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LUCAS SEQUENCES AND REPDIGITS

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Abstract. Let $(G_n)_{n\geqslant 1}$ be a binary linear recurrence sequence that is represented by the Lucas sequences of the first and second kind, which are $\{U_n\}$ and $\{V_n\}$, respectively. We show that the Diophantine equation $G_n = B\cdot(g^{lm}-1)/(g^l-1)$ has only finitely many solutions in $n, m \in \mathbb{Z}^+$, where $g \geqslant 2$, l is even and $1 \leqslant B \leqslant g^l - 1$. Furthermore, these solutions can be effectively determined by reducing such equation to biquadratic elliptic curves. Then, by a result of Baker (and its best improvement due to Hajdu and Herendi) related to the bounds of the integral points on such curves, we conclude the finiteness result. In fact, we show this result in detail in the case of $G_n = U_n$, and the remaining case can be handled in a similar way. We apply our result to the sequences of Fibonacci numbers $\{F_n\}$ and Pell numbers $\{P_n\}$. Furthermore, with the first application we determine all the solutions (n, m, g, B, l) of the equation $F_n = B\cdot(g^{lm}-1)/(g^l-1)$, where $2 \leqslant g \leqslant 9$ and $l = 1$.

Keywords: Diophantine equation; Lucas sequence; repdigit; elliptic curve

MSC 2020: 11D72, 11B37, 11B39, 11A63, 11J86

1. Introduction

Let P and Q be nonzero relatively prime integers. Let the sequences $\{U_n\} = \{U_n(P,Q)\}$ and $\{V_n\} = \{V_n(P,Q)\}$ be defined by the same recurrence relation

$$U_n = PU_{n-1} - QU_{n-2} \quad (\text{similarly, } V_n = PV_{n-1} - QV_{n-2})$$

for $n \geqslant 2$ with the initial terms $U_0 = 0$, $U_1 = 1$ and $V_0 = 2$, $V_1 = P$. The discriminant of these sequences is defined by $D = P^2 - 4Q$. The characteristic polynomial of the

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recurrence is given by

$$X^2 - PX + Q,$$

which has the roots

$$\alpha = \frac{P + \sqrt{D}}{2} \quad \text{and} \quad \beta = \frac{P - \sqrt{D}}{2}.$$

Thus, $\alpha \neq \beta$, $\alpha + \beta = P$, $\alpha \cdot \beta = Q$, and $(\alpha - \beta)^2 = D$. The sequences $\{U_n\}$ and $\{V_n\}$ are called the (first and second kind) Lucas sequences corresponding to the parameters (P, Q). The terms of Lucas sequences of the first and second kind satisfy the identity

$$V_n^2 = DU_n^2 + 4Q^n. \tag{1.1}$$

Moreover, if the ratio $\zeta = \alpha/\beta$ of the roots of the characteristic polynomial is a root of unity, then the sequences $\{U_n\}$ and $\{V_n\}$ are said to be degenerate, and non-degenerate otherwise. Describing all the degenerate Lucas sequences follows from the fact that

$$|\zeta + \zeta^{-1}| = \left|\frac{\alpha}{\beta} + \frac{\beta}{\alpha}\right| \leqslant 2.$$

Since $\alpha/\beta + \beta/\alpha = (P^2 - 2Q)/Q$, it follows that $P^2 - 2Q = 0, \pm Q, \pm 2Q$. This gives $P^2 = Q, 2Q, 3Q, 4Q$. Since $\gcd(P, Q) = 1$, we get $(P, Q) = (1, 1), (-1, 1), (2, 1)$ or $(-2, 1)$. Therefore, if $D = 0$ or $D = -3$, then the sequence is degenerate (for more details, see e.g. [21], pages 2–7). The most interesting Lucas sequences of the first and second kind are the sequences of the Fibonacci numbers, Pell numbers, Lucas numbers, and Pell-Lucas numbers, which are given by

$$\begin{aligned}
&F_0 = 0, \quad F_1 = 1, \quad F_n = F_{n-1} + F_{n-2} \qquad \forall\, n \geqslant 2,\\
&P_0 = 0, \quad P_1 = 1, \quad P_n = 2P_{n-1} + P_{n-2} \qquad \forall\, n \geqslant 2,\\
&L_0 = 2, \quad L_1 = 1, \quad L_n = L_{n-1} + L_{n-2} \qquad \forall\, n \geqslant 2,\\
&Q_0 = 2, \quad Q_1 = 2, \quad Q_n = 2Q_{n-1} + Q_{n-2} \qquad \forall\, n \geqslant 2,
\end{aligned}$$

respectively. Diophantine equations involving terms of such sequences have been investigated by several authors. For instance, there are many articles in the literature, which address Diophantine equations involving binary recurrence sequences with repdigits that are defined as follows. A natural number N is called a base g-repdigit for $g \geqslant 2$ if all of its base g-digits are equal; that is, if

$$N = b \cdot \frac{g^m - 1}{g - 1} \quad \text{for some } m \geqslant 1 \text{ and } b \in \{1, \ldots, g - 1\}.$$

The problem of finding all perfect powers among repdigits was presented in 1956 by Obláth (see [20]) and then solved by Bugeaud and Mignotte in 1999 (see [9]). Thereafter, many authors have started to study the solutions of such Diophantine equations. For instance, Luca in [16] used an elementary way to show that the largest number whose decimal expansion has only one distinct digit in the sequence of Fibonacci numbers or Lucas numbers is $F_{10} = 55$ or $L_5 = 11$, respectively. In addition, Díaz Alvarado and Luca in [10] found all Fibonacci numbers that are sums of two repdigits. Furthermore, a similar problem was investigated in the case of Lucas numbers by Adegbindin, Luca and Togbé in [1]. For other related results, we refer to [6], [12], [15], [18], [23] and the references given there. Luca's result (see [16]) was generalized by Marques and Togbé in [17], in which they determined all the solutions of the Diophantine equations

$$F_n = B \cdot \frac{10^{lm} - 1}{10^l - 1} \quad \text{and} \quad L_n = B \cdot \frac{10^{lm} - 1}{10^l - 1} \tag{1.2}$$

in positive integers m, n and l, with $m > 1$, $1 \leqslant l \leqslant 10$ and $1 \leqslant B \leqslant 10^l - 1$, which are $(m, n, l) = (2, 10, 1)$ and $(m, n, l) = (2, 5, 1)$ in the Fibonacci and Lucas cases, respectively. It is clear that these equations have solutions only with $l = 1$. In general, if $(G_n)_{n \geqslant 1}$ is an integer linear recurrence sequence, they gave a finiteness result for the equation

$$G_n = B \cdot \frac{g^{lm} - 1}{g^l - 1}, \tag{1.3}$$

where n, m, g, l and B are positive integers such that $m > 1$, $g > 1$, $1 \leqslant B \leqslant g^l - 1$. In fact, they proved their results using heavy computations followed by a result due to Matveev (see [19]) on the lower bound on linear forms of logarithms of algebraic numbers to obtain bounds for n and m. As these bounds could be very high, they used a result due to Dujella and Pethő (see [11]) on the Baker-Davenport reduction to reduce these bounds. With respect to these results, the following natural questions arise.

- ▷ Is there another approach that is easier to be applied to such concrete equations?
- ▷ Do the equations in (1.2) have solutions in any base g other than 10, say $g \geqslant 2$, in the case of $l = 1$?

In this paper, we answer the above questions positively. Indeed, we firstly use a different and direct approach to obtain a general finiteness result for the Diophantine equation (1.3) in which the sequence G_n is represented by the non-degenerate Lucas sequences of the first and second kind with $Q \in \{-1, 1\}$ and l is an even positive

integer. Our argument is based on combining equation (1.3) with the identity relationship between Lucas sequences of the first and second kind (1.1) to produce biquadratic elliptic curves of the form

$$y^2 = ax^4 + bx^2 + c, \tag{1.4}$$

with integer coefficients a, b, c and discriminant $\Delta = 16ac(b^2 - 4ac)^2 \neq 0$. The integral points of a biquadratic elliptic curve can be determined using an algorithm implemented in Magma (see [5]) as `SIntegralLjunggrenPoints()` (based on results obtained by Tzanakis, see [27]) or an algorithm described by Alekseyev and Tengely in [2] in which they gave an algorithmic reduction of the search for integral points on such a curve by solving a finite number of Thue equations. It is clear that such a biquadratic elliptic curve in the form (1.4) can be further written as an elliptic curve of the form

$$Y^2 = X^3 + bX^2 + acX, \tag{1.5}$$

where $X = ax^2$, $Y = axy$ and its discriminant is $a^2c^2(b^2 - 4ac) \neq 0$. For determining all integral points on a given elliptic curve, one can follow the so-called elliptic logarithm method developed by Stroeker and Tzanakis (see [26]) and independently by Gebel, Pethő and Zimmer (see [13]). There exists a number of software implementations for determining integral points on elliptic curves based on this technique such as an algorithm implemented in SageMath (see [25]) as `integral_points()`. In 1968, Baker (see [3]) gave an explicit bound for the size of all the integral solutions of any equation of the form

$$y^2 = ax^3 + bx^2 + cx + d,$$

where a, b, c, d denote rational integers with $a \neq 0$ and the polynomial on the right has three simple roots. Furthermore, one year later in [4], he extended this result to some further equations of the form

$$y^m = b_0x^n + b_1x^{n-1} + \ldots + b_n, \tag{1.6}$$

where $n \geqslant 3$ and $b_0 \neq 0$, $b_1, \ldots, b_n \in \mathbb{Z}$. Indeed, he gave bounds for the integral solutions of the given equations for all $m \geqslant 2$. Let us recall the result in the case of $m = 2$ in the following theorem and let us call it Baker's theorem:

Baker's theorem. *If the polynomial on the right of the Diophantine equation* (1.6)*, where* $m = 2$*, possesses at least three simple zeros, then all of its solutions in integers* x, y *satisfy*

$$\max(|x|, |y|) < \exp\exp\exp\{(n^{10n}H)^{n^2}\},$$

where $H = \max\limits_{0 \leqslant i \leqslant n} |b_i|$.

Such bounds have been improved and generalized by several authors including Brindza (see [7]), Shorey and Tijdeman (see [22]), Sprindžuk (see [24]), Bugeaud (see [8]), Hajdu and Herendi (see [14]). In fact, the best known bounds concerning the solutions of elliptic equations over $\mathbb{Q}$ are due to Hajdu and Herendi in 1998 (see [14]).

R e m a r k 1.1. Since a finiteness result for equation (1.3) in the case of $G_n = U_n$ or $G_n = V_n$ can be obtained in a similar way, we only present and prove this result in detail in the case of $G_n = U_n$ and omit the proof of the remaining case.

As applications of our result, we apply our method on the sequences of Fibonacci and Pell numbers that satisfy equation (1.3). Indeed, with the first application we also generalize the result of Marques and Togbé in [17] in the case of Fibonacci numbers by determining all the solutions (n, m, g, B, l) of $F_n = B \cdot (g^{lm} - 1)/(g^l - 1)$ in the case of $2 \leqslant g \leqslant 9$ and $l = 1$. Note that the case of Lucas numbers can be generalized similarly, therefore we omit the details of this case. More precisely, we use our approach in the case where l is even, otherwise we follow the technique of Marques and Togbé in [17] of using the result of Matveev on linear forms in three logarithms and the result of Dujella and Pethő on the method of Baker-Davenport reduction.

2. Auxiliary results

In this section, we recall some useful results that will be used in the proof of Theorem 3.2 (particularly in case of $l = 1$). Recall that the Binet's Fibonacci numbers formula is known as

$$F_n = \frac{\alpha^n - \beta^n}{\alpha - \beta}, \quad \text{where } (\alpha, \beta) = \Big(\frac{1 + \sqrt{5}}{2}, \frac{1 - \sqrt{5}}{2}\Big) \tag{2.1}$$

for all $n \geqslant 0$, where α is called the golden ratio and $\beta = -1/\alpha$. Moreover, it is also known that

$$\alpha^{n-2} \leqslant F_n \leqslant \alpha^{n-1} \quad \text{holds for all } n \geqslant 1. \tag{2.2}$$

Also, recall that the logarithmic height of an s-degree algebraic number α is defined as

$$h(\alpha) = \frac{1}{s}\bigg(\log|a| + \sum_{j=1}^{s}\log\max\{1, |\alpha^{(j)}|\}\bigg), \tag{2.3}$$

where a is the leading coefficient of the minimal polynomial of α (over $\mathbb{Z}$), $(\alpha^{(j)})_{1\leqslant j\leqslant s}$ are the conjugates of the algebraic number α, and the absolute value of a complex number $z = x + \mathrm{i}y$ is determined by $|z| = \sqrt{x^2+y^2}$.

In order to obtain some bounds for n and m, we need to use a lower bound for a linear form logarithms à la Baker, which was given by the following lemma due to Matveev, see [19] (also see Lemma 2 in [17]).

Lemma 2.1. *Define*

$$\Lambda = a_1\log\alpha_1 + a_2\log\alpha_2 + a_3\log\alpha_3, \tag{2.4}$$

where a_1, a_2 and a_3 are nonzero integers and α_1, α_2 and α_3 are nonzero algebraic numbers. Let d be the degree of the number field $\mathbb{Q}(\alpha_1,\alpha_2,\alpha_3)$ over $\mathbb{Q}$ and $\chi = [\mathbb{R}(\alpha_1,\alpha_2,\alpha_3):\mathbb{R}]$. If $\Lambda\neq 0$, then

$$\log|\Lambda| \geqslant -C_1 d^2 A_1A_2A_3\log(1.5\mathrm{e}dB'\log(\mathrm{e}d)),$$

where A_1, A_2 and A_3 are real numbers satisfying the condition

$$A_j \geqslant \max\{dh(\alpha_j), |\log(\alpha_j)|, 0.16\} \quad \forall\, j\in\{1,2,3\}, \tag{2.5}$$

$$B' \geqslant \max\{1, \max\{|a_j|A_j/A_1 :\ 1\leqslant j\leqslant 3\}\}, \tag{2.6}$$

and

$$C_1 = \frac{5.16^5}{6\chi}\cdot \mathrm{e}^3(7+2\chi)(20.2+\log(3^{5.5}d^2\log(\mathrm{e}d))). \tag{2.7}$$

After finding upper bounds for n and m, which could be very large, the next step is to reduce them. For that we use the following lemma, which is a variant of the Baker-Davenport lemma, due to the result of Dujella and Pethő (see Lemma 5 in [11]).

Lemma 2.2. *Suppose that M is a positive integer. Let p/q be the convergent of the continued fraction expansion of κ such that $q > 6M$ and let $\varepsilon = \|\mu q\| - M\|\kappa q\|$, where $\|\cdot\|$ denotes the distance from the nearest integer. If $\varepsilon > 0$, then there is no solution of the inequality*

$$0 < m\kappa - n + \mu < AK^{-m}$$

in $n, m\in\mathbb{Z}$ in the range

$$\frac{\log(Aq)-\log\varepsilon}{\log K} \leqslant m \leqslant M.$$

3. Main results

This section contains two main theorems. The first one presents our main theoretical result related to the finiteness result of equation (1.3) in the case when l is even, and in the second theorem we present our computational results regarding the applications of our method on the sequences of Fibonacci and Pell numbers and a generalization of the Fibonacci case in the result of Marques and Togbé in [17]. We also remark that in Theorem 3.1; $g, l \geqslant 2$ are fixed, l is even and $1 \leqslant B \leqslant g^l - 1$ are assumed. But in practice, determining the solutions of any equation of the form (1.3) (for a particular sequence and any even number l) is achieved similarly for every $g \in \{2, 3, \ldots\}$ and $B \in \{1, 2, \ldots, g^l - 1\}$. Therefore, to make the presentation simpler, in Theorem 3.2 we assume that $2 \leqslant g \leqslant 9$, $l \in \{1, 2, 4\}$ and $1 \leqslant B \leqslant \min\{10, g^l - 1\}$ in the case of $G_n = F_n$ and $2 \leqslant g \leqslant 9$, $l = 2$ and $1 \leqslant B \leqslant \min\{5, g^l - 1\}$ in the case of $G_n = P_n$.

Theorem 3.1. *Let P and Q be nonzero integers with $Q \in \{-1, 1\}$ and t be a positive integer. If $G_n = U_n(P, Q)$ is non-degenerate and $l = 2t$, then the Diophantine equation* (1.3) *has finitely many solutions of the form (n, m), which can be effectively determined.*

Theorem 3.2. *If $G_n = F_n$, then the Diophantine equation* (1.3) *has the following solutions with $2 \leqslant g \leqslant 9$, $l \in \{1, 2, 4\}$ and $1 \leqslant B \leqslant \min\{10, g^l - 1\}$:*

$$\begin{aligned}(n, m, g, B, l) \in \{&(4, 2, 2, 1, 1), (5, 2, 4, 1, 1), (6, 2, 3, 2, 1), (6, 2, 7, 1, 1),\\ &(7, 3, 3, 1, 1), (8, 2, 6, 3, 1), (8, 3, 4, 1, 1), (5, 2, 2, 1, 2),\\ &(8, 3, 2, 1, 2), (9, 2, 4, 2, 2), (9, 2, 2, 2, 4)\}.\end{aligned}$$

Furthermore, suppose that $2 \leqslant g \leqslant 9$, $l = 2$, $1 \leqslant B \leqslant \min\{5, g^l - 1\}$ and $G_n = P_n$. Then equation (1.3) *has no more solutions other than $(n, m, g, B, l) = (3, 2, 2, 1, 2)$.*

4. Proofs of theorems

4.1. Proof of Theorem 3.1. Since $G_n = U_n(P, Q) = U_n$ with $Q \in \{-1, 1\}$ and $l = 2t$ for an integer $t \geqslant 1$, we combine equation (1.3) with identity (1.1) to obtain

$$(g^{2t} - 1)^2 V_n^2 = DB^2(g^{2tm} - 1)^2 + 4(g^{2t} - 1)^2 Q^n,$$

which can be further written as biquadratic curves of the form

$$y^2 = DB^2(x^4 - 2x^2 + 1) + 4G^2 Q^n, \tag{4.1}$$

where $D = P^2 - 4Q$, $G = (g^{2t} - 1)$, $1 \leqslant B \leqslant (g^{2t} - 1)$, $x = g^{tm}$ and $y = GV_n$ such that $m, g \geqslant 2$. Next, we show that the given curves have nonzero discriminants in order to prove they present elliptic curves. Since $Q \in \{-1, 1\}$, we split the proof into two cases:

Case 1. If $Q = 1$, then equation (4.1) becomes

$$y^2 = D_1 B^2 (x^4 - 2x^2 + 1) + 4G^2, \tag{4.2}$$

where $D_1 = (P^2 - 4)$, whose discriminant is

$$\Delta_1 = 4096 D_1^3 G^4 B^6 (D_1 B^2 + 4G^2).$$

In addition to P being nonzero, we consider only non-degenerate Lucas sequences, i.e.

$$(P, Q) \notin \{(-2, 1), (-1, 1), (1, 1), (2, 1)\}.$$

Hence, $D_1 > 0$, which implies that $(D_1 B^2 + 4G^2) > 0$ as $B > 0$ and $G > 0$. Therefore it is clear that $\Delta_1 \neq 0$ (indeed, $\Delta_1 > 0$).

Case 2. Similarly, if $Q = -1$, we obtain the curves

$$y^2 = D_2 B^2 (x^4 - 2x^2 + 1) + 4G^2 \tag{4.3}$$

and

$$y^2 = D_2 B^2 (x^4 - 2x^2 + 1) - 4G^2, \tag{4.4}$$

where $D_2 = (P^2 + 4)$, and their discriminants are

$$\Delta_2 = 4096 D_2^3 G^4 B^6 (D_2 B^2 + 4G^2)$$

and

$$\Delta_3 = 4096 D_2^3 G^4 B^6 (D_2 B^2 - 4G^2),$$

respectively. Again, it is obvious that $\Delta_2 \neq 0$ as $D_2 > 0$, $B > 0$ and $G > 0$. For a contradiction we assume that $\Delta_3 = 0$, which is true if and only if

$$D_2 B^2 - 4G^2 = 0.$$

The latter equation is true if and only if D_2 is a square number; that is, if there exists a nonzero integer T such that

$$T^2 - P^2 = 4,$$

which has no more rational integer solutions other than $(T, P) \in \{(-2, 0), (2, 0)\}$, which contradicts that $P \neq 0$. Thus, $\Delta_3 \neq 0$.

Therefore, we conclude that the biquadratic curves (4.1) represent elliptic curves. Moreover, it was mentioned earlier that the biquadratic curves (4.1) can be written in the form (1.5); that is,

$$Y^2 = X^3 - 2DB^2X^2 + DB^2(DB^2 + 4G^2Q^n)X, \tag{4.5}$$

where $X = DB^2x^2$ and $Y = DB^2xy$. In a similar way, one can easily show that the latter curves have nonzero discriminants. Thus, curves (4.5) represent elliptic curves. Finally, by the result of Baker's theorem and its best improvement by Hajdu and Herendi, the number of the integral points of curves (4.1) or (4.5) is finite. Hence, these points can be effectively determined using the techniques mentioned earlier. The only problem that may appear here is that there is no known algorithm to determine the rank and generators of the Mordell-Weil groups of elliptic curves, there are techniques that work well in practice but there is no guarantee to succeed. If we have such an elliptic curve, then we may follow the previously mentioned argument of Alekseyev and Tengely. As a result, the number of the solutions (n, m) is finite, and they can be effectively determined. This completes the proof of Theorem 3.1. □

4.2. Proof of Theorem 3.2. We split the proof of this theorem into two cases regarding the sequences of Fibonacci numbers and Pell numbers in which they satisfy equation (1.3). The proof of Fibonacci case is divided into two subcases: the first one is if $l = 1$, in which we use the result of Matveev on linear forms in three logarithms and the result of Dujella and Pethő on the method of Baker-Davenport reduction, and the other is the case of $l = 2, 4$, in which we apply our approach presented in Theorem 3.1. On the other hand, the proof of the Pell case will be handled in a similar way using only the result of Theorem 3.1.

4.2.1. The Fibonacci case: $G_n = F_n$.
Case $l = 1$:
Step 1. Finding a bound for n in the equation

$$F_n = B \cdot \frac{g^m - 1}{g - 1}, \tag{4.6}$$

where $2 \leqslant g \leqslant 9, 1 \leqslant B \leqslant \min\{10, g-1\}$ and $m \geqslant 2$. Note that since $(g-1) < 10$ for all $2 \leqslant g \leqslant 9$, here we only use the range $1 \leqslant B \leqslant (g-1)$. Suppose that $n > 50$. By substituting Binet's Fibonacci numbers formula (2.1) in equation (4.6), we get that

$$\frac{\alpha^n - \beta^n}{\sqrt{5}} = B \cdot \frac{g^m - 1}{g - 1},$$

which can be further written as

$$\alpha^n - \frac{\sqrt{5}B}{g-1}g^m = \beta^n - \frac{\sqrt{5}B}{g-1}. \tag{4.7}$$

By taking the absolute value for both sides of the latter equation, we obtain that

$$\left|\alpha^n - \frac{\sqrt{5}B}{g-1}g^m\right| \leqslant \alpha^{-50} + \sqrt{5} < 2.3 \tag{4.8}$$

as $\beta = -1/\alpha$, $n > 50$ and $B \leqslant (g-1)$. Define

$$\Lambda = \log\frac{\sqrt{5}B}{g-1} - n\log\alpha + m\log g. \tag{4.9}$$

Since

$$\mathrm{e}^{\Lambda} = \frac{\sqrt{5}B}{g-1}\alpha^{-n}g^m,$$

we get (from inequality (4.8)) that

$$|\mathrm{e}^{\Lambda} - 1| < \frac{2.3}{\alpha^n} < \alpha^{-n+2}. \tag{4.10}$$

To apply Lemma 2.1, we first state and prove the following claim:

Claim 4.1. *Suppose that Λ is defined in equation* (4.9). *Then $\Lambda > 0$.*

P r o o f. From equation (4.7) and the fact that $\beta = -1/\alpha$, we deduce that

$$1 - \mathrm{e}^{\Lambda} = \frac{1}{\alpha^n}\Big(\beta^n - \frac{\sqrt{5}B}{g-1}\Big) = \frac{1}{\alpha^n}\Big((-1)^n\alpha^{-n} - \frac{\sqrt{5}B}{g-1}\Big). \tag{4.11}$$

Now, we consider the following cases regarding the values of n.

▷ If n is even, then equation (4.11) with the hypotheses $-n < -50$, $g \leqslant 9$ and $B \geqslant 1$, implies that

$$1 - \mathrm{e}^{\Lambda} = \frac{1}{\alpha^n}\Big(\alpha^{-n} - \frac{\sqrt{5}B}{g-1}\Big) < \frac{1}{\alpha^n}\Big(\alpha^{-50} - \frac{\sqrt{5}}{8}\Big) < 0,$$

which leads to $\Lambda > 0$.

▷ If n is odd, then for all $n > 50$, $2 \leqslant g \leqslant 9$ and $1 \leqslant B \leqslant g-1$ we have that $(\alpha^{-1})^n + \sqrt{5}B/(g-1) > 0$. Therefore equation (4.11) again gives

$$1 - \mathrm{e}^{\Lambda} = \frac{-1}{\alpha^n}\Big((\alpha^{-1})^n + \frac{\sqrt{5}B}{g-1}\Big) < 0,$$

which also implies that $\Lambda > 0$.

Thus, the claim is completely proved. □

From (4.10) and the fact that $\Lambda > 0$, we obtain that $\Lambda < \mathrm{e}^{\Lambda} - 1 < \alpha^{-n+2}$. Therefore

$$\log|\Lambda| < (-n+2)\log\alpha. \tag{4.12}$$

With respect to the notation of Lemma 2.1 and by comparing equations (2.4) and (4.9), we have that

$$\alpha_1 = \frac{\sqrt{5}B}{g-1},\ \alpha_2 = \alpha,\ \alpha_3 = g,\ a_1 = 1,\ a_2 = -n \quad \text{and} \quad a_3 = m.$$

We also note that the number field $\mathbb{Q}(\alpha_1, \alpha_2, \alpha_3) = \mathbb{Q}\big(\sqrt{5}\big)$ is of degree $d = 2$. Furthermore, the conjugates of α_1, α_2 and α_3 are $\alpha_1' = -\alpha_1$, $\alpha_2' = \beta$ and $\alpha_3' = \alpha_3$, respectively. Clearly, the minimal polynomial of α_1 is

$$(x-\alpha_1)(x-\alpha_1') = x^2 - \frac{5B^2}{(g-1)^2},$$

which is a divisor of $(g-1)^2x^2 - 5B^2$. Therefore, by using the definition of the logarithmic height of algebraic numbers in (2.3) we get that the logarithmic height of the 2-degree algebraic number α_1 is

$$h(\alpha_1) \leqslant \frac{1}{2}\big(2\log 8 + 2\log\sqrt{5}\big) < 2.89$$

as $g \leqslant 9$ and $B \leqslant g-1$. Similarly, $h(\alpha_2) = \frac{1}{2}\log\alpha < 0.25$ and $h(\alpha_3) = \log g < 2.2$ as $g \leqslant 9$. Hence, from inequality (2.5) we take $A_1 = 5.78$, $A_2 = 0.5$ and $A_3 = 4.4$. Furthermore, to obtain an estimated value for B' using inequality (2.6), let us first consider and prove the following claim:

Claim 4.2. *The values of n and m in equation* (4.6) *satisfy that $n > m$.*

P r o o f. First of all, from equation (4.6) we have that

$$F_n \geqslant \frac{g^m - 1}{g-1} \quad \text{as } B \geqslant 1. \tag{4.13}$$

On the other hand, we see that

$$\frac{g^m-1}{g-1} - g^{m-1} = g^m \cdot \Big(\frac{1}{g-1} - \frac{1}{g}\Big) - \frac{1}{g-1} = g^m \cdot \frac{1}{g(g-1)} - \frac{1}{g-1} > 0 \quad \text{as } m > 1,$$

which gives

$$\frac{g^m-1}{g-1} > g^{m-1}. \tag{4.14}$$

Combining inequalities (4.13), (4.14) and (2.2), we get that

$$g^{m-1} < F_n \leqslant \alpha^{n-1}. \tag{4.15}$$

Taking the logarithm for both sides, we obtain that $(m-1)\log g < (n-1)\log\alpha$, which leads to

$$m < (n-1)\frac{\log\alpha}{\log g} + 1. \tag{4.16}$$

Since $\log\alpha/\log g < 1$ as $g \geqslant 2$, we get that $m < (n-1)+1 = n$ with $m \geqslant 2$. This proves the claim. □

Therefore since $n > 50$, we have that

$$\max\{1, \max\{|a_j|A_j/A_1 :\ 1 \leqslant j \leqslant 3\}\} = \max\Big\{\frac{0.5}{5.78}n, \frac{4.4}{5.78}m\Big\},$$

and then it suffices to take $B' = \frac{5}{6}n$ as $n > m$. From (2.7) we get that $C_1 < 4.45 \cdot 10^9$ since $\chi = 1$ and $d = 2$. Therefore, Lemma 2.1 yields

$$\log|\Lambda| > -2.27 \cdot 10^{11}\log(11.51n). \tag{4.17}$$

Combining inequalities (4.12) and (4.17), we obtain that

$$2.27 \cdot 10^{11}\log(11.51n) > (n-2)\log\alpha,$$

which implies that $n < 10^{14}$.

Step 2. Finding a bound for m in equation (4.6). For that, we first give the following lemma:

Lemma 4.1. *The solutions of equation* (4.6) *satisfy*

$$(n-2)\frac{\log\alpha}{\log g} < m < (n-1)\frac{\log\alpha}{\log g} + 1. \tag{4.18}$$

P r o o f. The proof follows easily from combining fact (2.2) and equation (4.6). Indeed, for all $2 \leqslant g \leqslant 9$, $1 \leqslant B \leqslant g-1$ and $m > 1$ one can see that

$$\alpha^{n-2} \leqslant F_n < g^m.$$

Taking the logarithm for both sides, we obtain that

$$(n-2)\log\alpha < m\log g,$$

which leads to

$$(n-2)\frac{\log\alpha}{\log g} < m. \tag{4.19}$$

The upper bound follows from inequality (4.16). Hence, Lemma 4.1 is proved. □

Thus, from the upper bound of inequality (4.18) and the estimate of n (that is, $n < 10^{14}$) we obtain that

$$m < (10^{14} - 1)\frac{\log \alpha}{\log g} + 1 < 7 \cdot 10^{13} \quad \text{as } g \geqslant 2.$$

Step 3. Reducing the obtained bounds. We know that $0 < \Lambda < \alpha^{-n+2}$. From inequality (4.15) we also know that $\alpha^{n-1} > g^{m-1}$, which leads to

$$\alpha^{-n+2} < g^{-m+1}\alpha < g^{-m+2}$$

as $\alpha < g$ for all $2 \leqslant g \leqslant 9$. Hence,

$$0 < m \log \alpha_3 - n \log \alpha_2 + \log \alpha_1 < g^{-m+2}.$$

Dividing the latter inequality by $\log \alpha_2$, we get that

$$0 < m\frac{\log \alpha_3}{\log \alpha_2} - n + \frac{\log \alpha_1}{\log \alpha_2} < 3 \cdot g^2 \cdot g^{-m}. \tag{4.20}$$

Without loss of generality and to be more precise, since $\alpha_1 = \sqrt{5}B/(g-1)$, $\alpha_2 = \alpha$ and $\alpha_3 = g$ for all $g \in \{2, 3, \ldots, 9\}$ and $1 \leqslant B \leqslant g - 1$, respectively, we use the notation "n_g, m_g, B_g" instead of "n, m, B" for the rest of the proof of the bounds reduction step. Therefore we rewrite (4.20) in the form

$$0 < m_g\kappa_g - n_g + \mu_g < 3 \cdot g^2 \cdot g^{-m_g}, \tag{4.21}$$

where

$$\kappa_g = \frac{\log g}{\log \alpha} \quad \text{and} \quad \mu_g = \frac{\log\left(\sqrt{5}B_g\right) - \log(g-1)}{\log \alpha}.$$

It is clear that

$$\mu_g \geqslant \frac{\log \sqrt{5} - \log(g-1)}{\log \alpha}$$

as $B_g \geqslant 1$. Since α and g are multiplicatively independent, we have that κ_g is irrational. Thus, we may denote $P_{(k,g)}/Q_{(k,g)}$ to be the kth convergent of the continued fraction of κ_g. Now, we use Lemma 2.2 to reduce the upper bound of m_g (which is very large since $m_g < 7 \cdot 10^{13}$). That will lead to reducing the upper bound of n_g. Therefore we take $M = M_g = 7 \cdot 10^{13}$. Moreover, if the conditions of Lemma 2.2 are satisfied, that is, if $Q_{(k,g)} > 6M$ and $\varepsilon_g = \|\mu_g Q_{(k,g)}\| - M\|\kappa_g Q_{(k,g)}\| > 0$, then we take $A_g = 3 \cdot g^2$ and $K_g = g$. For $g = 2$, we have $B_2 = 1, \kappa_2 = \log 2/\log \alpha$ and $\mu_2 \geqslant \log \sqrt{5}/\log \alpha$. Therefore we obtain that

$$\frac{P_{(32,2)}}{Q_{(32,2)}} = \frac{2683806884597620}{1863211227378077},$$

from which we have $Q_{(32,2)} > 6M$ and

$$\varepsilon_2 \geqslant \left\| \frac{\log \sqrt{5}}{\log \alpha} Q_{(32,2)} \right\| - M \left\| \frac{\log 2}{\log \alpha} Q_{(32,2)} \right\| > 0.4 > 0.$$

Therefore we have that $A_2 = 3 \cdot 2^2$ and $K_2 = 2$. Hence, Lemma 2.2 tells us that there is no solution to inequality (4.21) and then to the Diophantine equation (4.6) in the case of $g = 2$; that is,

$$F_{n_2} = 2^{m_2} - 1 \tag{4.22}$$

in the range

$$\left[\left\lfloor \frac{\log(A_2 Q_{(32,2)}/\varepsilon_2)}{\log K_2} \right\rfloor + 1, M \right] = [56, 7 \cdot 10^{13}].$$

Therefore $m_2 \leqslant 56$ and inequality (4.19) gives us $n_2 < 83$. To finish, we use SageMath (see [25]) to print all the Fibonacci numbers in the range $50 < n_2 < 83$ of which we see that there are no Fibonacci numbers satisfying equation (4.22) with $2 \leqslant m_2 \leqslant 56$. However, in the range $3 \leqslant n_2 \leqslant 50$ we get the solution $(n_2, m_2, g, B_2, l) = (4, 2, 2, 1, 1)$. Let us now consider the case $g = 6$, which implies that $1 \leqslant B_6 \leqslant 5$, $\kappa_6 = \log 6/\log \alpha$ and $\mu_6 \geqslant \log\left(1/\sqrt{5}\right)/\log \alpha$. Thus, $Q_{(30,6)} = 1232281049712607 > 6M$ and $\varepsilon_6 > 0.1 > 0$. For that we take $A_6 = 3 \cdot 6^2$ and $K_6 = 6$, and Lemma 2.2 leads to unsolvablity of the equation

$$F_{n_6} = \frac{B_6}{5} \cdot (6^{m_6} - 1) \tag{4.23}$$

with $24 \leqslant m_6 \leqslant M$. Therefore $m_6 \leqslant 24$, which implies that $n_6 < 92$. Again, we get no solutions to equation (4.23) (in fact, to equation (4.6) in the case of $g = 6$) with $50 < n_6 < 92$, but we get the solution $(8, 2, 6, 3, 1)$, where $3 \leqslant n_6 \leqslant 50$. In a similar way, for all the remaining values of g, one can show that the hypotheses of Lemma 2.2 are satisfied and determine the other desired solutions in the theorem in the case of $l = 1$.

Case $l \in \{2, 4\}$:

▷ If $l = 2$, then quation (1.3) becomes

$$F_n = B \cdot \frac{g^{2m} - 1}{g^2 - 1}. \tag{4.24}$$

We consider in detail the case where we have $g = 2$, and the remaining values of g will be pursued in a similar way following the proof of Theorem 3.1. Since $P = 1$,

$Q = -1$, $l = 2$, $t = 1$ and $g = 2$, we have that $D_2 = 5$, $G = 3$ and $1 \leqslant B \leqslant 3$. Therefore, for all $B \in \{1, 2, 3\}$, equation (4.3) leads to the biquadratic elliptic curves

$$y^2 = 5x^4 - 10x^2 + 41, \tag{4.25}$$

$$y^2 = 20x^4 - 40x^2 + 56, \tag{4.26}$$

$$y^2 = 45x^4 - 90x^2 + 81, \tag{4.27}$$

and also equation (4.4) gives the curves

$$y^2 = 5x^4 - 10x^2 - 31, \tag{4.28}$$

$$y^2 = 20x^4 - 40x^2 - 16, \tag{4.29}$$

$$y^2 = 45x^4 - 90x^2 + 9, \tag{4.30}$$

respectively, where $x = 2^m$ and $y = 3L_n$. Let us consider curve (4.25) and by using the Magma function `SIntegralLjunggrenPoints()`, we get the following integral points with positive values for the x-coordinates

$$[[1, -6], [2, 9], [5, 54], [8, -141]].$$

Combining the values of x of these integral points with $x = 2^m$, we only obtain $m = 3$. Therefore, since $B = 1$, $g = 2$ and $m = 3$, equation (4.24) implies that $n = 8$. Hence, we get the solution $(n, m, g, B, l) = (8, 3, 2, 1, 2)$. Next, we consider (4.26), which has no integral points other than $[x, y] = [1, 6]$ in which the value of x is positive. Thus, we have no solution for equation (4.24). Similarly, we get no solution for (4.24) in the case of equations (4.27), (4.29) and (4.30). Finally, we deal with equation (4.28) and here we get $x = 4$, which leads to $m = 2$. Hence, we get the solution $(5, 2, 2, 1, 2)$. The other remaining values of g can be treated in a similar way. Indeed, we get only one solution, which is $(9, 2, 4, 2, 2)$ in the case of $B = 2$ and $g = 4$.

▷ If $l = 4$, then quation (1.3) implies that

$$F_n = B \cdot \frac{g^{4m} - 1}{g^4 - 1}. \tag{4.31}$$

In fact, it can be proven completely following the same approach as in the previous case, in which we had $l = 2$. But let us treat this case using the elliptic curve equation (4.5). For that we may consider the case where we have $g = 2$ and $B = 2$. Again, since $D = 5$, $t = 2$ and $G = 15$, equation (4.5) gives the curves

$$Y^2 = X^3 - 40X^2 - 17600X, \tag{4.32}$$

$$Y^2 = X^3 - 40X^2 + 18400X, \tag{4.33}$$

where $X = 20(16)^m$ and $Y = 300(4)^m L_n$. By considering equation (4.32) and using the SageMath function `integral_points()`, we get the integral points

$$[(176 : 1056 : 1), (220 : 2200 : 1), (225 : 2325 : 1), (320 : 4800 : 1), (529 : 11293 : 1),$$
$$(4400 : 290400 : 1), (5120 : 364800 : 1), (818620 : 740649800 : 1)],$$

in which we considered only the points with positive values for the X-coordinates. From these points, only $X = 5120$ leads to a solution of equation (4.31), that is $(n, m, g, B, l) = (9, 2, 2, 2, 4)$. On the other hand, the integral points of the elliptic curve (4.33) give no solution to equation (4.31). Furthermore, the other remaining cases for all the values of g can be handled in a similar way. More precisely, one can show that equation (4.31) has no more solutions.

The Fibonacci case is completely proved.

4.2.2. The Pell case: $G_n = P_n$. Here, we have $l = 2$. Thus, equation (1.3) becomes

$$P_n = B \cdot \frac{g^{2m} - 1}{g^2 - 1}. \tag{4.34}$$

Solving this equation completely is handled in a similar way in the case of Fibonacci numbers with l being even. Here, we have $D_2 = 8$ and $t = 1$. If we consider $g = 2$ and $B = 1$, then we get $G = 3$. These lead to the biquadratic elliptic curves

$$y^2 = 8x^4 - 16x^2 - 28, \tag{4.35}$$
$$y^2 = 8x^4 - 16x^2 + 44, \tag{4.36}$$

where $x = 2^m$ and $y = 3Q_n$. Equation (4.35) leads to the solution $(n, m, g, B, l) = (3, 2, 2, 1, 2)$, and equation (4.36) implies no solution to equation (4.34). The remaining cases are treated similarly. As a result, equation (4.34) does not have any more solutions. Hence, the Pell case is also proved.

Therefore, Theorem 3.2 is completely proved. □

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