Jorge Bustamante

A Marchaud type inequality

Commentationes Mathematicae Universitatis Carolinae, Vol. 63 (2022), No. 2, 159–167

Persistent URL: http://dml.cz/dmlcz/151083

A Marchaud type inequality

Jorge Bustamante

Abstract. We present a new Marchaud type inequality in $\mathbb{L}^p$ spaces.

Keywords: Marchaud type inequality

Classification: 26A15, 26D10

1. Introduction

For $1 \le p < \infty$, the Banach space $\mathbb{L}^p$ consists of all 2π-periodic, pth power Lebesgue integrable (class of) functions f on $\mathbb{R}$ with the norm

$$\|f\|_p = \left(\frac{1}{2\pi}\int_{-\pi}^{\pi} |f(x)|^p \, \mathrm{d}x\right)^{1/p}.$$

We also set $C_{2\pi}$ for the family of all 2π-periodic continuous functions on $\mathbb{R}$ with the norm

$$\|f\|_\infty = \sup_{x\in[-\pi,\pi]} |f(x)|.$$

Moreover

$$X^p = \begin{cases} \mathbb{L}^p, & \text{if } 1 \le p < \infty, \\ C_{2\pi}, & \text{if } p = \infty. \end{cases}$$

Notice that X^∞ is not the space of essentially bounded functions.

For $m \in \mathbb{N}$ and $1 \le r \le p \le \infty$, $r \ne \infty$, set

$$W^m_{p,r} = \{f \in X^p \colon f = \varphi \ \text{ a.e. } \varphi, \varphi^{(1)}, \dots, \varphi^{(m-1)} \in AC,\ \varphi^{(m)} \in X^r\}. \tag{1}$$

In the case $p = r = \infty$, we set $W^m_{\infty,\infty} = C^m_{2\pi}$ (functions m-times continuously differentiable).

For $m \in \mathbb{N}$, $f \in X^p$, $1 \le p \le \infty$, and $t > 0$ the usual modulus of continuity (smoothness) of order m of f is defined by

$$\omega_m(f,t)_p = \sup_{|h|\le t} \|\Delta^m_h f\|_p, \tag{2}$$

DOI 10.14712/1213-7243.2022.014

where

$$\Delta_h^m f(x) = \sum_{k=0}^{m} (-1)^k \binom{m}{k} f(x+kh).$$

The Marchaud inequality appeared for the first time in [7], but there are various extensions. Let us recall a few.

Theorem 1 (H. Johnen, [5, page 302])**.** *Assume $1 \le p < \infty$. If $f \in \mathbb{L}^p$ and*

$$\int_0^1 \frac{\omega_{m+n}(f,u)_p}{u^{k+1}}\,\mathrm{d}u < \infty \tag{3}$$

for some $k \in \mathbb{N}$, $1 \le k \le m-1$, then $f \in W^k_{p,p}$, and for $0 < t \le 1$ and $n \in \mathbb{N}$,

$$\omega_m(f^{(k)},t)_p \le C_{r,m}\left(t^m \int_t^2 \frac{\omega_{m+n}(f,u)_p}{u^{m+k+1}}\,\mathrm{d}u + \int_0^t \frac{\omega_{m+n}(f,u)_p}{u^{k+1}}\,\mathrm{d}u \right). \tag{4}$$

Another kind of estimate was given by H. Johnen and K. Scherer in [6]. If $1 \le k \le m-1$ and (3) holds, then $f \in W^k_{p,p}$ and

$$\omega_{m-k}(f^{(k)},t)_p \le C \int_0^t \frac{\omega_m(f,u)_p}{u^{k+1}}\,\mathrm{d}u, \qquad f \in X^p.$$

A proof was also included in [3, pages 178–179].

Recall that if $1 \le r < p \le \infty$ and $f \in X^p$, then $f \in X^r$ and $\|f\|_r \le \|f\|_p$. Therefore $\omega_m(f,t)_r \le \omega_m(f,t)_p$.

In this note we show that a result similar to (4) holds, if we replace $\omega_m(f,u)_p$ by $\omega_m(f,u)_r$, with $1 \le r < p$. In fact, we prove the following theorem:

Theorem 2. *Assume $1 \le r < p < \infty$ and $m \in \mathbb{N}$, $m > 2$. There exists a constant C such that, if $f \in \mathbb{L}^p$, $1 \le k < m-1$, and*

$$\int_0^1 \frac{\omega_m(f,s)_r}{s^{1+k+1/r-1/p}}\,\mathrm{d}s < \infty, \tag{5}$$

then $f \in W^k_{p,p}$ and for $0 < t \le 1/2$,

$$\omega_{m-k}(f^{(k)},t)_p \le C\left(t^{m-1-k}\|f\|_r + \int_0^t \frac{\omega_m(f,s)_r}{s^{1+k+1/r-1/p}}\,\mathrm{d}s + \frac{\omega_m(f,t)_r}{t^{1+k}} \right). \tag{6}$$

Notice that the case $k = m-1$ is not included in Theorem 2. It is done because in such a case the term corresponding to $\|f\|_r$ does not go to zero when $t \to 0$. The result can be improved. As we show in Proposition 2, if we assume condition (5), then $f \in W^k_{r,r}$. A stronger property holds, but it requires to consider fractional derivatives and fractional moduli of smoothness (for definitions see [1]). In fact, it can be proved that under condition (5) there exists $\beta \in (0,1)$ such that the fractional derivative $D^{k+\beta}f$ exists a.e. This kind of problems goes beyond the scope of the paper.

2. Known results

In this section we recall some known facts.

The classes $W^m_{p,p}$ can be described in terms of strong derivatives. A function $f \in \mathbb{L}^p$ has a strong derivative, if there exists $g \in \mathbb{L}^p$ such that

$$\lim_{h\to 0+} \left\| \frac{f(\cdot + h) - f(\cdot)}{h} - g \right\|_p = 0.$$

In such a case we denote $g = D^{(1)}_s f$. For $m \in \mathbb{N}$, the strong derivative is defined by $D^{(m)}_s(f) = D^{(1)}_s(D^{(m-1)}_s f)$. It is known that $f \in W^m_{p,p}$ if and only if f has a strong derivative $D^{(m)}_s f$, see [2, Theorem 10.1.12]. Moreover, if $f \in W^m_{p,p}$, then $D^{(m)}_s f = \varphi^{(m)}$, where φ is associated to f as in (1). In what follows we identify f and φ. Moreover, if $f \in W^m_{p,p}$, all strong derivatives of lower order exist, see [2, Theorem 10.1.6].

It is known that $D^m \colon W^m_{p,p} \to \mathbb{L}^p$, defined by $D^m f = D^{(m)}_s f$ is a closed linear operator, see [1, Lemma 2].

In the next result, for $1 \le s < \infty$, $\mathbb{L}^s[a,b]$ denotes the usual Lebesgue space.

Theorem 3 (V. N. Gabushin, [4])**.** *Assume $p, q, r \geq 1$ are real numbers, $0 \leq k < m$, $k, n \in \mathbb{N}$, and*

$$\frac{m-k}{q} + \frac{k}{r} \geq \frac{m}{p}. \tag{7}$$

There exists a constant A such that, if $f \in \mathbb{L}^q[a,b]$ and $f^{(m)} \in \mathbb{L}^r[a,b]$, and $0 < \delta \leq (b-a)$, then

$$\delta^k \|D^k f\|_p \leq A(\delta^{1/p-1/q}\|f\|_q + \delta^{m+1/p-1/r}\|D^m f\|_r).$$

Proposition 1 (see [3, page 45])**.** *If $1 \leq p \leq \infty$, $f \in \mathbb{L}^p$, $m, n \in \mathbb{N}$ and $s > 0$, then*

$$\omega_m(f, ns)_p \leq n^m\, \omega_m(f, s)_p, \tag{8}$$

and

$$\omega_m(f, t)_p \leq 2\, \omega_{m-1}(f, t)_p. \tag{9}$$

Theorem 4 (see [3, page 177]). *If $1 \le r < \infty$ and $m \in \mathbb{N}$, there exist positive constants M_1 and M_2 such that for $f \in \mathbb{L}^r$ and $0 < t$*

$$M_1 \omega_m(f,t)_r \le K_m(f,t)_r \le M_2 \omega_m(f,t)_r, \tag{10}$$

where

$$K_m(f,t)_r = \inf_{g \in W^m_{p,p}} \{\|f-g\|_r + t^m \|D^m g\|_r\}.$$

For $n \in \mathbb{N}_0$, let $\mathbb{T}_n$ be the family of all trigonometric polynomials of degree not greater than n. For $f \in \mathbb{L}^r$ define

$$E_{n,r}(f) = \inf_{T \in \mathbb{T}_n} \|f - T\|_r.$$

3. Two results related with strong derivatives

Theorem 5. *Let r be a real number, $1 \le r < \infty$, and $k, m \in \mathbb{N}$. If $f \in \mathbb{L}^r$ and*

$$\int_0^1 \frac{\omega_m(f,s)_r}{s^{1+k}}\, \mathrm{d}s < \infty, \tag{11}$$

then $f \in W^j_{r,r}$ for every j, $0 \le j \le k$. Moreover if $\{g_n\}_{n\in\mathbb{N}}$ is a sequence satisfying $g_n \in W^m_{r,r}$ and

$$\|f - g_n\|_r + \frac{1}{n^m}\|D^m g_n\|_r \le C\omega_m\Big(f, \frac{1}{n}\Big)_r, \qquad n \in \mathbb{N}, \tag{12}$$

with a constant C which depends not on f or n (whose existence is guaranteed by Theorem 4), then

$$(D^j f)(x) = (D^j g_n)(x) + \sum_{i=1}^{\infty} (D^j(g_{n2^i} - g_{n2^{i-1}}))(x) \qquad \text{a.e.} \tag{13}$$

for each j, $0 \le j \le k$, and every $n \in \mathbb{N}$.

Proof: Let $\{g_n\}_{n\in\mathbb{N}} \subset W^m_{r,r}$ be a sequence such that (12) holds.

Fix j, $0 \le j \le k$, and for $n, \nu \in \mathbb{N}$ set

$$G_n = \sum_{i=1}^{\infty}(g_{n2^i} - g_{n2^{i-1}}), \qquad S_{\nu,n} = \sum_{i=1}^{\nu}(g_{n2^i} - g_{n2^{i-1}}),$$

and

$$H_{\nu,n,j} = \sum_{i=1}^{\nu}(D^j g_{n2^i} - D^j g_{n2^{i-1}}) = D^j(S_{\nu,n}).$$

If $j=0$, it follows from (12) that

$$\lim_{\nu\to\infty}\|f-g_n-S_{\nu,n}\|_r=\lim_{\nu\to\infty}\|f-g_{n2^\nu}\|_r=0.$$

Hence $f-g_n=G_n$ a.e. for each $n\in\mathbb{N}$. This proves (13) for $j=0$.

In what follows we assume $1\le j\le k$.

First, we prove that $H_{\nu,n,j}$ is a Cauchy sequence in $\mathbb{L}^r$. If $\nu,\sigma\in\mathbb{N}$, since $k<m$, from Theorem 3 (with $\delta=1/(n2^i)$ and $p=q=r$) and (12), we obtain

$$\begin{aligned}
\|H_{\nu+\sigma,n,j}-H_{\nu,n,j}\|_r &\le \sum_{i=\nu+1}^{\nu+\sigma}\|D^j(g_{n2^i}-g_{n2^{i-1}})\|_r\\
&\le A\sum_{i=\nu+1}^{\nu+\sigma}(n2^i)^j\|g_{n2^i}-g_{n2^{i-1}}\|_r+A\sum_{i=\nu+1}^{\nu+\sigma}\frac{1}{(n2^i)^{m-j}}\|D^m(g_{n2^i}-g_{n2^{i-1}})\|_r\\
&\le A\sum_{i=\nu+1}^{\nu+\sigma}(n2^i)^j(\|g_{n2^i}-f\|_r+\|g_{n2^{i-1}}-f\|_r)\\
&\quad +AC\sum_{i=\nu+1}^{\nu+\sigma}\frac{(n2^i)^m}{(n2^i)^{m-j}}\,\omega_m\Big(f,\frac{1}{n2^i}\Big)_r\\
&\quad +AC\sum_{i=\nu+1}^{\nu+\sigma}\frac{(n2^i)^m}{(n2^i)^{m-j}}\frac{1}{2^m}\,\omega_m\Big(f,\frac{1}{n2^{i-1}}\Big)_r\\
&\le 2AC\sum_{i=\nu}^{\nu+\sigma}(n2^{i+1})^j\omega_m\Big(f,\frac{1}{n2^i}\Big)_r+2AC\sum_{i=\nu}^{\nu+\sigma}(n2^i)^j\omega_m\Big(f,\frac{1}{n2^i}\Big)_r
\end{aligned}$$

(here we have used (8) and it will be used again in the next inequality)

$$\begin{aligned}
&\le 2^{m+1}AC\sum_{i=\nu}^{\nu+\sigma}(n2^{i+1})^j\omega_m\Big(f,\frac{1}{n2^{i+1}}\Big)_r+2AC\sum_{i=\nu}^{\nu+\sigma}(n2^i)^j\omega_m\Big(f,\frac{1}{n2^i}\Big)_r\\
&\le C_1\sum_{i=\nu}^{\nu+\sigma+1}(n2^i)^j\omega_m\Big(f,\frac{1}{n2^i}\Big)_r\le C_2\sum_{i=\nu}^{\nu+\sigma+1}\int_{1/n2^i}^{1/(n2^{i-1})}\frac{\omega_m(f,s)_r}{s^{j+1}}\,\mathrm{d}s\\
&\le C_2\int_0^{1/(n2^\nu)}\frac{\omega_m(f,s)_r}{s^{j+1}}\,\mathrm{d}s.
\end{aligned}$$

Therefore, it follows from (11) that $\{H_{\nu,n,j}\}_{\nu=1}^\infty$ is a Cauchy sequence in $\mathbb{L}^r$. Thus there exists $g\in\mathbb{L}^r$ such that $\|H_{\nu,n,j}-g\|_r\to0$, as $\nu\to\infty$.

Since $S_{\nu,n}\to f-g_n=G_n$, $H_{\nu,n}(f)=D^k(S_{\nu,n})\to g$ in $\mathbb{L}^r$, as $\nu\to\infty$, and D^j is a closed linear operator, then $f-g_n=G_n\in W^j_{r,r}$ and $D^j(f-g_n)=$

$D^j f - D^j g_n = g$. Hence

$$D^j g_n + \sum_{i=1}^{\infty} (D^j g_{n2^i} - D^j g_{n2^{i-1}}) = g + D^j g_n = D^j f \qquad \text{a.e.}$$

□

Proposition 2. *If* $1 \leq r < p \leq \infty$, *and* $f \in \mathbb{L}^r$ *satisfies* (11) *with* $k \in \mathbb{N}$, *then* $f \in W^k_{r,r}$.

Proof: It is known, see [9, page 334], that if

$$\sum_{i=1}^{\infty} i^{k-1} E_{i,r}(f) < \infty,$$

then f is equivalent to a function $g \in X^k_{r,r}$. On the other hand, there exists a constant C such that for each $f \in \mathbb{L}^r$, see [9, page 325],

$$E_{n,r}(f) \leq C\omega_m\Big(f, \frac{1}{n+1}\Big)_r. \tag{14}$$

Hence

$$\begin{aligned}\sum_{i=1}^{\infty} i^{k-1} E_{i,r}(f) &\leq C_1 \sum_{i=1}^{\infty} i^{k-1} \omega_m\Big(f, \frac{1}{i+1}\Big)_r \leq C_1 \sum_{i=1}^{\infty} \int_i^{i+1} s^{k-1} \omega_m\Big(f, \frac{1}{s}\Big)_r \,\mathrm{d}s \\ &= C_1 \int_1^{\infty} s^{k-1} \omega_m\Big(f, \frac{1}{s}\Big)_r \,\mathrm{d}s = C_1 \int_0^1 \frac{\omega_m(f,t)_r}{t^{k+1}} \,\mathrm{d}t < \infty.\end{aligned}$$

□

4. Proof of Theorem 2

Given $t \in (0, 1/2]$, choose $n \in \mathbb{N}$ such that

$$\frac{1}{1+n} < t \leq \frac{1}{n} < 2\pi.$$

Set $\tau(j) = (1+n)2^j$ and $\lambda_j = 1/\tau(j)$. If $f \in \mathbb{L}^p$, taking into account (10) for each $j \in \mathbb{N}_0$ we can fix $h_j \in W^m_{r,r}$ such that

$$\|f - h_j\|_r + \lambda_j^m \|D^m h_j\|_r \leq 2M_2 \omega_m(f, \lambda_j)_r. \tag{15}$$

Step 1: From Proposition 2 we know that $f \in W^k_{r,r}$. Hence there exists φ such that $f = \varphi$ a.e., $\varphi, \varphi^{(1)}, \dots, \varphi^{(m-1)} \in AC$, and $\varphi^{(m)} \in \mathbb{L}^r$. Thus we only need to find $G \in \mathbb{L}^p$, such that

$$\varphi^{(m)} = G \qquad \text{a.e.} \tag{16}$$

Of course, we can assume that $f = \varphi$.

The conditions $h_j \in W^m_{r,r}$ and $0 \le k < m$ imply $D^k h_j \in \mathbb{L}^p$. In fact $D^k h_j$ is an (absolutely) continuous function.

Taking into account that $h_i \to f$ in the norm of $\mathbb{L}^r$, see (15),

$$f = h_0 + \sum_{i=0}^{\infty} (h_{i+1} - h_i) \qquad \text{a.e.} \tag{17}$$

Since $(D^k h_{i+1} - D^k h_i) \in \mathbb{L}^p$ for each $i \in \mathbb{N}$, if

$$\sum_{i=0}^{\infty} \|D^k h_{i+1} - D^k h_i\|_p < \infty \tag{18}$$

then the series

$$D^k h_0 + \sum_{i=0}^{\infty} (D^k h_{i+1} - D^k h_i) \tag{19}$$

converges in $\mathbb{L}^p$, see [8, page 109].

Let us verify that (18) holds. In order to simplify the notations we set $\alpha = 1/r - 1/p$.

In the case $q = r < p$, condition (7) holds. Since $\lambda_i^k < 2\boldsymbol{\pi}$, Theorem 3 can be used (with $\delta = \lambda_i$, recall that we consider $q = r$) and it follows from (15) that

$$\begin{aligned}
\lambda_i^k \|D^k h_{i+1} - D^k h_i\|_p &\le A(\lambda_i^{-\alpha} \|h_{i+1} - h_i\|_r + \lambda_i^{m-\alpha} \|D^m h_{i+1} - D^m h_i\|_r) \\
&\le A\lambda_i^{-\alpha} (\|h_{i+1} - f\|_r + \|f - h_i\|_r + \lambda_i^m \|D^m h_i\|_r + 2^m \lambda_{i+1}^m \|D^m h_{i+1}\|_r) \\
&\le 2M_2 A \lambda_j^{-\alpha} (\omega_m(f, \lambda_i)_r + 2^m \omega_m(f, \lambda_{i+1})_r) \\
&\le 2AM_2(1 + 2^m) \lambda_i^{-\alpha} \omega_m(f, \lambda_i)_r.
\end{aligned}$$

Note that, since $\lambda_i = 2\lambda_{i+1}$,

$$1 \le 2 \int_{\lambda_{i+1}}^{\lambda_i} \frac{\mathrm{d}s}{s}.$$

Therefore, using (8), one has (recall that $k + \alpha > 0$)

$$\begin{aligned}
\sum_{j=0}^{\infty} \|D^k h_{j+1} - D^k h_j\|_p &\le 2(1 + 2^m) AM_2 \sum_{i=0}^{\infty} \lambda_i^{-\alpha} \frac{\omega_m(f, \lambda_i)_r}{\lambda_i^k} \\
&\le 4(1 + 2^m) AM_2 \sum_{i=0}^{\infty} \frac{2^m \omega_m(f, \lambda_{i+1})_r}{\lambda_i^{k+\alpha}} \int_{\lambda_{i+1}}^{\lambda_i} \frac{\mathrm{d}s}{s}
\end{aligned}$$

$$\begin{aligned}
&\le 4(1+2^m)\, A\, M_2\, 2^m \sum_{i=0}^{\infty} \int_{\lambda_{i+1}}^{\lambda_i} \frac{\omega_m(f,s)_r}{s^{1+k+\alpha}}\, \mathrm{d}s \\
&= 4(1+2^m) A M_2 2^m \int_0^{\lambda_0} \frac{\omega_m(f,s)_r}{s^{1+k+\alpha}}\, \mathrm{d}s \qquad (20)\\
&\le 4(1+2^m) A M_2 2^m \int_0^{t} \frac{\omega_m(f,s)_r}{s^{1+k+1/r-1/p}}\, \mathrm{d}s < \infty,
\end{aligned}$$

where we use condition (5). Thus the series (19) converges in $\mathbb{L}^p$. The limit of the partial sums of the series (19) will be denoted by M_k.

We will show that in (16) we can take $G = M_k$. Since h_0 can be chosen as g_{n+1} in (12) and h_i as $g_{(n+1)2^i}$, it follows from (13) that

$$M_k = D^k g_{n+1} + \sum_{i=0}^{\infty} D^k\big(g_{(n+1)2^{i+1}} - g_{(n+1)2^i}\big) = D^k f \qquad \text{a.e.}$$

We have proved that $f \in W^k_{p,p}$.

Step 2: We will need an estimate of $\|D^{m-1}h_0\|_p$ in terms of $\|f\|_r$ and $\omega_m(f,t)_r$. Take into account that

$$\frac{1}{\lambda_0} = 1 + n \le 2n \le \frac{2}{t}.$$

We use again Theorem 3 (with $\delta = 1$) and (15) to obtain

$$\begin{aligned}
\|D^{m-1}h_0\|_p &\le A(\|h_0\|_r + \|h_0^{(m)}\|_r) \le A(\|f\|_r + \|f-h_0\|_r + \|D^m h_0\|_r) \\
&\le A\Big(\|f\|_r + 2M_2\omega_m(f,\lambda_0)_r + \frac{1}{\lambda_0^m} 2M_2\omega_m(f,\lambda_0)_r\Big) \qquad (21)\\
&\le A\Big(\|f\|_r + 2M_2\frac{\omega_m(f,t)_r}{t^m} + 2\frac{2^m}{t^m} M_2\omega_m(f,t)_r\Big) \\
&\le A\Big(\|f\|_r + 2M_2(1+2^m)\frac{\omega_m(f,t)_r}{t^m}\Big).
\end{aligned}$$

Step 3: Let us verify (6). From (8), (9) and (10), we know that

$$\begin{aligned}
\omega_{m-k}(D^k f, t)_p &\le 2^{m-k}\omega_{m-k}\Big(D^k f, \frac{1}{1+n}\Big)_p \le 2^{1+m-k}\omega_{m-1-k}\Big(D^k f, \frac{1}{1+n}\Big)_p \\
&\le \frac{2^{1+m-k}}{M_1}\Big(\|D^k f - D^k h_0\|_p + \frac{1}{(1+n)^{m-1-k}}\|D^{m-1}h_0\|_p\Big) \qquad (22)\\
&\le \frac{2^{1+m-k}}{M_1}\Big(\|D^k f - D^k h_0\|_p + t^{m-1-k}\|D^{m-1}h_0\|_p\Big).
\end{aligned}$$

From (22), (17), (20), and (15) we obtain

$$\begin{aligned}\omega_{m-k}(D^k f,t)_p &\le C_2\Big(\sum_{j=0}^{\infty}\|D^k h_{j+1}-D^k h_j\|_p+t^{m-1-k}\|D^{m-1}h_0\|_p\Big)\\ &\le C_2\Big(\int_0^t \frac{\omega_m(f,s)_r}{s^{1+k+\alpha}}\,\mathrm{d}s+t^{m-1-k}\|f\|_r+\frac{\omega_m(f,t)_r}{t^{1+k}}\Big).\end{aligned}$$

□

References

[1] Butzer P. L., Dyckhoff H., Görlich E., Stens R. L., *Best trigonometric approximation, fractional order derivatives and Lipschitz classes*, Canadian J. Math. **29** (1977), no. 4, 781–793.

[2] Butzer P. L., Nessel R. J., *Fourier Analysis and Approximation. Volume 1: One-dimensional Theory*, Pure and Applied Mathematics, 40, Academic Press, New York, 1971.

[3] DeVore R. A., Lorentz G. G., *Constructive Approximation*, Grundlehren der mathematischen Wissenschaften, 303, Springer, Berlin, 1993.

[4] Gabushin V. N., *Inequalities for the norms of a function and its derivatives in metric L_p-metrics*, Mat. Zametki **1** (1967), 291–298 (Russian).

[5] Johnen H., *Inequalities connected with the moduli of smoothness*, Mat. Vesnik **9(24)** (1972), 289–303.

[6] Johnen H., Scherer K., *On the equivalence of the K-functional and moduli of continuity and some applications*, Constructive Theory of Functions of Several Variables, Proc. Conf., Math. Res. Inst., Oberwolfach, 1976, Lecture Notes in Math., 571, Springer, Berlin, 1977, pages 119–140.

[7] Marchaud A., *Sur les dérivées et sur les différences des fonctions de variables réelles*, Thèses de l'entre-deux-guerres (1927), no. 78, 98 pages (French).

[8] Rudin W., *Functional Analysis*, McGraw–Hill Series in Higher Mathematics, McGraw–Hill Book Co., New York, 1973.

[9] Timan A. F., *Theory of Approximation of Functions of Real Variable*, Translated from the Russian by J. Berry, English translation edited and editorial preface by J. Cossar, Pergamon Press Book International Series of Monographs in Pure and Applied Mathematics, 34, The Macmillan Company, Pergamon Press, New York, 1963.

J. Bustamante:

Benemérita Universidad Autónoma de Puebla,

Facultad de Ciencias Físico-Matemáticas, Avenida San Claudio y 18 Sur, Colonia San Manuel, Edificio FM1-101B, Ciudad Universitaria, C.P. 72570, Puebla, México

E-mail: jbusta@fcfm.buap.mx

(*Received* January 8, 2021, *revised* November 14, 2021)