma'_y/S'_y) \oplus \mathrm{Hom}(V \otimes R, \Sigma'_y/S'_y)/\mathrm{Hom}(S^2(V), \Sigma'_y/S'_y) \oplus \mathrm{Hom}(V \otimes R, \Sigma'_y/S'_y) \simeq \mathrm{Hom}(\Lambda^2(V), \Sigma'_y/S'_y)$, where $T_{\mathcal{Z}_n(Y), w_y}/\mathbf{D}_{n,k,w_y} \simeq \Sigma'_y/S'_y$. □

The isomorphism also follows after noticing that $\Theta(w, V)$ is surjective. However, inasmuch as we can only see a less elegant proof of this fact, we do not include it here.

Let $(X, J_X)$ be an $n$-dimensional compact almost-complex manifold and $F\colon (X, J_X) \hookrightarrow \mathcal{Z}_n(Y)$ be a universal embedding. Consider the map $\tilde{F}\colon (X, J_X) \to \mathbf{Gr}^{\mathrm{o}}_{n,k}$, $\tilde{F}(x) = \mathrm{Im}(\bar{\partial}_{J_X} F(x))$, and the following diagram

$$\begin{array}{ccccc} (\rho\circ\tilde{F})^*(\Lambda^2\gamma^*_{n,k}\otimes\pi^*_{n,k}(T_{\mathcal{Z}_n(Y)}/\mathbf{D}_{n,k})) & \longrightarrow & \Lambda^2\gamma^*_{n,k}\otimes\pi^*_{n,k}(T_{\mathcal{Z}_n(Y)}/\mathbf{D}_{n,k}) & & \\ \downarrow & & \uparrow{\scriptstyle\tilde{\Theta}} & \searrow & \\ (X, J_X) & \xrightarrow{\tilde{F}} & \mathbf{Gr}^{\mathrm{o}}_{n,k} & \xrightarrow{\rho} & \mathrm{Gr}^{\mathbb{C}}(\boldsymbol{\Delta}_{n,k}, n)\,, \end{array}$$

where $\tilde{\Theta}$ is defined as $\tilde{\Theta}(\Gamma(F)) = \theta|_{\Gamma(F)\times\Gamma(F)}$, and note that based on the discussion preceding Lemma 5, we have that $\mathbf{Gr}^{\mathrm{o}}_{n,k} = \{\Gamma(F) \mid F \in \mathrm{Hom}\big(\gamma_{n,k}, \pi^*_{n,k}(T_{\mathcal{Z}_n(Y)/Y})\big)\}$. Indeed, by Proposition 2, $(\rho\circ\tilde{F})^*\big(\Lambda^2\gamma^*_{n,k}\otimes\pi^*_{n,k}(T_{\mathcal{Z}_n(Y)}/\mathbf{D}_{n,k})\big) \simeq (\rho\circ\tilde{F})^*(\mathbf{Gr}^{\mathrm{o}}_{n,k}/\mathbf{I}^{\mathrm{o}}_{n,k})$. Now we can see that the Nijenhuis tensor $N_{J_X}$ of $J_X$ is essentially the pullback of the bundle morphism $\tilde{\Theta}$ by the lift $\tilde{F}$ of $F$ to $\mathbf{Gr}^{\mathrm{o}}_{n,k}$.

**Proof of Proposition 3.** This is a direct application of Proposition 5.1 [9], since for any $x \in X$,

$$(4\tilde{\Theta} \circ \tilde{F})(x) = 4\tilde{\Theta}(\mathrm{Im}(\bar{\partial}_{J_X} F(x))) = 4\theta(F(x))|_{\mathrm{Im}(\bar{\partial}_{J_X} F(x))\times\mathrm{Im}(\bar{\partial}_{J_X} F(x))} = N_{J_X}(x)\,.$$

□

4.3. **The homotopy YC through universal bundles.** Our next goal will be to understand the relationship between the geometry of the vector bundle $\mathbf{Gr}^{\circ}_{n,k}/\mathbf{I}^{\circ}_{n,k}$, and the (non-)integrability of almost-complex structures on a manifold $X$. This will give rise to a strategy to study the HYC, which is covered below.

Suppose that $J_X$ is a non-integrable almost-complex structure on $X$, and let $J$ be a hypothetical, smoothly homotopical to $J_X$ integrable structure. Thereom 2 produces transverse to $\mathbf{D}_{n,k}$ isotopic embeddings $F_{J_X}: (X, J_X) \hookrightarrow \mathscr{Z}_n(Y)$ and $F_J: (X, J) \hookrightarrow \mathscr{Z}_n(Y)$ such that $\mathrm{Im}(\bar{\partial}_{J_X} F_{J_X}) \subset \mathrm{Gr}^{\mathbb{C}}(\mathbf{D}_{n,k}, n)$, $\mathrm{Im}(\bar{\partial}_J F_J) \subset \mathbf{I}_{n,k}$. The idea was to study the topology of $\mathbf{I}_{n,k}$ relative to $\mathrm{Gr}^{\mathbb{C}}(\mathbf{D}_{n,k}, n)$, or in other words of the quotient $\mathrm{Gr}^{\mathbb{C}}(\mathbf{D}_{n,k}, n)/\mathbf{I}_{n,k}$, independently in hopes of detecting an obstruction to the existence of $J$. It is unclear if such an obstruction can exist at all, but we will partially address that below. A point of concern was that $\mathrm{Gr}^{\mathbb{C}}(\mathbf{D}_{n,k}, n)/\mathbf{I}_{n,k}$ is not an easily recognizable geometric object, and so it would not be so simple to speak of its invariants. We remedied this by replacing the latter quotient with the vector bundle $\mathbf{Gr}^{\circ}_{n,k}/\mathbf{I}^{\circ}_{n,k}$: the injectivity of $d\pi_Y(F_t(x)) \circ \bar{\partial}_{J_t} F_t(x)$ ensures that the replacement by Zariski open subsets $\mathbf{Gr}^{\circ}_{n,k} \subset \mathrm{Gr}^{\mathbb{C}}(\mathbf{D}_{n,k}, n)$, $\mathbf{I}^{\circ}_{n,k} \subset \mathbf{I}_{n,k}$ has no effect on the strategy (cf. Remark 3).

Put $\widetilde{F}_{J_X} := \mathrm{Im}(\bar{\partial}_{J_X} F_{J_X}(x))$, $\widetilde{F}_J := \mathrm{Im}\big(\bar{\partial}_J F_J(x)\big)$. These lifts are homotopic. By Theorem 3, $\mathbf{Gr}^{\circ}_{n,k}$ is a holomorphic affine bundle over $\mathrm{Gr}^{\mathbb{C}}(\boldsymbol{\Delta}_{n,k}, n)$. Let the bundle projection be denoted by $\rho$. One then has homotopic maps $G_{J_X} := \rho \circ \widetilde{F}_{J_X}: (X, J_X) \to \mathrm{Gr}^{\mathbb{C}}(\boldsymbol{\Delta}_{n,k}, n)$, $G_J := \rho \circ \widetilde{F}_J: (X, J) \to \mathrm{Gr}^{\mathbb{C}}(\boldsymbol{\Delta}_{n,k}, n)$. Say that we had a (refined enough) invariant, given as a mapping $m\colon \mathrm{Vect}_{\mathbb{C}}\big(\mathrm{Gr}^{\mathbb{C}}(\boldsymbol{\Delta}_{n,k}, n)\big) \to \mathcal{R}$, where $\mathrm{Vect}_{\mathbb{C}}\big(\mathrm{Gr}^{\mathbb{C}}(\boldsymbol{\Delta}_{n,k}, n)\big)$ is the collection of all complex vector bundles with base $\mathrm{Gr}^{\mathbb{C}}(\boldsymbol{\Delta}_{n,k}, n)$, and $\mathcal{R}$ is some category. Assume that $m$ satisfied the following conditions: (i) if $\mathbf{J}$ is integrable, then $m\big(G^*_{\mathbf{J}}(\mathbf{Gr}^{\circ}_{n,k}/\mathbf{I}^{\circ}_{n,k})\big) = 0$; and (ii) if $\mathbf{J}$, $\mathbf{J}'$ are homotopic, then $m\big(G^*_{\mathbf{J}}(\mathbf{Gr}^{\circ}_{n,k}/\mathbf{I}^{\circ}_{n,k})\big) = m\big(G^*_{\mathbf{J}'}(\mathbf{Gr}^{\circ}_{n,k}/\mathbf{I}^{\circ}_{n,k})\big)$. Suppose that we could show that $m\big(G^*_{J_X}(\mathbf{Gr}^{\circ}_{n,k}/\mathbf{I}^{\circ}_{n,k})\big) \neq 0$. This would lead to the contradiction

$$0 = m\big(G^*_J(\mathbf{Gr}^{\circ}_{n,k}/\mathbf{I}^{\circ}_{n,k})\big) = m\big(G^*_{J_X}(\mathbf{Gr}^{\circ}_{n,k}/\mathbf{I}^{\circ}_{n,k})\big) \neq 0\,.$$

The conclusion would be that in such cases there can be no complex structure that is homotpical to $J_X$, in this way shedding light upon the HYC. The wider applicability of invariants such as $m$ could be a topic worth exploring.

Ordinary Chern classes satisfy condition (ii), and they do so rather strongly, via vector bundle isomorphism, but they need not satisfy (i).They are most likely too coarse for the strategy. For example, with the universal embedding $F_{J_{\mathbb{O}}}$ built from the standard octonion embedding of $S^6$ (cf. Example 1), the strategy appears to be inconclusive as it can be shown that for any almost-complex structure $J$ on $S^6$, $c_j\big(G^*_J(\mathrm{Gr}^{\circ}_{3,4}/\mathcal{I}^{\circ}_{3,4})\big) = 0$ for all $j$. For a detailed computation of these Chern classes, see the subsection 4.3.1 below. On that note, although the isomorphism of Proposition 2 might be useful, the isomorphism between the pulled back by $G_J$, for any $J$, vector bundles overlooks key information. The moral to be extracted from this discussion should be that an invariant that perceives at the homotopy level could be more efficient. Typically, vector bundles and their isomorphism classes are understood to be one and the same; here, if there was an identification, it would have to be homotopical in nature. The success of the strategy thus depends mostly on whether one can find a good invariant that works well with a given set of embedding data. More fundamentally, perhaps what is missing in the first place is an adequate notion of equivalence relation on $\mathrm{Vect}_{\mathbb{C}}\big(\mathrm{Gr}^{\mathbb{C}}(\boldsymbol{\Delta}_{n,k}, n)\big)$.

We are now able to state an advantage of Proposition 2 over Theorem 1: it provides great flexibility when it comes to the choice of vector bundle $\mathbf{Gr}^{\mathrm{o}}_{n,k}/\mathbf{I}^{\mathrm{o}}_{n,k}$ to be employed in the strategy. In a way, the more complicated the choice of $Y$ is the more likely the strategy will be successful. This is because one is looking for obstructions, and so the less trivial $\mathbf{Gr}^{\mathrm{o}}_{n,k}/\mathbf{I}^{\mathrm{o}}_{n,k}$ is, the better.

4.3.1. *The Chern classes of* $Gr^{\mathrm{o}}_{3,4}/\mathcal{I}^{\mathrm{o}}_{3,4}$. Recall the complex, quasi-projective manifold $\mathcal{Z}_3(\mathbb{C}^8) = \mathbb{C}^8 \times Q$ of complex dimension $N_{3,4} = 46$, where

$$Q = \{(S', S'', \Sigma', \Sigma'') \in F_{(1,4)}(\mathbb{C}^8) \times F_{(1,4)}(\mathbb{C}^8) \mid \Sigma' \oplus \Sigma'' = \mathbb{C}^8\}\,,$$

and where $F_{(1,4)}(\mathbb{C}^8)$ is the complex projective manifold of flags of signature $(1,4)$. For convenience, we denote the relative tangent bundle $T_{\mathcal{Z}_3(\mathbb{C}^8)/\mathbb{C}^8}$ by $T_Q$ so that

$$T_{\mathcal{Z}_3(\mathbb{C}^8)} = \pi^*(T_{\mathbb{C}^8}) \oplus T_Q,$$

and the affine bundle projection $\rho\colon \mathrm{Gr}^{\mathrm{o}}_{3,4} \to \mathrm{Gr}^{\mathbb{C}}(\Delta_{3,4}, 3)$ by $d\pi$.

Now, let $J$ be a hypothetical integrable almost-complex structure on $S^6$, and $J_t\colon [0,1] \times S^6 \to \mathrm{End}_{\mathbb{R}}(T_{S^6})$ be a homotopy of almost-complex structures from $J_0 = J_{\mathbb{O}}$ to $J_1 = J$. For each $t \in [0,1]$, we get a universal embedding $F_t\colon (S^6, J_t) \hookrightarrow \mathcal{Z}_3(\mathbb{C}^8)$ in the following way: if $J_t^{\mathbb{C}} \in \mathrm{End}_{\mathbb{C}}(T_{\mathbb{O}}|^{\mathbb{C}}_{S^6})$ is the complexification of $J_t$, $\Sigma'_t(u) = \mathrm{Eig}(J_t^{\mathbb{C}}(u), i)$, $\Sigma''_t(u) = \mathrm{Eig}(J_t^{\mathbb{C}}(u), -i)$, $S'_u(t) = \mathrm{Eig}(J_t^{\mathbb{C}}(u)|_{S_u}, i)$, and $S''_u(t) = \mathrm{Eig}(J_t^{\mathbb{C}}(u)|_{S_u}, -i)$, where $S_u = \{0\} \oplus N^{\mathbb{C}}_{S^6/\mathbb{O},u}$, then

$$F_t(u) = \big(u, S'_u(t), S''_u(t), \Sigma'_t(u), \Sigma''_t(u)\big)\,.$$

See Proposition 1. So, $J_t$ defines the isotopy $F_t$ of universal embeddings, which in turn gives rise to a homotopy of uniquely defined for $J_t$ lifts $\tilde{F}_t\colon [0,1] \times (S^6, J_t) \to Gr^{\mathrm{o}}_{3,4}$, where $\tilde{F}_t(u) = \mathrm{Im}(\bar{\partial}_{J_t} F_t(u))$. Let $G_t = d\pi \circ \tilde{F}_t$. It turns out that

**Proposition 4.** *All of the Chern classes of* $G_t^*(Gr^{\mathrm{o}}_{3,4}/\mathcal{I}^{\mathrm{o}}_{3,4})$ *vanish.*

Let us first observe that there is a vector bundle isomorphism

**Lemma 6.** $G_t^*(Gr^{\mathrm{o}}_{3,4}/\mathcal{I}^{\mathrm{o}}_{3,4}) \simeq \Lambda^2\overline{T^*_{S^6}} \otimes T_{S^6}$.

**Proof.** From Proposition 2, $\mathrm{Gr}^{\mathrm{o}}_{3,4}/\mathcal{I}^{\mathrm{o}}_{3,4} \simeq \Lambda^2\gamma^*_{3,4} \otimes \pi^*_{3,4}(T_{\mathcal{Z}_3(\mathbb{C}^8)}/\mathcal{D}_{3,4})$. By the transversality of the universal embedding $F_t\colon (S^6, J_t) \hookrightarrow \mathcal{Z}_3(\mathbb{C}^8)$, $F_{t*}(T_{S^6}) \simeq T_{\mathcal{Z}_3(\mathbb{C}^8)}/\mathcal{D}_{3,4}$. Then, since $F_t = \pi_{3,4} \circ G_t$, $G_t^*\big(\pi^*_{3,4}(T_{\mathcal{Z}_3(\mathbb{C}^8)}/\mathcal{D}_{3,4})\big) \simeq T_{S^6}$.

Let $u \in S^6$, $\zeta \in T_{S^6,u}$, and note that

$$\begin{aligned}\bar{\partial}_{J_t} F_t(u)(\zeta) &= \frac{1}{2}\big(dF_t(u)(\zeta) + J_{\mathcal{Z}_3(\mathbb{C}^8)}(F_t(u)) \circ dF_t(u) \circ J_t(u)(\zeta)\big)\\ &= \frac{1}{2}\big(dF_t(u)(\zeta) + i dF_t(u)(J_t(u)(\zeta))\big)\,,\end{aligned}$$

where $J_{\mathcal{Z}_3(\mathbb{C}^8)}$ is the given complex structure on $\mathcal{Z}_3(\mathbb{C}^8)$, which we may assume acts by multiplication by $i$ on fibers. The push-forward of $F_t$ at $u \in S^6$ is an $\mathbb{R}$-linear map

$$dF_t(u) = \mathbf{Id}_{T_{S^6,u}} \oplus \phi^t_u \colon T_{S^6,u} \to T_{\mathbb{C}^8,u} \oplus T_{Q,(S'_u(t),S''_u(t),\Sigma'_t(u),\Sigma''_t(u))}\,,$$

where $\mathbf{Id}_{T_{S^6,u}}$ is the $\mathbb{C}^8$-component, and $\phi^t_u$ is the $Q$-component of the push-forward of $F_t$ at $u$. Now, since the push-forward of $\pi\colon \mathcal{Z}_3(\mathbb{C}^8) \to \mathbb{C}^8$ is essentially the identity map, meaning that at any $w \in \mathcal{Z}_3(\mathbb{C}^8)$, $d\pi(w) = I_8 \oplus \mathbf{0}_{8\times 38}$, we obtain that

$$\begin{aligned} d\pi\big(F_t(u)\big)\big(\bar\partial_{J_t} F_t(u)(\zeta)\big) &= \frac{1}{2}\big(d\pi(F_t(u))\big(\zeta + \phi^t_u(\zeta)\big) \\ &\quad + d\pi(F_t(u))\big(J_t(u)(\zeta) + i\phi^t_u(J_t(u)(\zeta))\big)\big) \\ &= \frac{1}{2}\big(\zeta + iJ_t(u)(\zeta)\big)\,. \end{aligned}$$

Therefore, $G_t(u) = \{\frac{1}{2}(\zeta + iJ_t(u)(\zeta)) \mid \zeta \in T_{S^6,u}\} = T^{0,1}_{S^6,u}$, $G_t^*\gamma_{3,4} = T^{0,1}_{S^6} \simeq \overline{T_{S^6}}$, and so $G_t^*\gamma^*_{3,4} \simeq \overline{T^*_{S^6}}$, where $\overline{T_{S^6}}$ is the conjugate bundle to $T_{S^6}$, i.e. $T_{S^6}$ endowed with the complex structure $-J_{\mathbb{O}}$. And then,

$$\begin{aligned} G_t^*(\mathrm{Gr}^{\mathrm{o}}_{3,4}/\mathcal{I}^{\mathrm{o}}_{3,4}) &\simeq \Lambda^2 G_t^*\gamma^*_{3,4} \otimes G_t^*\big(\pi^*_{3,4}(T_{\mathcal{Z}_3(\mathbb{C}^8)}/\mathcal{D}_{3,4})\big) \\ &\simeq \Lambda^2\overline{T^*_{S^6}} \otimes T_{S^6}\,. \end{aligned}$$ □

The proof of Proposition 4 is straightforward once we review some basic facts about Chern classes of complex vector bundles. Let $E \to X$ be a rank $m$ complex vector bundle with Chern roots $\alpha_i := c_1(L_i)$, for $1 \le i \le m$, and $V \to X$ be a rank $n$ complex vector bundle with Chern roots $\beta_j := c_1(L'_j)$, for $1 \le j \le n$. For any set of indeterminates $x_1, \dots, x_r$, let $\sigma_k(x_1, \dots, x_r)$ be the $k$-th elementary symmetric polynomial in the $x_i$. The Chern roots of $E \otimes V$ are $\alpha_i + \beta_j$, where $1 \le i \le m$ and $1 \le j \le n$. By definition, the $k$-th Chern class of $E \otimes V$ is

$$\begin{aligned} c_k(E \otimes V) &= \sigma_k(\alpha_1 + \beta_1, \dots, \alpha_m + \alpha_n) \\ &= \sum_{1 \le i_1 < \dots < i_k \le m}\ \sum_{1 \le j_1 < \dots < j_k \le n} (\alpha_{i_1} + \beta_{j_1}) \dots (\alpha_{i_k} + \beta_{j_k})\,, \end{aligned}$$

and we can write

$$(\alpha_{i_1} + \beta_{j_1}) \dots (\alpha_{i_k} + \beta_{j_k}) = \alpha_{i_1} \dots \alpha_{i_k} + \beta_{j_1} \dots \beta_{j_k} + p^{i_1,\dots,i_k}_{j_1,\dots,j_k}\,,$$

for some $p^{i_1,\dots,i_k}_{j_1,\dots,j_k} \in \mathbb{Z}[\alpha_{i_1},\dots,\alpha_{i_k},\beta_{j_1},\dots,\beta_{j_k}]$. Then, since

$$\begin{aligned}
\sigma_k(\alpha_1+\beta_1,\dots,\alpha_m+\alpha_n) &= \sum_{1\le j_1<\dots<j_k\le n}\ \sum_{1\le i_1<\dots<i_k\le m} \alpha_{i_1}\dots\alpha_{i_k} \\
&\quad + \sum_{1\le i_1<\dots<i_k\le m}\ \sum_{1\le j_1<\dots<j_k\le n} \beta_{j_1}\dots\beta_{j_k} \\
&\quad + \sum_{1\le i_1<\dots<i_k\le m}\ \sum_{1\le j_1<\dots<j_k\le n} p^{i_1,\dots,i_k}_{j_1,\dots,j_k} \\
&= \sum_{1\le j\le n}\ \sum_{1\le i_1<\dots<i_k\le m} \alpha_{i_1}\dots\alpha_{i_k} + \sum_{1\le i\le m}\ \sum_{1\le j_1<\dots<j_k\le n} \beta_{j_1}\dots\beta_{j_k} \\
&\quad + \sum_{1\le i_1<\dots<i_k\le m}\ \sum_{1\le j_1<\dots<j_k\le n} p^{i_1,\dots,i_k}_{j_1,\dots,j_k} \\
&= n\sigma_k(\alpha_1,\dots,\alpha_m) + m\sigma_k(\beta_1,\dots,\beta_n) + P\,,
\end{aligned}$$

where $P = \sum_{1\le i_1<\dots<i_k\le m}\sum_{1\le j_1<\dots<j_k\le n} p^{i_1,\dots,i_k}_{j_1,\dots,j_k} \in \mathbb{Z}[c_1(E),\dots,c_{k-1}(E),c_1(V),\dots,c_{k-1}(V)]$, it follows that

$$c_k(E\otimes V) = nc_k(E) + mc_k(V) + P\,. \tag{1}$$

Moreover, if $m=3$,

$$c_3(\Lambda^2 E) = -c_3(E) + c_1(E)c_2(E)\,. \tag{2}$$

This is because $c_1(E)=\alpha_1+\alpha_2+\alpha_3$, $c_2(E)=\alpha_1\alpha_2+\alpha_1\alpha_3+\alpha_2\alpha_3$, $c_3(E)=\alpha_1\alpha_2\alpha_3$, and the Chern roots of $\Lambda^2E$ are $\alpha_1+\alpha_2,\alpha_1+\alpha_3,\alpha_2+\alpha_3$, implying that

$$\begin{aligned}
c_3(\Lambda^2E) &= \sigma_3(\alpha_1+\alpha_2,\alpha_1+\alpha_3,\alpha_2+\alpha_3) \\
&= (\alpha_1+\alpha_2)(\alpha_1+\alpha_3)(\alpha_2+\alpha_3) \\
&= (\alpha_1\alpha_3\alpha_2+\alpha_2\alpha_1\alpha_3) + (\alpha_1^2\alpha_2+\alpha_1^2\alpha_3) \\
&\quad + (\alpha_2\alpha_1\alpha_2+\alpha_2\alpha_3\alpha_2) + (\alpha_1\alpha_3^2+\alpha_2\alpha_3^2) \\
&= 2c_3(E) + (\alpha_1^2\alpha_2+\alpha_1^2\alpha_3+\alpha_2\alpha_1\alpha_2+\alpha_2\alpha_3\alpha_2+\alpha_1\alpha_3^2+\alpha_2\alpha_3^2) \\
&= 2c_3(E) + (c_1(E)c_2(E) - 3c_3(E)) \\
&= -c_3(E) + c_1(E)c_2(E)\,.
\end{aligned}$$

**Proof of Proposition 4.** Since $H^{2k}(S^6;\mathbb{Z})=0$ for all $k\neq 3$, the only not automatically trivial Chern classes of $G_t^*(\mathrm{Gr}^{\mathrm{o}}_{3,4}/\mathcal{I}^{\mathrm{o}}_{3,4})$ are the 3rd Chern classes. However, from Lemma 6 alongside with equations (1) and (2), we conclude that $c_3(G_t^*(\mathrm{Gr}^{\mathrm{o}}_{3,4}/\mathcal{I}^{\mathrm{o}}_{3,4})) = -3c_3(\overline{T^*_{S^6}}) + 3c_3(T_{S^6}) = 0$. □

## 5. Conclusion

Let us recapitulate the main contributions made by this article. We began with a generalization of Theorem 1 to universal embedding spaces associated to complex manifolds (Theorem 2). This was achieved through an intermediate result (Proposition 1) that was key in the correct formulation of Demailly's proposed idea to tackle the $S^6$ problem (see section 3.2.1). In order to check if the idea works, one would need to understand if there is anything stopping $\mathrm{Im}(\overline{\partial}_{J_{\mathbb{O}}} F_{J_{\mathbb{O}}})$ from being null-homotopic. In addition, one would need to check if $\pi_6(\mathcal{I}^{\mathrm{o}}_{3,4})$ is trivial. Then, we described the geometry of bundle constructions on our generalized universal embedding spaces that are related to the integrability of almost-complex structures ($\mathbf{Gr}^{\mathrm{o}}_{n,k}$, and $\mathbf{I}^{\mathrm{o}}_{n,k}$). We proved that these fiber bundles are actually holomorphic affine bundles over a Grassmannian (Theorem 3), implying that the quotient $\mathbf{Gr}^{\mathrm{o}}_{n,k}/\mathbf{I}^{\mathrm{o}}_{n,k}$ is a holomorphic vector bundle. And then, we showed that this quotient bundle is isomorphic to the vector bundle $\Lambda^2\gamma^*_{n,k} \otimes \pi^*_{n,k}(T_{\mathcal{Z}_n(Y)}/\mathbf{D}_{n,k})$ (Proposition 2). Additionally, we provided a linearization formula for the Nijenhuis tensor of an almost-complex structure (Proposition 3). The motivation for studying the geometry of the vector bundles $\mathbf{Gr}^{\mathrm{o}}_{n,k}/\mathbf{I}^{\mathrm{o}}_{n,k}$ was their potential use to investigate the non-existence of a complex structure up to homotopy. This potential could be realized if one could find a vector bundle invariant fitting the description of $m$ (see Section 4.3).

Looking ahead to the possibility of continuing this line of work, we believe that the main techniques presented here may apply to other kinds of geometric structures, besides almost-complex structures. For instance, one could try to explore extensions of our universal embedding theory to other $G$-structures viewed as vector bundle sections on the underlying manifold. Perhaps, the best suited structures for such an extension are almost-CR-structures, and almost-contact structures. In the future, it could be interesting to see if such embedding results are attainable, and if they give rise to extrinsic tools for the study of (non-)integrability as they do in the almost-complex case.

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