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GLOBAL WELL-POSEDNESS FOR THE KLEIN-GORDON-SCHRÖDINGER SYSTEM WITH HIGHER ORDER COUPLING

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Abstract. Global well-posedness for the Klein-Gordon-Schrödinger system with generalized higher order coupling, which is a system of PDEs in two variables arising from quantum physics, is proven. It is shown that the system is globally well-posed in $(u,n)\in L^2\times L^2$ under some conditions on the nonlinearity (the coupling term), by using the L^2 conservation law for u and controlling the growth of n via the estimates in the local theory. In particular, this extends the well-posedness results for such a system in Miao, Xu (2007) for some exponents to other dimensions and in lower regularity spaces.

Keywords: low regularity; global well-posedness; Klein-Gordon-Schrödinger equation; higher order coupling

MSC 2020: 35Q40, 35G55

1. Introduction

The initial value problem for the d-dimensional Klein-Gordon-Schrödinger system with the Yukawa coupling is

$$\begin{cases} \mathrm{i}\partial_t u + \Delta u = -nu, \\ \partial_t^2 n + (1-\Delta)n = |u|^2, \\ u(x,0)=u_0(x),\ n(x,0)=n_0(x),\ \partial_t n(x,0)=n_1(x), \end{cases} \tag{1.1}$$

where $u\colon [0,T^*)\times\mathbb{R}\to\mathbb{C}$ and $n\colon [0,T^*)\times\mathbb{R}\to\mathbb{R}$. This model arises in many physical situations, such as the interaction between quantum mechanical meson and nucleon fields (see [7]), the interaction between short and long dispersive waves in fluid mechanics (see [2]), and other physical systems.

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In [6], it is proven that the system (1.1) is globally well-posed in dimension $d=3$ for $(u_0,n_0,n_1)\in L^2\times L^2\times H^{-1}$. This result is further sharpened in [11] where the local well-posedness is established in dimension $d=2$ for $(u_0,n_0,n_1)\in H^s\times H^k\times H^{k-1}$ with $s>-\frac14$, $k\geqslant -\frac12$, $k-2s<\frac32$, $k-2<s<k+1$, and the global well-posedness for $s\geqslant 0$ and $s-\frac12\leqslant k<s+\frac32$. In dimension $d=3$, the global well-posedness is proven for $s\geqslant 0$ and $s-\frac12<k\leqslant s+1$.

As in [5], it is natural to generalize the system (1.1) into the system

$$\begin{cases} \mathrm{i}\partial_t u+\Delta u=-nf(|u|^2)u,\\ \partial_t^2 n+(1-\Delta)n=F(|u|^2),\\ u(x,0)=u_0(x),\ n(x,0)=n_0(x),\ \partial_t n(x,0)=n_1(x), \end{cases}\tag{1.2}$$

where $F'=f$ and $F(0)=f(0)=0$. The particular case $F(s):=s^m$ is denoted by KGS_m, which reads

$$\begin{cases} \mathrm{i}\partial_t u+\Delta u=-mn|u|^{2(m-1)}u,\\ \partial_t^2 n+(1-\Delta)n=|u|^{2m},\\ u(x,0)=u_0(x),\ n(x,0)=n_0(x),\ \partial_t n(x,0)=n_1(x). \end{cases}\tag{1.3}$$

This generalization follows [5], where the term $n|u|^2$ is replaced by the more general term $nF(|u|^2)$ in the Hamiltonian of (1.1). A reason that higher-order powers are introduced to the nonlinearity (or the coupling term) in (1.1) is to adjust the strength of the nonlinearity relative to the dispersion and study the balance between these two effects. They are also relevant in various physical models (see [1], [3], [8], [10] and references therein).

Following the above remarks, it is well-known that the smooth solution of (1.3) satisfies conservation of mass (see [10], [11], [13])

$$M[u](t):=\|u(t)\|_{L^2}=\|u_0\|_{L^2}=:M[u_0]\tag{1.4}$$

and conservation of the Hamiltonian

$$\begin{aligned}H[u,n](t)&:=\|\nabla u(t)\|_{L^2}^2+\frac12(\|(1-\Delta)^{1/2}n(t)\|_{L^2}+\|\partial_t n(t)\|_{L^2}^2)-\int_{\mathbb{R}^d} n(t)|u(t)|^{2m}\,\mathrm{d}x\\ &=H[u_0,n_0].\end{aligned}$$

The well-posedness for the system (1.3), on the other hand, is much less developed. For dimension $d=1$, this is studied in [10], where global well-posedness is proven in $(u,n)\in L^2\times H^{1/2}$ for $1\leqslant m<2$. Here, we want to provide global well-posedness results at lower regularity level.

We show that KGS_m is (locally or globally) well-posed for $(u_0, n_0, n_1) \in L^2 \times L^2 \times H^{-1}$, dimension $d = 1, 2, 3$, and some range of m using Strichartz estimates. Similar results also hold in higher dimension. More precisely, our main result is the following theorem.

Theorem 1.1. KGS_m (1.3) *is locally well-posed for* $(u_0, n_0, n_1) \in L^2(\mathbb{R}^d) \times L^2(\mathbb{R}^d) \times H^{-1}(\mathbb{R}^d)$,

(1) *for* $1 \leqslant m < \frac{5}{2}$ *in dimension* $d = 1$,
(2) *for* $1 \leqslant m < \frac{3}{2}$ *in dimension* $d = 2$,
(2) *for* $1 \leqslant m < \frac{7}{6}$ *in dimension* $d = 3$.

Moreover, it is globally well-posed

(1) *for* $1 \leqslant m \leqslant \frac{7}{4}$ *in dimension* $d = 1$,
(2) *for* $1 \leqslant m \leqslant \frac{5}{4}$ *in dimension* $d = 2$,
(3) *for* $m = 1$ *in dimension* $d = 3$,

and in this case the solution $(u, n) \in C(\mathbb{R}; L^2) \times C(\mathbb{R}; L^2)$ *satisfies* (1.4) *and*

$$\|n(t)\|_{L^2} + \|\partial_t n(t)\|_{H^{-1}} \leqslant \exp(c|t|\,\|u_0\|_{L^2}^{\beta}) \max(\|n_0\|_{L^2} + \|n_1\|_{H^{-1}}, \|u_0\|_{L^2}^{2m}) \tag{1.5}$$

for some $\beta := \beta(m, d) > 0$.

In particular, we extend the result of [10] for some exponents to rougher data, matching the level of regularity provided in [6]. We also cover a range of m for which KGS_m is $L^2 \times L^2$-'subcritical' in the sense of scaling (depending on the notion of scaling used). Note that KGS_m does not admit the usual notion of criticality based on scaling, but a notion of criticality based on the kind of scaling done in [9] for the Zakharov system will be useful in our case too. Refer to Section 3 for details.

We remark that our global well-posedness results use the conservation of mass only (but not the conservation of the Hamiltonian) and so they are still true regardless of the sign of the nonlinearity. Furthermore, as a note, by the local well-posedness, we mean that the solution exists in a small time interval and is unique, that the solution has the same regularity as the initial data in that time interval, and the solution depends continuously on the initial data. By global well-posedness, we mean that the above properties hold for all time $t > 0$.

This paper is organized in the following way. The notations used in the paper are introduced in Section 2, followed by a discussion on scaling and criticality in Section 3. Some linear and nonlinear estimates are established in Section 4. Finally, main Theorem 1.1 is proven in Section 5.

2. Notations

Given $A, B \geqslant 0$, we write $A \lesssim B$ if $A \leqslant k \cdot B$ for some universal constant k. We write $A \approx B$ or $A \sim B$ which means both $A \lesssim B$ and $B \lesssim A$. The notation $A \ll B$ means $B > k \cdot A$. We introduce the notation $\langle x \rangle := \sqrt{1+|x|^2}$, and $\langle \nabla \rangle$ for the operator with the Fourier multiplier $\langle \xi \rangle$. We also use the notation $\alpha+$ for $\alpha \in \mathbb{R}$ that means a number slightly larger than α, i.e., $\alpha + \varepsilon$ for some $\varepsilon > 0$.

Given Lebesgue space exponents q, r and a function $f(x,t)$ on $\mathbb{R} \times \mathbb{R}$, we define the mixed (space-time) Lebesgue norm

$$\|f\|_{L_t^q L_x^r} := \biggl(\int_{\mathbb{R}} \biggl(\int_{\mathbb{R}} |f(x,t)|^r \,\mathrm{d}x\biggr)^{q/r} \mathrm{d}t\biggr)^{1/q}.$$

Let H^s be the usual Sobolev spaces equipped with the norm

$$\|f\|_{H^s} = \biggl(\int_{\xi} (1+|\xi|^2)^s |\hat{f}(\xi)|^2 \,\mathrm{d}\xi\biggr)^{1/2} =: \|\langle \xi \rangle^s \hat{f}\|_{L_\xi^2},$$

where $\hat{f}$ is the Fourier transform of f. Denote also by $f^\vee$ the inverse Fourier transform of f.

A pair of exponents (q,r) is called *admissible* for $\mathbb{R}^d \times \mathbb{R}$ if $2 \leqslant q$, $r \leqslant \infty$ and $(q,r,d) \neq (2,\infty,2)$, and

$$\frac{2}{q} + \frac{d}{r} = \frac{d}{2}.$$

Finally, we introduce the norm

$$\|(n_0, n_1)\|_{\mathcal{G}} := (\|n_0\|_{L_x^2}^2 + \|n_1\|_{H_x^{-1}}^2)^{1/2}$$

and the shorthand notation $\|n(t)\|_{\mathcal{G}} := \|(n(t), \partial_t n(t))\|_{\mathcal{G}}$.

3. Criticality and scaling heuristics

First, in order to discuss the useful notion of criticality for KGS_m, we transform the system into an equivalent system of the first order in t (following [9], [10]). Put

$$n_\pm := \frac{1}{2}\Bigl(n \pm \frac{1}{\mathrm{i}}(1-\Delta)^{1/2}\partial_t n\Bigr).$$

Then, we have $n = n_+ + n_-$, $\partial_t n = \mathrm{i}(1-\Delta)^{1/2}(n_+ - n_-)$, and $n_+ = \overline{n}_-$. The equivalent first order system is

$$
\begin{cases}
\mathrm{i}\partial_t u + \Delta u = -m(n_+ + n_-)|u|^{2(m-1)}u, \\
\mathrm{i}\partial_t n_\pm \pm (1-\Delta)^{1/2} n_\pm = \pm\dfrac{1}{2}(1-\Delta)^{-1/2}(|u|^{2m}), \\
u(x,0) = u_0(x),\ n_\pm(x,0) = \dfrac{1}{2}\Big(n_0 \pm \dfrac{1}{\mathrm{i}}(1-\Delta)^{1/2} n_1\Big).
\end{cases} \tag{3.1}
$$

Consider a similar system

$$
\begin{cases}
\mathrm{i}\partial_t u + \Delta u = -m(n_+ + n_-)|u|^{2(m-1)}u, \\
\mathrm{i}\partial_t n_\pm \pm (-\Delta)^{1/2} n_\pm = \pm\dfrac{1}{2}(-\Delta)^{-1/2}(|u|^{2m}), \\
u(x,0) = u_0(x),\ n_\pm(x,0) = \dfrac{1}{2}\Big(n_0 \pm \dfrac{1}{\mathrm{i}}(1-\Delta)^{1/2} n_1\Big).
\end{cases} \tag{3.2}
$$

This system does not possess any true scaling, but as in [9], [10], if it were not for the term Δu in the left-hand side of the first equation in (3.2), which means that the linear effect of the wave equation is stronger than that of the Schrödinger equation, then (3.2) would be invariant under the scaling

$$
u \mapsto u_\lambda := \lambda^{3/(4m-2)} u(\lambda t, \lambda x), \quad n \mapsto n_\lambda := \lambda^{(2-m)/(2m-1)} n(\lambda t, \lambda x)
$$

and also would be critical for $(u_0, n_\pm(0)) \in H_x^s \times H_t^k$, where $s = d/2 - 3/(4m-2)$ and $k = d/2 - (2-m)/(2m-1)$.

If it were not for the term $\pm(-\Delta)^{1/2} n_\pm$ in the left-hand side of the second equation in (3.2), which means that the linear effect of the Schrödinger equation is stronger than that of the wave equation, then (3.2) would be invariant under the scaling

$$
u \mapsto u_\lambda := \lambda^{5/(4m-2)} u(\lambda^2 t, \lambda x), \quad n \mapsto n_\lambda := \lambda^{(3-m)/(2m-1)} n(\lambda^2 t, \lambda x)
$$

and also would be critical for $(u_0, n_\pm(0)) \in H_x^s \times H_t^k$, where $s = d/2 - 5/(4m-2)$ and $k = d/2 - (3-m)/(2m-1)$.

Similarly, if it were not for the term $\mathrm{i}\partial_t u$ in the left-hand side of the first equation in (3.2), then (3.2) would be invariant under the scaling

$$
u \mapsto u_\lambda := \lambda^{2/(2m-1)} u(\lambda t, \lambda x), \quad n \mapsto n_\lambda := \lambda^{2/(2m-1)} n(\lambda t, \lambda x)
$$

and also would be critical for $(u_0, n_\pm(0)) \in H_x^s \times H_t^k$, where $s = k = d/2 - 5/(4m-2)$.

If it were not for the term $i\partial_t n_\pm$ in the left-hand side of the second equation in (3.2), then (3.2) would be invariant under the scaling

$$u \mapsto u_\lambda := \lambda^{2/(2m-1)} u(\lambda^2 t, \lambda x), \quad n \mapsto n_\lambda := \lambda^{2/(2m-1)} n(\lambda^2 t, \lambda x)$$

and also would be critical for $(u_0, n_\pm(0)) \in H_x^s \times H_t^k$, where $s = k = d/2 - 2/(2m-1)$.

From these heuristics, we see that our local and global well-posedness results cover the values of m in the $L^2 \times L^2$-subcritical region under some or all conditions above for the dimension $d = 1, 2, 3$. The $L^2 \times L^2$-critical problem is more difficult to establish, and will be the subject of future research.

4. Estimates for the group and the Duhamel terms

Let $U(t) := e^{it\Delta}$ be the free linear Schrödinger group and let $G(t)$ be the free linear Klein-Gordon group, i.e.,

$$G(t)(n_0, n_1) = \cos(t(1-\Delta)^{1/2}) n_0 + \frac{\sin(t(1-\Delta)^{1/2})}{(1-\Delta)^{1/2}} n_1.$$

We begin with the following well-known estimates on the linear term.

Proposition 4.1 (Linear estimates). *We have:*

(1) $\|U(t)u_0\|_{C(\mathbb{R};L^2)} = \|u_0\|_{L^2}$.

(2) (Strichartz estimates) *If (q, r) is an admissible pair, then* $\|U(t)u_0\|_{L_t^q L_x^r} \lesssim \|u_0\|_{L^2}$.

(3) $\|G(t)(n_0, n_1)\|_{C(\mathbb{R};\mathcal{G})} = \|(n_0, n_1)\|_{\mathcal{G}}$.

P r o o f. Refer to Remark 2.2.1 and Theorem 2.3.3 in [4] for the detailed proof of (1) and (2). For (3), let n solve the linear Klein-Gordon equation, i.e.,

$$\partial_t^2 n - \Delta n + n = 0 \tag{4.1}$$

with the initial data $(n(0), \partial_t n(0)) = (n_0, n_1)$. We apply the operator $\langle\nabla\rangle^{-1}$ to (4.1), then multiply by $\langle\nabla\rangle^{-1}\partial_t n$. On the Fourier side, we have

$$\int_\xi \hat{n}(\xi)\partial_t \hat{n}(\xi) + \langle\xi\rangle^{-2}\partial_t \hat{n}(\xi)\partial_t^2 \hat{n}(\xi)\,d\xi = 0$$

or equivalently the conservation identity

$$\partial_t(\|n(t)\|_{L^2}^2 + \|\partial_t n(t)\|_{H^{-1}}^2) = 0,$$

which implies the asserted statement. □

Next, we need the following estimates for the nonlinear term.

Proposition 4.2 (Nonlinear estimates). *We have:*

(1) (Strichartz estimates) *If both* (q,r) *and* $(\tilde{q},\tilde{r})$ *are admissible pairs, then*

$$\Big\|\int_0^t U(t-t')f(x,t')\,\mathrm{d}t'\Big\|_{C([0,T];L^2)} \lesssim \|f\|_{L^{\tilde{q}'}_{[0,T]}L^{\tilde{r}'}_x},$$

$$\Big\|\int_0^t U(t-t')f(x,t')\,\mathrm{d}t'\Big\|_{L^q_{[0,T]}L^r_x} \lesssim \|f\|_{L^{\tilde{q}'}_{[0,T]}L^{\tilde{r}'}_x},$$

where $\tilde{p}'$, $\tilde{q}'$ *are Hölder dual to* p, q, *respectively.*

(2) *Also,*

$$\Big\|\int_0^t \frac{\sin((t-t')(1-\Delta)^{1/2})}{(1-\Delta)^{1/2}}f(x,t')\,\mathrm{d}t'\Big\|_{C([0,T];\mathcal{G})} \lesssim \|f\|_{L^1_{[0,T]}H^{-1}_x}.$$

P r o o f. Refer to Theorem 2.3.3 in [4] for the proof of (1). The proof of (2) is in [6], Lemma 2.3. □

The estimates above will be used to establish our main theorem.

5. Proof of the main theorem

The goal of this section is to prove Theorem 1.1. We mainly use the standard space-time norms, Strichartz estimates, and Hölder inequality. We prove the case $d=1$ in detail and then make a remark about the cases $d=2,3$. For convenience, as in [12], introduce the Strichartz space $S^0(I\times\mathbb{R}^d)$ (or just S^0) for any time interval I as the closure of the Schwartz functions under the norm

$$\|u\|_{S^0(I\times\mathbb{R}^d)} := \sup_{(q,r)\ \text{admissible}} \|u\|_{L^q_I L^r_x}.$$

In particular, the S^0 norm controls the $C(I;L^2)$ norm.

P r o o f of Theorem 1.1 for $d=1$. First, let $1\leqslant m<\frac{5}{2}$. Define the Duhamel maps Λ_S and Λ_G as

$$\Lambda_S(u,n) := U(t)u_0 - \mathrm{i}\int_0^t U(t-t')(mn|u|^{2(m-1)}u)(x,t')\,\mathrm{d}t', \tag{5.1}$$

$$\Lambda_G(u) := G(t)(n_0,n_1) + \int_0^t \frac{\sin((t-t')(1-\Delta)^{1/2})}{(1-\Delta)^{1/2}}(|u|^{2m})(x,t')\,\mathrm{d}t'. \tag{5.2}$$

We look for a fixed point $(u(t), n(t)) = (\Lambda_S(u,n), \Lambda_G(u))$ in a closed ball of $S^0([0,T] \times \mathbb{R}) \times C([0,T];\mathcal{G})$. Estimating Λ_S in S^0 by applying Strichartz estimates with $(\tilde{q}, \tilde{r}) = (\infty, 2)$ and the Hölder inequality, we have

$$
\begin{aligned}
\|\Lambda_S(u,n)\|_{S^0([0,T]\times\mathbb{R})} &\lesssim \|u_0\|_{L^2} + \|n|u|^{2(m-1)}u\|_{L^1_{[0,T]}L^2_x} \\
&\leqslant \|u_0\|_{L^2} + T^{(5-2m)/4}\|n|u|^{2(m-1)}u\|_{L_t^{4/(2m-1)}L^2_x} \\
&\leqslant \|u_0\|_{L^2} + T^{(5-2m)/4}\|n\|_{L^\infty_t L^2_x}\|u\|^{2m-1}_{L^4_t L^\infty_x} \\
&\leqslant \|u_0\|_{L^2} + T^{(5-2m)/4}\|n\|_{C([0,T];\mathcal{G})}\|u\|^{2m-1}_{S^0([0,T]\times\mathbb{R})}.
\end{aligned}
$$

Also, estimating Λ_G in $C([0,T];\mathcal{G})$ by applying Proposition 4.1, the Sobolev embedding, and Hölder inequality,

$$
\begin{aligned}
\|\Lambda_G(u)\|_{C([0,T];\mathcal{G})} &\leqslant \|(n_0,n_1)\|_{\mathcal{G}} + c\||u|^{2m}\|_{L^1_{[0,T]}H^{-1}_x} \\
&\leqslant \|(n_0,n_1)\|_{\mathcal{G}} + c\||u|^{2m}\|_{L^1_{[0,T]}L^1_x} \\
&\leqslant \|(n_0,n_1)\|_{\mathcal{G}} + cT^{(3-m)/2}\|u\|^{2m}_{L^{4m/(m-1)}_{[0,T]}L^{2m}_x} \\
&\leqslant \|(n_0,n_1)\|_{\mathcal{G}} + cT^{(3-m)/2}\|u\|^{2m}_{S^0([0,T]\times\mathbb{R})}.
\end{aligned}
$$

Similar estimates hold for the differences $\Lambda_S(u_1,n_1) - \Lambda_S(u_2,n_2)$ and $\Lambda_G(u_1) - \Lambda_G(u_2)$, namely

$$
\begin{aligned}
&\|\Lambda_S(u_1,n_1) - \Lambda_S(u_2,n_2)\|_{S^0} \\
&\qquad \lesssim T^{(5-2m)/4}(\|n_1\|_{C([0,T];\mathcal{G})}(\|u_1\|^{2(m-1)}_{S^0} + \|u_2\|^{2(m-1)}_{S^0})\|u_1 - u_2\|_{S^0} \\
&\qquad\quad + (\|u_1\|^{2m-1}_{S^0} + \|u_2\|^{2m-1}_{S^0})\|n_1 - n_2\|_{C([0,T];\mathcal{G})})
\end{aligned}
$$

and

$$
\|\Lambda_G(u_1) - \Lambda_G(u_2)\|_{C([0,T];\mathcal{G})} \lesssim T^{(3-m)/2}(\|u_1\|^{2m-1}_{S^0} + \|u_2\|^{2m-1}_{S^0})\|u_1 - u_2\|_{S^0}.
$$

Standard argument then implies that if T is such that

$$T^{(3-m)/2}\|u_0\|^{2m-1}_{L^2} \lesssim 1, \tag{5.3}$$

$$T^{(5-2m)/4}\|u_0\|^{2m-2}_{L^2}\|(n_0,n_1)\|_{\mathcal{G}} \lesssim 1, \tag{5.4}$$

$$T^{(3-m)/2}\|u_0\|^{2m}_{L^2} \lesssim \|(n_0,n_1)\|_{\mathcal{G}}, \tag{5.5}$$

then the mapping (Λ_S, Λ_G) is a contraction, yielding a fixed point (u,n) which is a solution to (1.3) on $[0,T]$ such that

$$\|u\|_{S^0([0,T]\times\mathbb{R})} \lesssim \|u_0\|_{L^2}, \tag{5.6}$$

$$\|n\|_{C([0,T];\mathcal{G})} \leqslant \|(n_0,n_1)\|_{\mathcal{G}} + cT^{(3-m)/2}\|u_0\|^{2m}_{L^2}. \tag{5.7}$$

Now, let $1 \leqslant m \leqslant \frac{7}{4}$. By the conservation of mass, $\|u(t)\|_{L^2} = \|u_0\|_{L^2}$ and so we are only concerned with the possibility of the growth of $\|n(t)\|_{\mathcal{G}}$ from one time step to the next. Suppose that after a number of iterations, we have $\|n(t)\|_{\mathcal{G}} \gg \|u(t)\|_{L^2}^{2m} = \|u_0\|_{L^2}^{2m}$. Take this time as the initial time $t = 0$ so that $\|(n_0, n_1)\|_{\mathcal{G}} \gg \|u_0\|_{L^2}^{2m}$. Then (5.5) is satisfied automatically. Also, by (5.4), we may take the time increment of size

$$T \sim (\|u_0\|_{L^2}^{2m-2}\|(n_0, n_1)\|_{\mathcal{G}})^{-4/(5-2m)}. \tag{5.8}$$

We see from (5.7) that we can perform K iterations on time intervals, each of length (5.8), where

$$K \sim \frac{\|(n_0, n_1)\|_{\mathcal{G}}}{T^{(3-m)/2}\|u_0\|_{L^2}^{2m}}, \tag{5.9}$$

before the quantity $\|n(t)\|_{\mathcal{G}}$ doubles. The total time we advance after these K iterations is

$$\begin{aligned} KT &\sim \frac{T^{(m-1)/2}\|(n_0, n_1)\|_{\mathcal{G}}}{\|u_0\|_{L^2}^{2m}} \sim \frac{(\|u_0\|_{L^2}^{2m-2}\|(n_0, n_1)\|_{\mathcal{G}})^{-4(m-1)/2(5-2m)}\|(n_0, n_1)\|_{\mathcal{G}}}{\|u_0\|_{L^2}^{2m}} \\ &= \frac{\|(n_0, n_1)\|_{\mathcal{G}}^{(7-4m)/(5-2m)}}{\|u_0\|_{L^2}^{(2m+4)/(5-2m)}} \gtrsim \frac{\|u_0\|_{L^2}^{(7-4m)/(5-2m)}}{\|u_0\|_{L^2}^{(2m+4)/(5-2m)}} = \frac{1}{\|u_0\|_{L^2}^{(6m-3)/(5-2m)}} \end{aligned}$$

by (5.9) and (5.8), where in the last line we used the fact that $(7-4m)/(5-2m) \geqslant 0$ and $\|(n_0, n_1)\|_{\mathcal{G}} \gg \|u_0\|_{L^2}^{2m}$ by assumption at this time. The last expression is independent of $\|n(t)\|_{\mathcal{G}}$. We can now repeat this entire procedure, each time advancing the time interval of length $\sim \|u_0\|_{L^2}^{-(6m-3)/(5-2m)}$. Upon each iteration, the size of $\|n(t)\|_{\mathcal{G}}$ is at most doubled, giving the exponential-in-time upper bound stated in the theorem. This completes the proof. □

For the dimension $d = 2, 3$, the proof is similar as soon as we have the following estimates.

Remark 5.1. For $d = 2$, applying the Strichartz estimates with $(\tilde{q}, \tilde{r}) = (4/(2m-1), 4/(3-2m))$ and Hölder inequality, we have

$$\|\Lambda_S(u, n)\|_{S^0([0,T]\times\mathbb{R}^2)} \lesssim \|u_0\|_{L^2} + T^{(3-2m)/2}\|n\|_{C([0,T];\mathcal{G})}\|u\|_{S^0([0,T]\times\mathbb{R}^2)}^{2m-1}$$

and, similarly, $\|\Lambda_G(u)\|_{C([0,T];\mathcal{G})} \lesssim \|(n_0, n_1)\|_{\mathcal{G}} + T^{2-m}\|u\|_{S^0([0,T]\times\mathbb{R}^2)}^{2m}$.

For $d = 3$, applying the Strichartz estimates with $(\tilde{q}, \tilde{r}) = (2/(2m-1), 6/(5-4m))$ and Hölder inequality,

$$\|\Lambda_S(u, n)\|_{S^0([0,T]\times\mathbb{R}^3)} \lesssim \|u_0\|_{L^2} + T^{(7-6m)/4}\|n\|_{C([0,T];\mathcal{G})}\|u\|_{S^0([0,T]\times\mathbb{R}^3)}^{2m-1}$$

and, similarly, $\|\Lambda_G(u)\|_{C([0,T];\mathcal{G})} \lesssim \|(n_0, n_1)\|_{\mathcal{G}} + T^{(9-6m)/4}\|u\|_{S^0([0,T]\times\mathbb{R}^3)}^{2m}$.

The rest of the proof follows as before.

R e m a r k 5.2. In [10], the proof of the global well-posedness for (1.3) in $L^2 \times H^{1/2}$ is proven using Bourgain's restriction norm method. We remark that the results can be proven using more elementary methods (standard space-time Lebesgue spaces, the Sobolev embedding and Hölder inequality) as outlined in the proof of Theorem 1.1 above.

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