Mathematica Bohemica

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Mathematica Bohemica, Vol. 148 (2023), No. 2, 197–210

Persistent URL: `http://dml.cz/dmlcz/151684`

MONOTONE AND CONE PRESERVING MAPPINGS ON POSETS

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Received February 25, 2021. Published online May 16, 2022.
Communicated by Sándor Radeleczki

Abstract. We define several sorts of mappings on a poset like monotone, strictly monotone, upper cone preserving and variants of these. Our aim is to study in which posets some of these mappings coincide. We define special mappings determined by two elements and investigate when these are strictly monotone or upper cone preserving. If the considered poset is a semilattice then its monotone mappings coincide with semilattice homomorphisms if and only if the poset is a chain. Similarly, we study posets which need not be semilattices but whose upper cones have a minimal element. We extend this investigation to posets that are direct products of chains or an ordinal sum of an antichain and a finite chain. We characterize equivalence relations induced by strongly monotone mappings and show that the quotient set of a poset by such an equivalence relation is a poset again.

Keywords: poset; directed poset; semilattice; chain; monotone; strictly monotone; upper cone preserving; strictly upper cone preserving; strongly upper cone preserving; ordinal sum; induced equivalence relation

MSC 2020: 06A11, 06A06, 06A12

1. Introduction

Partially ordered sets, shortly posets, are relational structures which occur frequently both in various areas of mathematics and in applications. Posets have been studied from numerous points of view depending on their application. One possible approach is to consider various mappings on a given poset and check when they coincide. Examples of such mappings are monotone mappings, cone preserving mappings, filter preserving mappings, etc. If the poset in question is of a particular form, e.g., if it is a semilattice or lattice, we can consider also homomorphisms. If a poset

Support of the research by the Austrian Science Fund (FWF), project I 4579-N, and the Czech Science Foundation (GAČR), project 20-09869L entitled "The many facets of orthomodularity", is gratefully acknowledged.

DOI: 10.21136/MB.2022.0026-21

is directed then it can be converted into a so-called directoid, i.e., a groupoid with one binary operation. Homomorphisms of such directed posets were already investigated by the first author in [2]. For a bit more general relational structures, so-called quasiordered sets, cone preserving mappings were studied in [4]. Homomorphisms of semilattices were investigated by Berrone in [1].

Based on the mentioned results, we introduce a list of interesting mappings on posets and find out how the fact, that some of the mappings from this list coincide or satisfy some special assumptions, influences the structure of the poset.

We do not consider the research on this topic to be finished. We rather consider our paper as a starting point which could inspire other authors to go on in this direction. We are convinced that the algebraic theory of posets is of fundamental importance in the whole of mathematics.

2. Elementary concepts and results

Let $\mathbf{P} := (P, \leqslant)$ be a poset, $A, B \subseteq P$ and $a, b \in P$. Then $A \leqslant B$ should mean $x \leqslant y$ for all $(x, y) \in A \times B$. Instead of $A \leqslant \{b\}$, $\{a\} \leqslant B$ and $\{a\} \leqslant \{b\}$ we simply write $A \leqslant b$, $a \leqslant B$ and $a \leqslant b$, respectively. The sets

$$\begin{aligned} L(A) &:= \{x \in P \colon\ x \leqslant A\}, \\ U(A) &:= \{x \in P \colon\ A \leqslant x\} \end{aligned}$$

are called the *lower* and *upper cone* of A, respectively. Instead of $L(A \cup B)$, $L(A \cup \{b\})$, $L(\{a\} \cup B)$, $L(\{a, b\})$ and $L(\{a\})$ we simply write $L(A, B)$, $L(A, b)$, $L(a, B)$, $L(a, b)$ and $L(a)$, respectively. In a similar way we proceed for U and in analogous cases. Moreover, put $L^*(a) := (L(a)) \setminus \{a\}$ and $U^*(a) := (U(a)) \setminus \{a\}$. $\mathbf{P}$ is called *up-directed* if $U(x, y) \neq \emptyset$ for all $x, y \in P$. The subset A of P is called a *filter* of $\mathbf{P}$ if $x \in A$ and $x \leqslant y$ imply $y \in A$. Let $\operatorname{Fil}\mathbf{P}$ denote the set of all filters of $\mathbf{P}$. For every $a \in P$, the set $[a) := \{x \in P \colon\ a \leqslant x\}$ is a filter of $\mathbf{P}$, the so-called *principal filter* generated by a. Note that $[a) = U(a)$.

Remark 2.1. If $\mathbf{P}$ is a poset then $(\operatorname{Fil}\mathbf{P}, \subseteq)$ is a complete lattice with the smallest element $\emptyset$ and greatest element P and

$$\bigvee_{i \in I} F_i = \bigcup_{i \in I} F_i, \quad \bigwedge_{i \in I} F_i = \bigcap_{i \in I} F_i$$

for every family F_i, $i \in I$, of filters of $\mathbf{P}$.

A mapping $f \colon\ P \to P$ is called

(i) *monotone* if $x \leqslant y$ implies $f(x) \leqslant f(y)$,
(ii) *strictly monotone* if $x < y$ implies $f(x) < f(y)$,
(iii) *upper cone preserving* if $f(U(x,y)) = U(f(x), f(y))$ for all $x, y \in P$,
(iv) *strictly upper cone preserving* if $f(U(x,y)) = U(f(x), f(y))$ for all $x, y \in P$ with $x \neq y$,
(v) *strongly upper cone preserving* if $f(U(x,y)) = U(f(x), f(y))$ for all $x, y \in P$ with $f(x) \neq f(y)$.

Observe that for a monotone f we have $f(L(x,y)) \subseteq L(f(x), f(y))$ and $f(U(x,y)) \subseteq U(f(x), f(y))$ for all $x, y \in P$.

Throughout the paper, we consider only non-void posets.

In the following, for every poset $(P, \leqslant)$ and every element $a \in P$, let f_a denote the constant mapping from P to P with value a.

Using the mapping f_a which is evidently monotone, we can characterize up-directed posets having a maximal element as follows.

Lemma 2.2. *Let $\mathbf{P} = (P, \leqslant)$ be a poset and $a \in P$. Then $\mathbf{P}$ is up-directed and a is the maximal element of $\mathbf{P}$ if and only if f_a is upper cone preserving.*

Proof. Let $b, c \in P$. If $\mathbf{P}$ is up-directed and a maximal then

$$f_a(U(b,c)) = \{a\} = U(a) = U(a,a) = U(f_a(b), f_a(c))$$

showing that f_a is upper cone preserving. Conversely, if f_a is upper cone preserving then

$$f_a(U(b,c)) = U(f_a(b), f_a(c)) = U(a,a) = U(a) \supseteq \{a\} \neq \emptyset,$$
$$U(a) = U(a,a) = U(f_a(a), f_a(a)) = f_a(U(a,a)) = \{a\}$$

showing that $\mathbf{P}$ is up-directed and that a is maximal. $\square$

Several elementary facts on cone preserving mappings are stated in the next lemma.

Lemma 2.3. *Let $\mathbf{P} = (P, \leqslant)$ be a poset and $f\colon P \to P$. Then the following properties hold:*

(i) *f is monotone if and only if $f(U(x)) \subseteq U(f(x))$ for all $x \in P$,*
(ii) *if f is upper cone preserving then it is monotone and $f(F) \in \operatorname{Fil}\mathbf{P}$ for all $F \in \operatorname{Fil}\mathbf{P}$,*
(iii) *if every monotone mapping from P to P is upper cone preserving then $|P| = 1$,*
(iv) *if f is monotone then $f(L(A)) \subseteq L(f(A))$ and $f(U(A)) \subseteq U(f(A))$ for all $A \subseteq P$.*

P r o o f. (i) This is obvious.

(ii) Assume f to be upper cone preserving. Then

$$f(U(x)) = f(U(x,x)) = U(f(x), f(x)) = U(f(x)) \quad \text{for all } x \in P$$

and hence f is monotone according to (i). Moreover, if $F \in \operatorname{Fil}\mathbf{P}$ then

$$\begin{aligned} f(F) &= f\Big(\bigcup_{x\in F} U(x)\Big) = \bigcup_{x\in F} f(U(x)) = \bigcup_{x\in F} f(U(x,x)) \\ &= \bigcup_{x\in F} U(f(x), f(x)) = \bigcup_{x\in F} U(f(x)) \in \operatorname{Fil}\mathbf{P}. \end{aligned}$$

(iii) This follows from Lemma 2.2 by observing that every constant mapping is monotone.

(iv) If f is monotone and $a \in f(L(A))$ then there exists some $b \in L(A)$ with $f(b) = a$ and since f is monotone we have $a = f(b) \in L(f(A))$. The statement for U follows by duality. $\square$

E x a m p l e 2.4. Consider the poset depicted in Figure 1.

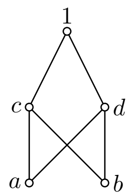


Figure 1.

Then $f\colon P \to P$ defined by

$$f(x) := \begin{cases} c & \text{if } x \in \{a,b,c\}, \\ 1 & \text{otherwise} \end{cases}$$

is upper cone preserving and hence monotone according to Lemma 2.3 (ii). This poset is not a singleton and thus there exists a monotone mapping which is not upper cone preserving according to Lemma 2.3 (iii). The mapping $g\colon P \to P$ defined by

$$g(x) := \begin{cases} b & \text{if } x = a, \\ x & \text{otherwise} \end{cases}$$

is monotone, but not upper cone preserving since

$$g(U(a,b)) = g(\{c,d,1\}) = \{c,d,1\} \neq \{b,c,d,1\} = U(b) = U(b,b) = U(g(a), g(b)).$$

In the next proposition we prove that injective mappings preserving principal filters are upper cone preserving.

Proposition 2.5. *Let $(P, \leqslant)$ be a poset. Then every injective mapping $f\colon P \to P$ satisfying $f([x)) = [f(x))$ for all $x \in P$ is upper cone preserving.*

Proof. If $a, b \in P$ and $f\colon P \to P$ is injective and satisfies $f([x)) = [f(x))$ for all $x \in P$ then $f(U(a)) = U(f(a))$ and

$$\begin{aligned} f(U(a,b)) &= f(U(a) \cap U(b)) = f(U(a)) \cap f(U(b)) \\ &= U(f(a)) \cap U(f(b)) = U(f(a), f(b)). \end{aligned}$$

□

3. Mappings determined by two elements

In the following, for every poset $(P, \leqslant)$ and every $a, b \in P$ with $a \neq b$, let f_{ab} denote the mapping from P to P defined by

$$f_{ab}(x) := \begin{cases} b & \text{if } x = a, \\ x & \text{otherwise.} \end{cases}$$

The question when the mapping f_{ab} is strictly monotone is answered in the next proposition.

Proposition 3.1. *Let $(P, \leqslant)$ be a poset and $a, b \in P$ with $a \neq b$. Then f_{ab} is strictly monotone if and only if $a \parallel b$, $L^*(a) \subseteq L^*(b)$ and $U^*(a) \subseteq U^*(b)$.*

Proof. Obviously, f_{ab} is strictly monotone if and only if $a \parallel b$ (since $b < b$ is impossible) and for all $x \in P$ the following implications hold:

$$a < x \Rightarrow b < x, \tag{1}$$

$$x < a \Rightarrow x < b. \tag{2}$$

Now (1) and (2) are equivalent to $U^*(a) \subseteq U^*(b)$ and $L^*(a) \subseteq L^*(b)$, respectively.

□

Similarly, we can ask when the mapping f_{ab} is upper cone preserving. The answer is as follows.

Theorem 3.2. *Let $\mathbf{P} = (P, \leqslant)$ be a poset and $a, b \in P$ with $a \neq b$. Then f_{ab} is upper cone preserving if and only if a is a minimal element of $\mathbf{P}$ and $U^*(a) = U(b)$.*

P r o o f. Let $c, d \in P \setminus \{a\}$. First assume f_{ab} to be upper cone preserving. Then $b \leqslant a$ would imply

$$a \in U(b) = U(b, b) = U(f_{ab}(a), f_{ab}(a)) = f_{ab}(U(a, a)) = f_{ab}(U(a)) = U^*(a) \cup \{b\}$$

and hence $a = b$, a contradiction. Therefore $b \nleqslant a$. Now

$$b \in U(b) = U(b, b) = U(f_{ab}(a), f_{ab}(b)) = f_{ab}(U(a, b)) = U(a, b) \subseteq U(a)$$

since $a \notin U(a, b)$ and hence $a \leqslant b$, i.e., $a < b$. Now $c < a$ would imply

$$a \in U(c) = U(c, c) = U(f_{ab}(c), f_{ab}(c)) = f_{ab}(U(c, c)) = f_{ab}(U(c)) = ((U(c)) \setminus \{a\}) \cup \{b\}$$

and hence $a = b$, a contradiction. This shows that a is minimal. Moreover,

$$U^*(a) = f_{ab}(U(a)) = f_{ab}(U(a, a)) = U(f_{ab}(a), f_{ab}(a)) = U(b, b) = U(b)$$

since $b \in U^*(a)$. Conversely, assume a to be minimal and $U^*(a) = U(b)$. Then $a < b$ and $c, d \nleqslant a$. Therefore $a \notin U(a, c)$ and $a \notin U(c, d)$. We have

$$\begin{aligned}
&f_{ab}(U(a, a)) = f_{ab}(U(a)) = U^*(a) = U(b) = U(b, b) = U(f_{ab}(a), f_{ab}(a)),\\
&f_{ab}(U(a, c)) = U(a, c) = U(b, c) = U(f_{ab}(a), f_{ab}(c)),\\
&f_{ab}(U(c, d)) = U(c, d) = U(f_{ab}(c), f_{ab}(d))
\end{aligned}$$

and hence f_{ab} is upper cone preserving. $\square$

It should be remarked that $U^*(a) = U(b)$ implies $a \prec b$. Namely, from $b \in U(b) = U^*(a)$ we conclude $a < b$. If there would exist some $c \in P$ with $a < c < b$ then $c \in U^*(a) = U(b)$, a contradiction. This shows $a \prec b$.

Some posets with strictly monotone mappings which are not upper cone preserving are in the next assertion. One of them is depicted in Figure 1.

Corollary 3.3. *Let $(P, \leqslant)$ be a poset containing two elements a and b with $a \parallel b$ satisfying $L^*(a) \subseteq L^*(b)$ and $U^*(a) \subseteq U^*(b)$. Then f_{ab} is strictly monotone and it is not upper cone preserving.*

P r o o f. The mapping f_{ab} is strictly monotone by Proposition 3.1 and it is not upper cone preserving by Theorem 3.2. $\square$

In Theorem 3.2 we characterized when the mapping f_{ab} is upper cone preserving. Now we show when this mapping is strictly upper cone preserving.

Theorem 3.4. *Let $(P, \leqslant)$ be a poset and $a, b \in P$ with $a \neq b$. Then f_{ab} is strictly upper cone preserving if and only if $|L(a)| \leqslant 2$ and $U^*(a) = U(b)$.*

Proof. First assume f_{ab} to be strictly upper cone preserving. Then $b \leqslant a$ would imply

$$a \in U(b) = U(b, b) = U(f_{ab}(a), f_{ab}(b)) = f_{ab}(U(a, b)) = f_{ab}(U(a)) = U^*(a) \cup \{b\}$$

and hence $a = b$, a contradiction. Therefore $b \nleqslant a$. Now

$$b \in U(b) = U(b, b) = U(f_{ab}(a), f_{ab}(b)) = f_{ab}(U(a, b)) = U(a, b) \subseteq U(a)$$

since $a \notin U(a, b)$, and hence $a \leqslant b$, i.e., $a < b$ and therefore $U(b) \subseteq U^*(a)$. If $c \in U^*(a)$ then

$$c \in U(c) = f_{ab}(U(c)) = f_{ab}(U(a, c)) = U(f_{ab}(a), f_{ab}(c)) = U(b, c) \subseteq U(b)$$

showing $U^*(a) \subseteq U(b)$. Altogether, we obtain $U^*(a) = U(b)$. Now $|L(a)| > 2$ would imply that there exist $d, e \in P$ with $d \neq e$ and $d, e < a$, and hence

$$f_{ab}(U(d, e)) = (U(d, e)) \setminus \{a\} \neq U(d, e) = U(f_{ab}(d), f_{ab}(e))$$

contradicting the fact that f_{ab} is strongly upper cone preserving. Hence $|L(a)| \leqslant 2$. If, conversely, $|L(a)| \leqslant 2$ and $U^*(a) = U(b)$ and $g, h \in P \setminus \{a\}$ then

$$\begin{aligned}
f_{ab}(U(a, g)) &= \begin{cases} f_{ab}(U(a)) = U^*(a) = U(b) = U(b, g) = U(f_{ab}(a), f_{ab}(g)) & \text{if } g \leqslant a, \\ U(a, g) = U(b, g) = U(f_{ab}(a), f_{ab}(g)) & \text{if } g \nleqslant a, \end{cases} \\
f_{ab}(U(g, h)) &= U(g, h) = U(f_{ab}(g), f_{ab}(h)) \qquad \text{if } g \neq h
\end{aligned}$$

and hence f_{ab} is strictly upper cone preserving. □

4. Chains

The following statement asserts that a monotone mapping on a chain is upper cone preserving if and only if the range of this mapping is a filter of this chain.

Proposition 4.1. *Let $\mathbf{C} = (C, \leqslant)$ be a chain and $f\colon C \to C$ monotone. Then f is upper cone preserving if and only if $f(C) \in \operatorname{Fil} \mathbf{C}$.*

Proof. Observe that $f(C) \in \operatorname{Fil} \mathbf{C}$ if and only if $U(f(x)) \subseteq f(C)$ for all $x \in C$. Let $a \in C$. If f is upper cone preserving then

$$U(f(a)) = U(f(a), f(a)) = f(U(a, a)) \subseteq f(C).$$

Conversely, assume $U(f(x)) \subseteq f(C)$ for all $x \in C$. Since f is monotone, we have $f(U(a)) \subseteq U(f(a))$ according to Lemma 2.3 (i). Now let $b \in U(f(a))$. If $b = f(a)$ then $b \in f(U(a))$. Now assume $b > f(a)$. Since $b \in U(f(a)) \subseteq f(C)$, there exists some $c \in C$ with $f(c) = b$. Now $c \leqslant a$ would imply $b = f(c) \leqslant f(a)$, a contradiction. Hence $c \in U(a)$ and therefore $b = f(c) \in f(U(a))$. This shows $U(f(a)) \subseteq f(U(a))$ and hence $f(U(a)) = U(f(a))$. Now, for $x, y \in C$ we have

$$U(f(x), f(y)) = \begin{cases} U(f(y)) = f(U(y)) = f(U(x,y)) & \text{if } x \leqslant y, \\ U(f(x)) = f(U(x)) = f(U(x,y)) & \text{otherwise,} \end{cases}$$

i.e., f is upper cone preserving. □

Another interesting question concerns posets which are semilattices. Because every semilattice homomorphism is a monotone mapping, we can ask when every monotone mapping of a given semilattice into itself is a homomorphism. Using the method developed by Berrone (see [1]), we can prove the following result.

Theorem 4.2. *A join-semilattice $(P, \vee)$ is a chain if and only if every monotone mapping from P to P is a homomorphism.*

Proof. Let $\mathbf{P} = (P, \vee)$ be a join-semilattice and $a, b \in P$. If $\mathbf{P}$ is a chain and f a monotone mapping from P to P then

$$f(a \vee b) = \begin{cases} f(b) = f(a) \vee f(b) & \text{if } a \leqslant b, \\ f(a) = f(a) \vee f(b) & \text{otherwise} \end{cases}$$

and hence f is a homomorphism. Now assume $\mathbf{P}$ is not a chain. Then there exist $c, d \in P$ with $c \parallel d$. Define $g\colon P \to P$ by

$$g(x) := \begin{cases} c & \text{if } x < c \vee d, \\ c \vee d & \text{otherwise.} \end{cases}$$

Assume $a \leqslant b$. If $a < c \vee d$ then $g(a) = c \leqslant g(b)$. If $a \not< c \vee d$ then $b \not< c \vee d$ and hence $g(a) = c \vee d = g(b)$. This shows that g is monotone. But g is not a homomorphism since

$$g(c \vee d) = c \vee d \neq c = c \vee c = g(c) \vee g(d).$$

We have proved that there exists a monotone mapping from P to P that is not a homomorphism. □

By duality, Theorem 4.2 also holds for meet-semilattices and hence also for lattices. In addition, the theorem allows us to determine some monotone mappings of direct products of chains that are homomorphisms of this product. To see them, we need the following concepts.

For $i = 1, 2$ let $f_i\colon A_i \to B_i$. Then $f_1 \times f_2$ denotes the mapping from $A_1 \times A_2$ to $B_1 \times B_2$ defined by

$$(f_1 \times f_2)(x_1, x_2) := (f_1(x_1), f_2(x_2)) \quad \text{for all } (x_1, x_2) \in A_1 \times A_2.$$

A mapping $g\colon A_1 \times A_2 \to B_1 \times B_2$ is called *directly decomposable* if there exist $g_1\colon A_1 \to B_1$ and $g_2\colon A_2 \to B_2$ with $g_1 \times g_2 = g$. The *direct product of* two *posets* $(P_1, \leqslant_1)$ and $(P_2, \leqslant_2)$ is the poset $(P_1 \times P_2, \leqslant)$ defined by

$$(x_1, x_2) \leqslant (y_1, y_2) \Leftrightarrow x_1 \leqslant_1 y_1 \text{ and } x_2 \leqslant_2 y_2 \quad ((x_1, x_2), (y_1, y_2) \in P_1 \times P_2).$$

Let P be a direct product of chains. As proved in [3], every lattice homomorphism from P to P is directly decomposable since P is a lattice and the variety of lattices is congruence distributive. We can ask if monotone directly decomposable mappings from P to P are lattice homomorphisms. The following corollary of Theorem 4.2 gives a positive answer.

Corollary 4.3. *Let $(C_1, \leqslant)$, $(C_2, \leqslant)$ be chains, $(P, \leqslant) := (C_1, \leqslant) \times (C_2, \leqslant)$ and f a monotone directly decomposable mapping from P to P. Then f is a lattice homomorphism.*

P r o o f. If $f = f_1 \times f_2$ with $f_i\colon C_i \to C_i$ for $i = 1, 2$ then f_1, f_2 are monotone and, by Theorem 4.2, also (semi-)lattice homomorphisms which implies that f is a (semi-)lattice homomorphism, too. $\square$

Direct decomposability of homomorphisms was investigated by the authors and Goldstern in [3]. For mappings which need not be homomorphisms we cannot use methods involved in congruence distributive varieties. A simple characterization of directly decomposable mappings is formulated in the following lemma.

Lemma 4.4. *Let A_1, A_2, B_1, B_2 be non-void sets and $f\colon A_1 \times A_2 \to B_1 \times B_2$ and for $i = 1, 2$, let p_i denote the projection of $B_1 \times B_2$ onto B_i. Then the following statements are equivalent:*

(i) *f is decomposable,*

(ii) *$p_1(f(x_1, x_2)) = p_1(f(x_1, y_2))$ and $p_2(f(x_1, x_2)) = p_2(f(y_1, x_2))$ for all $x_1, y_1 \in A_1$ and $x_2, y_2 \in A_2$.*

P r o o f. (i) $\Rightarrow$ (ii): If $f = f_1 \times f_2$ then

$$p_1(f(x_1,x_2)) = p_1(f_1(x_1), f_2(x_2)) = f_1(x_1) = p_1(f_1(x_1), f_2(y_2)) = p_1(f(x_1,y_2)),$$
$$p_2(f(x_1,x_2)) = p_2(f_1(x_1), f_2(x_2)) = f_2(x_2) = p_2(f_1(y_1), f_2(x_2)) = p_2(f(y_1,x_2))$$

for all $x_1, y_1 \in A_1$ and $x_2, y_2 \in A_2$.

(ii) $\Rightarrow$ (i): Let $a_1 \in A_1$ and $a_2 \in A_2$ and for $i = 1,2$ define $f_i\colon A_i \to B_i$ by

$$f_1(x_1) := p_1(f(x_1,a_2)) \quad \text{for all } x_1 \in A_1,$$
$$f_2(x_2) := p_2(f(a_1,x_2)) \quad \text{for all } x_2 \in A_2.$$

Because of (ii), f_1 and f_2 are well-defined and

$$\begin{aligned} f(x_1,x_2) &= (p_1(f(x_1,x_2)), p_2(f(x_1,x_2))) \\ &= (p_1(f(x_1,a_2)), p_2(f(a_1,x_2))) = (f_1(x_1), f_2(x_2)) \end{aligned}$$

for all $(x_1,x_2) \in A_1 \times A_2$, i.e., $f = f_1 \times f_2$. $\square$

Instead of join-semilattices we can investigate posets whose upper cones $U(x,y)$ have a minimal element. Of course, every join-semilattice has this property, but there are many other examples of such posets, e.g. all finite up-directed posets.

Theorem 4.5. *If $(P,\leqslant)$ is a poset, $a,b \in P$, $a \parallel b$ and $U(a,b)$ has a minimal element then there exists a monotone mapping f from P to P with $f(U(a,b)) \neq U(f(a),f(b))$ and hence there exists a monotone mapping from P to P which is not strictly upper cone preserving.*

P r o o f. Let $(P,\leqslant)$ be a poset and $a,b,c \in P$, and assume $a \parallel b$ and that c is the minimal element of $U(a,b)$. Define $f\colon P \to P$ by

$$f(x) := \begin{cases} a & \text{if } x < c, \\ c & \text{otherwise.} \end{cases}$$

Let $d,e \in P$ with $d \leqslant e$. If $d < c$ then $f(d) = a \leqslant f(e)$. If $d \not< c$ then $e \not< c$ and hence $f(d) = c = f(e)$. This shows that f is monotone. We have $a,b \leqslant c$. Since $a = c$ would imply $b \leqslant c = a$ and $b = c$ would imply $a \leqslant c = b$, we have $a,b < c$ and therefore $a \in U(a) = U(a,a) = U(f(a),f(b))$. Now assume $f(U(a,b)) = U(f(a),f(b))$. Then $a \in f(U(a,b))$ and hence there exists some $g \in U(a,b)$ with $f(g) = a$. Since c is a minimal element of $U(a,b)$ we have $g \not< c$ and hence $a = f(g) = c$, a contradiction. Therefore $f(U(a,b)) \neq U(f(a),f(b))$. $\square$

Corollary 4.6. *If $(P, \leqslant)$ is an up-directed poset which is not a chain and which satisfies the Descending Chain Condition then there exists a monotone mapping from P to P which is not strictly upper cone preserving and hence not upper cone preserving.*

On the other hand, if the poset in question is a chain, we can give a necessary and sufficient condition for a monotone mapping to be upper cone preserving.

5. Ordinal sums and equivalence relations

We have seen that the Descending Chain Condition together with the property that every monotone mapping is strictly upper cone preserving forces an up-directed poset to be a chain. It seems that our conditions are too restrictive. In fact, if we replace monotone mappings by strictly monotone ones, we can obtain a richer structure of posets in which strictly monotone mappings are strongly upper cone preserving.

The *ordinal sum* of two posets $(A, \leqslant)$ and $(B, \leqslant)$ with $A \cap B = \emptyset$ is the poset with the base set $A \cup B$ where the order inside A and inside B coincides with the original one and $A < B$, i.e., every element of A is below every element of B. Now, we can state the following result.

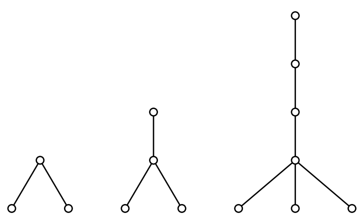
Proposition 5.1. *Every strictly monotone mapping on the ordinal sum of an antichain and a finite chain is strongly upper cone preserving.*

Proof. If f is a strictly monotone mapping on the ordinal sum $(P, \leqslant)$ of an antichain $(A, \leqslant)$ and a finite chain $(C, \leqslant)$, $a, b \in P$ and $f(a) \neq f(b)$ then $f(A) \subseteq A$, $f(x) = x$ for all $x \in C$ and

$$f(U(a,b)) = \begin{cases} f(C) = C = U(f(a), f(b)) & \text{if } a, b \in A, \\ f(U(b)) = U(b) = U(f(a), b) = U(f(a), f(b)) & \text{if } a \in A \text{ and } b \in C, \\ U(a,b) = U(f(a), f(b)) & \text{if } a, b \in C. \end{cases}$$

$\square$

Example 5.2. Examples of such ordinal sums are visualized in Figure 2.



(a) (b) (c)

Figure 2.

Every mapping $f\colon A \to B$ induces an equivalence relation Θ on A by defining $(x, y) \in \Theta$ if $f(x) = f(y)$. This equivalence relation is called the *kernel* of f, usually denoted by $\ker f$. The question when for a given poset $(P, \leqslant)$ and a given mapping $f\colon P \to P$ the quotient set $P/(\ker f)$ is again a poset is answered in the next theorem.

Let $(P, \leqslant)$ and $(Q, \leqslant)$ be posets and $f\colon P \to Q$. Recall that f is called *strongly monotone* if it is monotone, and $a, b \in P$ and $f(a) \leqslant f(b)$ imply that there exist $a', b' \in P$ with $f(a') = f(a)$, $f(b') = f(b)$ and $a' \leqslant b'$.

Definition 5.3. Let $\mathbf{P} = (P, \leqslant)$ be a poset. An equivalence relation Θ on P is called an *S-equivalence* on $\mathbf{P}$ if it satisfies the following two conditions:

(i) If $a, b, b', c \in P$, $a \leqslant b$, $b' \leqslant c$ and $(b, b') \in \Theta$ then there exist $a' \in [a]\Theta$ and $c' \in [c]\Theta$ with $a' \leqslant c'$,

(ii) if $a, a', b, b' \in P$, $a \leqslant b$, $b' \leqslant a'$ and $(a, a'), (b, b') \in \Theta$ then $(a, b) \in \Theta$.

Theorem 5.4. *Let $\mathbf{P} = (P, \leqslant)$ be a poset, $f\colon P \to P$ strongly monotone and Θ an S-equivalence on $\mathbf{P}$ and define $[a]\Theta \leqslant [b]\Theta$ if there exist $a' \in [a]\Theta$ and $b' \in [b]\Theta$ with $a' \leqslant b'$. Then*

(i) $\ker f$ *is an S-equivalence on $\mathbf{P}$,*

(ii) $(P/\Theta, \leqslant)$ *is a poset and $x \mapsto [x]\Theta$ is strongly monotone.*

Proof. (i) Put $\Phi := \ker f$ and assume $a, b, b', c \in P$, $a \leqslant b$, $b' \leqslant c$ and $(b, b') \in \Phi$. Then

$$f(a) \leqslant f(b) = f(b') \leqslant f(c).$$

Since f is strongly monotone there exist $a', c' \in P$ with $f(a') = f(a)$, $f(c') = f(c)$ and $a' \leqslant c'$. Hence $a' \in [a]\Phi$, $c' \in [c]\Phi$ and $a' \leqslant c'$ proving (i) of Definition 5.3. Next assume $a, a', b, b' \in P$, $a \leqslant b$, $b' \leqslant a'$ and $(a, a'), (b, b') \in \Phi$. Then

$$f(a) \leqslant f(b) = f(b') \leqslant f(a') = f(a)$$

and hence $f(a) = f(b)$, i.e., $(a, b) \in \Phi$ proving (ii) of Definition 5.3.

(ii) We consider the binary relation $\leqslant$ on P/Θ. Obviously, $\leqslant$ is reflexive. Assume $a, b \in P$, $[a]\Theta \leqslant [b]\Theta$ and $[b]\Theta \leqslant [a]\Theta$. Then there exist $a', a'' \in [a]\Theta$ and $b', b'' \in [b]\Theta$ with $a' \leqslant b'$ and $b'' \leqslant a''$. Since $(a', a''), (b', b'') \in \Theta$, we conclude by (ii) of Definition 5.3 that $(a', b') \in \Theta$. This shows $[a]\Theta = [a']\Theta = [b']\Theta = [b]\Theta$ proving antisymmetry of $\leqslant$. Now assume $a, b, c \in P$, $[a]\Theta \leqslant [b]\Theta$ and $[b]\Theta \leqslant [c]\Theta$. Then there exist $a' \in [a]\Theta$, $b', b'' \in [b]\Theta$ and $c' \in [c]\Theta$ with $a' \leqslant b'$ and $b'' \leqslant c'$. Since $(b', b'') \in \Theta$, we conclude by (i) of Definition 5.3 that there exist $a'' \in [a']\Theta$ and $c'' \in [c']\Theta$ with $a'' \leqslant c''$. Now $a'' \in [a]\Theta$ and $c'' \in [c]\Theta$ which shows $[a]\Theta \leqslant [c]\Theta$ proving transitivity of $\leqslant$. Altogether, $(P/\Theta, \leqslant)$ is a poset. Clearly, $x \mapsto [x]\Theta$ is monotone and by the definition of $\leqslant$ on P/Θ, this mapping is strongly monotone. $\square$

E x a m p l e 5.5. Consider the poset $\mathbf{P} = (P, \leqslant)$ visualized in Figure 3.

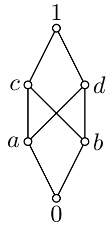


Figure 3.

Let $f \colon P \to P$ be defined by

x	0	a	b	c	d	1
$f(x)$	0	a	a	c	c	1

and put $\Theta := \ker f$. Then f is strongly monotone, $\Theta = \{0\}^2 \cup \{a, b\}^2 \cup \{c, d\}^2 \cup \{1\}^2$ is an S-equivalence on $\mathbf{P}$ and $(P/\Theta, \leqslant) = (\{[0]\Theta, [a]\Theta, [c]\Theta, [1]\Theta\}, \leqslant)$ is again a poset where $[0]\Theta < [a]\Theta < [c]\Theta < [1]\Theta$.

6. Conclusion

Posets were and are studied from various points of view and by means of a number of methods. In our paper we compared several mappings used in posets and investigated when they coincide in dependence of the structure of the poset $\mathbf{P}$ in question. In particular, we investigated monotone mappings, strictly monotone mappings, upper cone preserving mappings, strictly upper cone preserving mappings, principal filter preserving mappings and mappings determined by two elements. We showed that every injective principal filter preserving mapping is upper cone preserving (Proposition 2.5). We characterized when the mapping determined by two elements is upper cone preserving (Theorem 3.2) or strictly upper cone preserving (Theorem 3.4). In particular, if $\mathbf{P}$ is a chain then every monotone mapping whose range is a filter is upper cone preserving (Proposition 4.1) and a join-semilattice is a chain if and only if every monotone mapping is a homomorphism (Theorem 4.2). As special cases we treated posets which are ordinal sums of an antichain and a finite chain. We also determined a relationship between strongly monotone mappings and S-equivalences. In future, it should be of some interest to classify posets where some other kinds of mappings coincide with those mentioned above.

A c k n o w l e d g e m e n t . The authors are grateful to the anonymous referee whose valuable comments helped to increase the quality of the paper.

References

[1] *L. R. Berrone*: The homomorphism equation on semilattices. Aequationes Math. *94* (2020), 803–816. zbl MR doi
[2] *I. Chajda*: Homomorphisms of directed posets. Asian-Eur. J. Math. *1* (2008), 45–51. zbl MR doi
[3] *I. Chajda, M. Goldstern, H. Länger*: A note on homomorphisms between products of algebras. Algebra Universalis *79* (2018), Paper No. 25, 7 pages. zbl MR doi
[4] *I. Chajda, Š. Hošková*: A characterization of cone preserving mappings of quasiordered sets. Miskolc Math. Notes *6* (2005), 147–152. zbl MR doi

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