Mathematica Bohemica

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Mathematica Bohemica, Vol. 147 (2022), No. 3, 285–299

Persistent URL: http://dml.cz/dmlcz/151007

ON STABILITY, BOUNDEDNESS, AND SQUARE INTEGRABILITY OF SOLUTIONS OF CERTAIN THIRD ORDER NEUTRAL DIFFERENTIAL EQUATIONS

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Received May 28, 2019. Published online July 13, 2021.
Communicated by Josef Diblík

Abstract. The authors establish some new sufficient conditions under which all solutions of a certain class of nonlinear neutral delay differential equations of the third order are stable, bounded, and square integrable. Illustrative examples are given to demonstrate the main results.

Keywords: stability; boundedness; square integrability; Lyapunov functional; neutral differential equation of third order

MSC 2020: 34K20, 34K40

1. Introduction

The study of qualitative behavior of solutions of nonlinear differential equations, such as their stability, boundedness, square integrability, etc., without explicitly determining the solutions has attracted the attention of researchers for many years going back to the pioneering work of Lyapunov (see [18]). The aim of this paper is to study the asymptotic stability of solutions to a class of third-order neutral equations of the form

$$(x''(t)+\Omega(x''(t-r)))'+\Psi(x(t))x''(t)+\Phi(x(t))x'(t)+h(x(t-\sigma))=0, \tag{1.1}$$

J. R. Graef's research was supported in part by a University of Tennessee at Chattanooga SimCenter–Center of Excellence in Applied Computational Science and Engineering (CEACSE) grant.

DOI: 10.21136/MB.2021.0081-19

and the boundedness and square integrability of solutions of the corresponding forced equation

$$(x''(t)+\Omega(x''(t-r)))'+\Psi(x(t))x''(t)+\Phi(x(t))x'(t)+h(x(t-\sigma))=e(t), \tag{1.2}$$

for all $t\geqslant t_1\geqslant t_0+\varrho$, where $\varrho=\max\{r,\sigma\}$.

The asymptotic behavior of solutions of equations of the form of (1.1) with $\Omega(x)=0$ has been studied by many authors utilizing various methods. For example, in 1953, Simanov (see [29]) investigated the global stability of the zero solution of the equation

$$x'''+\psi(x,x')x''+bx'+cx=0,$$

where b and c are constants. Later, Ezeilo (see [8]) discussed the global stability of the zero solution of the equation of the form

$$x'''+\psi(x,x')x''+\varphi(x')+g(x)=0.$$

Swick in [30] studied the asymptotic behavior of solutions of the nonlinear differential equations

$$x'''+ax''+g(x)x'+h(x)=e(t)$$

and

$$x'''+p(t)x''+q(t)g(x')+h(x)=e(t).$$

Nakashima in [20] considered the perturbed versions of the these equations. In 1972, Hara (see [14]) investigated the asymptotic behavior of solutions of differential equations of the form

$$x'''+a(t)x''+b(t)g(x,x')+c(t)h(x)=p(t,x,x',x'')$$

and showed that all solutions are uniformly bounded and satisfy the conditions

$$x(t)\to 0,\quad x'(t)\to 0,\quad \text{and}\quad x''(t)\to 0\quad \text{as } t\to\infty.$$

Hara in [15] also considered the third order equations

$$x'''+a(t)x+b(t)x'+c(t)x=p(t)$$

and

$$x'''+a(t)x''+b(t)x'+c(t)h(x)=p(t,x,x',x'')$$

and established conditions under which all solutions of the above equations are uniformly bounded and tend to zero as $t\to\infty$.

More recently, Qian (see [23], [24]) and Omeike (see [21]) discussed the global stability and asymptotic behavior of solutions of the differential equations

$$x''' + \psi(x, x')x'' + f(x, x') = 0$$

and

$$x''' + \psi(x, x')x'' + f(x, x') = p(t, x, x', x'')$$

with particular attention paid to the boundedness of solutions. In 2007, Zhang and Si (see [36]) investigated the asymptotic stability of solutions of

$$x'''(t) + g(x'(t), x''(t)) + f(x(t), x'(t)) + h(x(t)) = 0.$$

By defining a Lyapunov functional, Tunç in [32] investigated the stability and boundedness of solutions to nonlinear third order differential equations with constant delay, r, of the form

$$\begin{aligned} x'''(t) + g(x(t), x'(t))x''(t) + f(x(t-r), x'(t-r)) + h(x(t-r)) \\ = p(t, x(t), x'(t), x(t-r), x'(t-r), x''(t)). \end{aligned}$$

Ademola and Arawomo (see [1]) obtained criteria for uniform stability, uniform boundedness, and uniform ultimate boundedness of solutions for the more general third order nonlinear delay differential equation

$$x''' + f(x, x', x'')x'' + g(x(t-r(t)), x'(t-r(t))) + h(x(t-r(t))) = p(t, x, x', x'').$$

In [11], Graef et al. obtain sufficient conditions that guarantee the existence of square integrability and asymptotic stability of the zero solution of the non-autonomous third-order delay differential equation

$$(x(t) + \beta x(t-r))''' + a(t)(Q(x(t))x'(t))' + b(t)R(x(t))x'(t) + c(t)f(x(-r)) = 0,$$

and the boundedness and square integrability of solutions of the corresponding forced equation

$$(x(t) + \beta x(t-r))''' + a(t)(Q(x(t))x'(t))' + b(t)R(x(t))x'(t) + c(t)f(x(t-r)) = e(t),$$

which are different from equations (1.1) and (1.2) above.

Our motivation for the present work has come from the papers mentioned above as well as from many others in the literature. To the best of our knowledge, there do not appear to be any results of the type obtained in this paper for third order neutral differential equations of the form (1.1) by using Lyapunov functionals. It is clear that equations (1.1) and (1.2) are different from those mentioned above due to the presence of the neutral term Ω in our equations. Hence, our results include and generalize theirs. We note that many results concerning the theory of neutral functional differential equations are given in the monographs by Hale and Lunel (see [12], [13]). These equations find numerous applications in natural sciences and technology, but they are characterized by specific properties which make their study difficult in both concepts and techniques. For additional results related to those in this paper, we refer the reader to references [2]–[7], [9], [10], [16], [17], [19], [22], [25]–[28], [31], [33]–[35].

Here we assume that in equations (1.1) and (1.2) the functions $\Phi, \Psi, h, \Omega\colon \mathbb{R} \to \mathbb{R}$ are continuous and $h(0) = 0$. In addition, it is also assumed that Φ', Ψ', h', and Ω' exist and are continuous.

By a *solution* of (1.1) we mean a continuous function $x\colon [t_x, \infty) \to \mathbb{R}$ for $t_x \geqslant t_0+\varrho$ such that $x(t) \in C^2([t_x - \varrho, \infty), \mathbb{R})$, $x(t)$ satisfies equation (1.1) on $[t_x, \infty)$, and $Z(t) \in C^1([t_x, \infty), \mathbb{R})$. Without further mention, we will assume throughout that every solution $x(t)$ of (1.1) under consideration here is continuable to the right and nontrivial, i.e., $x(t)$ is defined on some ray $[t_x, \infty)$ and $\sup\{|x(t)|\colon t \geqslant T\} > 0$ for every $T \geqslant t_x$. Moreover, we tacitly assume that (1.1) possesses such solutions.

We set, for a solution x of (1.1),

$$\begin{aligned} Z(t) &= x''(t) + \Omega(x''(t-r)) \quad \forall\, t \geqslant t_0 + \varrho, \\ y(t) &= x'(t), \\ \Delta(t) &= x^2(t) + y^2(t) + Z^2(t), \end{aligned}$$

and

$$\Gamma(t) = x^2(t) + x'^2(t) + x''^2(t).$$

Equation (1.1) can be written as the system

$$\tag{1.3} \begin{cases} x'(t) = y(t), \\ y'(t) = z(t), \\ (z(t) + \Omega(z(t-r)))' = -\Psi(x(t))z(t) - \Phi(x(t))y(t) - h(x(t)) \\ \qquad\qquad\qquad\qquad\qquad + \displaystyle\int_{t-\sigma}^{t} h'(x(s))y(s)\,\mathrm{d}s. \end{cases}$$

It easy to see from (1.3) that

$$\tag{1.4} Z(t) = x''(t) + \Omega(x''(t-r)) = z(t) + \Omega(z(t-r)).$$

2. Stability

Before stating and proving our main results, we introduce the following hypotheses. Assume that there are positive constants d, φ_0, φ_1, ψ_0, ψ_1, n, δ_0, δ_1, and K such that the following conditions on the functions that appear in equation (1.1) are satisfied:

(i) $\psi_0 \leqslant \Psi(x) - d \leqslant \psi_1$ and $\varphi_0 \leqslant \Phi(x) \leqslant \varphi_1$;
(ii) $|\Omega(x)| \leqslant K|x|$ for all x;
(iii) $|h'(x)| \leqslant \delta_0$ for all x, and $h(x)/x \geqslant \delta_1$ for all $x \neq 0$;
(iv) $\int_{-\infty}^{\infty}(|\Psi'(u)| + |\Phi'(u)|)\,\mathrm{d}u \leqslant n$.

For ease of exposition we adopt the following notation:

$$\begin{aligned}\theta_1(t) &= \Psi'(x(t))x'(t),\\ \theta_2(t) &= \Phi'(x(t))x'(t),\\ \omega(t) &= |\theta_1(t)| + |\theta_2(t)|.\end{aligned}$$

Theorem 2.1. *In addition to conditions* (i)–(iv), *assume that there are positive constants A, B, C, and ε such that:*

(v) $-d\delta_1 + \frac{1}{2}(\delta_0 K + d\psi_1) = -A$;
(vi) $\delta_0 - d\varphi_0 + \frac{1}{2}(d + K(d + \varphi_1)) = -B$;
(vii) $(d - \psi_0) + \frac{1}{2}d(\psi_1 + 1) + \frac{1}{2}K(3d + \varphi_1 + 2\delta_0 + 2\psi_1) + \varepsilon = -C$;
(viii) $\delta_0/\varphi_0 < d < \min\{\frac{1}{3}\psi_0, \frac{1}{2}\varphi_0\}$.

Then, the zero solution of (1.3) *is uniformly asymptotically stable provided that*

$$\sigma < \frac{2}{\delta_0}\min\Big\{\frac{A}{d}, \frac{B}{(3d + 1 + K)}, C, \frac{\varepsilon}{K}\Big\}.$$

P r o o f. The proof of this theorem depends on properties of the continuously differentiable functional $W(t, x_t, y_t, z_t) = W$ defined by

$$W = V\mathrm{e}^{-\eta^{-1}\int_{t_1}^{t}\omega(s)\,\mathrm{d}s}, \tag{2.1}$$

where

$$\begin{aligned}V &= V(t, x_t, y_t, z_t) = V_1 + V_2 + V_3,\\ V_1 &= d\int_0^x h(u)\,\mathrm{d}u + h(x)y + \frac{1}{2}\Phi(x)y^2,\\ V_2 &= \frac{1}{2}(dy + Z)^2 + \frac{d}{2}(\Psi(x) - d)y^2 + dxZ + \frac{d}{2}\Phi(x)x^2,\\ V_3 &= \mu\int_{-\sigma}^{0}\int_{t+s}^{t} y^2(u)\,\mathrm{d}u\,\mathrm{d}s + \gamma\int_{t-r}^{t} z^2(s)\,\mathrm{d}s.\end{aligned}$$

Here, γ, μ and η are positive constants to be specified later in the proof.

Since $h(0) = 0$, it is easy to verify that

$$2\int_0^x h'(u)h(u)\,\mathrm{d}u = h^2(x),$$

and since $|h'(x)| \leqslant \delta_0$, we see that $|h(x)| \leqslant \delta_0|x|$.

From conditions (i), (iii), and (viii),

$$\begin{aligned}
V_1 &= d\int_0^x h(u)\,\mathrm{d}u + \frac{\Phi(x)}{2}\Big(y + \frac{h(x)}{\Phi(x)}\Big)^2 - \frac{1}{2\Phi(x)}h^2(x) \\
&\geqslant d\int_0^x h(u)\,\mathrm{d}u - \frac{1}{\varphi_0}\int_0^x h'(u)h(u)\,\mathrm{d}u \geqslant \int_0^x \Big(d - \frac{\delta_0}{\varphi_0}\Big)h(u)\,\mathrm{d}u \geqslant \Big(d - \frac{\delta_0}{\varphi_0}\Big)\frac{\delta_1}{2}x^2.
\end{aligned}$$

Also, using (i) and (viii) we have

$$\begin{aligned}
V_2 &= \frac{1}{4}(Z^2 + 4dxZ + 2d\Phi(x)x^2) + \frac{1}{4}(Z^2 + 4dyZ + 2d(\Psi(x) - d)y^2)) + \frac{1}{2}d^2y^2 \\
&= \frac{1}{8}(Z + 2dx)^2 + \frac{1}{4}d\Phi(x)\Big(x + \frac{1}{\Phi(x)}Z\Big)^2 + (\Phi(x) - 2d)\Big(\frac{1}{4}dx^2 + \frac{1}{8\Phi(x)}Z^2\Big) \\
&\quad + \frac{1}{8}(Z + 2dy)^2 + \frac{1}{4}d(\Psi(x) - d)\Big(y + \frac{1}{\Psi(x) - d}Z\Big)^2 \\
&\quad + (\Psi(x) - 3d)\Big(\frac{dy^2}{4} + \frac{Z^2}{8(\Psi(x) - d)}\Big) + \frac{1}{2}d^2y^2 \\
&\geqslant (\varphi_0 - 2d)\Big(\frac{d}{4}x^2 + \frac{1}{8\varphi_1}Z^2\Big) + (\psi_0 - 3d)\Big(\frac{d}{4}y^2 + \frac{1}{8\psi_1}Z^2\Big) \\
&\geqslant k_0(x^2 + Z^2) + k_1(y^2 + Z^2),
\end{aligned}$$

where $k_0 = \frac{1}{4}(\varphi_0 - 2d)\min\{d, 1/(2\varphi_1)\}$, and $k_1 = \frac{1}{4}(\psi_0 - 3d)\min\{d, 1/(2\psi_1)\}$.

Hence, there exists a positive constant λ_0, small enough so that

$$V \geqslant \lambda_0\Delta(t). \tag{2.2}$$

Therefore, from (2.2) and (2.1), we obtain

$$W \geqslant \lambda_1(x^2(t) + y^2(t) + Z^2(t)) = \lambda_1\Delta(t), \tag{2.3}$$

where $\lambda_1 = \lambda_0\mathrm{e}^{-n/\eta}$.

To obtain an upper estimate on V, note that by using Schwarz's inequality, we have

$$V_1 \leqslant \frac{\delta_0}{2}(d + \delta_0)x^2 + \frac{1}{2}(1 + \varphi_1)y^2,$$

and

$$V_2 \leqslant \frac{d}{2}(1 + \varphi_1)x^2 + \frac{d}{2}(1 + d + \psi_1)y^2 + \Big(d + \frac{1}{2}\Big)Z^2.$$

Thus,

$$V \leqslant \lambda_2 \Delta(t) + \mu \int_{-\sigma}^{0} \int_{t+s}^{t} y^2(u)\,\mathrm{d}u\,\mathrm{d}s + \gamma \int_{t-r}^{t} z^2(s)\,\mathrm{d}s, \tag{2.4}$$

where

$$\lambda_2 = \max\Big\{\frac{\delta_0}{2}(d+\delta_0) + \frac{d}{2}(1+\varphi_1), \frac{1}{2}(1+\varphi_1) + \frac{d}{2}(1+d+\psi_1), \Big(d+\frac{1}{2}\Big)\Big\}.$$

Since

$$\mathrm{e}^{-n/\eta} < \mathrm{e}^{-\eta^{-1}\int_{t_1}^{t} \omega(s)\,\mathrm{d}s} < 1, \tag{2.5}$$

from (2.2), (2.4) and (2.1), we see that

$$\lambda_1 \Delta(t) \leqslant W \leqslant V \leqslant \lambda_2 \Delta(t) + \mu \int_{-\sigma}^{0} \int_{t+s}^{t} y^2(u)\,\mathrm{d}u\,\mathrm{d}s + \gamma \int_{t-r}^{t} z^2(s)\,\mathrm{d}s,$$

where $\lambda_1 = \lambda_0 \mathrm{e}^{-n/\eta}$.

Now taking the derivative of V along the trajectories of system (1.3), we obtain

$$V'_{(1.3)} = U_1 + U_2 + U_3 + \frac{d}{2}\theta_1(t)y^2 + \frac{1}{2}\theta_2(t)(y^2 + dx^2)$$

where

$$\begin{aligned} U_1 = &\ -dxh(x(t)) + (h'(x) - d\Phi(x) + \mu\sigma)y^2(t) + (d - \Psi(x(t)) + \gamma)z^2(t),\\ U_2 = &\ dyz(t) - d\Psi(x(t))xz(t) - \gamma z^2(t-r)\\ &+ (d-\Phi(x))y - h(x(t)) + (d - \Psi(x(t)))z(t)\Omega(z(t-r)), \end{aligned}$$

and

$$U_3 = (dx + dy + Z)\int_{t-\sigma}^{t} h'(x(s))y(s)\,\mathrm{d}s - \mu \int_{t-\sigma}^{t} y^2(s)\,\mathrm{d}s.$$

Conditions (i) and (iii) imply that

$$U_1 \leqslant -d\delta_1 x^2 + (\delta_0 - d\varphi_0 + \mu\sigma)y^2(t) + (d - \psi_0 + \gamma)z^2(t).$$

Using Schwarz's inequality again, together with conditions (i)–(iii), we obtain

$$\begin{aligned} U_2 \leqslant &\ \frac{1}{2}(\delta_0 K + d\psi_1)x^2(t) + \frac{1}{2}(d + K(d+\varphi_1))y^2\\ &+ \frac{1}{2}(d\psi_1 + d + K(d+\psi_1) + \delta_0 K)z^2(t)\\ &+ \Big(\frac{K}{2}(2d + \varphi_1 + \delta_0 + \psi_1) - \gamma\Big)z^2(t-r). \end{aligned}$$

Notice that

$$
\begin{aligned}
dx(t)\int_{t-\sigma}^{t} h'(x(s))y(s)\,\mathrm{d}s &\leqslant \frac{d\delta_0}{2}\sigma x^2(t) + \frac{d\delta_0}{2}\int_{t-\sigma}^{t} y^2(s)\,\mathrm{d}s,\\
dy(t)\int_{t-\sigma}^{t} h'(x(s))y(s)\,\mathrm{d}s &\leqslant \frac{d\delta_0}{2}\sigma y^2(t) + \frac{d\delta_0}{2}\int_{t-\sigma}^{t} y^2(s)\,\mathrm{d}s,\\
z(t)\int_{t-\sigma}^{t} h'(x(s))y(s)\,\mathrm{d}s &\leqslant \frac{\delta_0}{2}\sigma z^2(t) + \frac{\delta_0}{2}\int_{t-\sigma}^{t} y^2(s)\,\mathrm{d}s,\\
\Omega(z(t-r))\int_{t-\sigma}^{t} h'(x(s))y(s)\,\mathrm{d}s &\leqslant \frac{\delta_0 K}{2}\sigma z^2(t-r) + \frac{\delta_0 K}{2}\int_{t-\sigma}^{t} y^2(s)\,\mathrm{d}s.
\end{aligned}
$$

With some rearrangement of terms and using the estimates above, we can easily obtain

$$
\begin{aligned}
V'_{(1.3)} \leqslant{}& \Big(-d\delta_1 + \frac{1}{2}(\delta_0 K + d\psi_1) + \frac{d\delta_0}{2}\sigma\Big)x^2(t)\\
&+ \Big(\delta_0 - d\varphi_0 + \frac{1}{2}(d + K(d+\varphi_1)) + \Big(\mu + \frac{d\delta_0}{2}\Big)\sigma\Big)y^2(t)\\
&+ \Big(d - \psi_0 + \frac{1}{2}(d\psi_1 + d + K(d+\psi_1+\delta_0)) + \gamma + \frac{\delta_0}{2}\sigma\Big)z^2(t)\\
&+ \Big(\frac{K}{2}(2d+\varphi_1+\delta_0+\psi_1) - \gamma + \frac{K\delta_0}{2}\sigma\Big)z^2(t-r)\\
&+ \frac{d}{2}\theta_1(t)y^2 + \frac{1}{2}(dx^2+y^2)\theta_2(t) + \Big(\frac{\delta_0}{2}(2d+1+K) - \mu\Big)\int_{t-\sigma}^{t} y^2(s)\,\mathrm{d}s.
\end{aligned}
$$

If we now choose

$$
\frac{\delta_0}{2}(2d+1+K) = \mu \quad \text{and} \quad \frac{K}{2}(2d+\varphi_1+\delta_0+\psi_1) + \varepsilon = \gamma,
$$

then

$$
\begin{aligned}
V'_{(1.3)} \leqslant{}& \lambda_3\omega(t)(x^2(t)+y^2(t)) + \Big(-A + \frac{d\delta_0}{2}\sigma\Big)x^2(t)\\
&+ \Big(-B + \frac{\delta_0}{2}(3d+1+K)\sigma\Big)y^2(t) + \Big(-C + \frac{\delta_0\sigma}{2}\Big)z^2(t)\\
&+ \Big(-\varepsilon + \frac{\delta_0 K}{2}\sigma\Big)z^2(t-r),
\end{aligned}
$$

where $\lambda_3 = \frac{1}{2}(d+1)$. Now

$$
Z^2 = z^2 + \Omega^2(z(t-r)) + 2z\Omega(z(t-r)), \tag{2.6}
$$

so applying Schwarz's inequality and condition (ii) gives

$$
Z^2 \leqslant 2(z^2 + \Omega^2(z(t-r))) \leqslant 2(z^2 + K^2 z^2(t-r)). \tag{2.7}
$$

If we now take

$$\sigma < \frac{2}{\delta_0} \min\Big\{\frac{A}{d}, \frac{B}{(3d+1+K)}, C, \frac{\varepsilon}{K}\Big\},$$

then we can write

$$\begin{aligned}
(2.8)\quad V'_{(1.3)} &\leqslant \lambda_3\omega(t)(x^2(t)+y^2(t)) - \alpha_1 x^2(t) - \alpha_2 y^2(t) - \alpha_3 z^2(t) - \alpha_4 z^2(t-r),\\
&\leqslant \lambda_3\omega(t)(x^2(t)+y^2(t)) - \alpha_1 x^2(t) - \alpha_1 y^2(t) - \alpha_5(z^2(t) + K^2 z^2(t-r))\\
&\leqslant \lambda_3\omega(t)(x^2(t)+y^2(t)) - \alpha_1 x^2(t) - \alpha_2 y^2(t) - \frac{\alpha_5}{2} Z^2,
\end{aligned}$$

where

$$\alpha_1 = A - \frac{\delta_0}{2}\sigma, \quad \alpha_2 = B - \frac{\delta_0}{2}(3d+1+K)\sigma, \quad \alpha_3 = C - \frac{\delta_0\sigma}{2},$$
$$\alpha_4 = \varepsilon - \frac{\delta_0 K}{2}\sigma, \quad \alpha_5 = \min\Big\{\alpha_3, \frac{\alpha_4}{K^2}\Big\},$$

and

$$\alpha_i > 0 \quad \text{for } i = 1, 2, \ldots, 5.$$

Hence,

$$V'_{(1.3)} \leqslant \lambda_3\omega(t)(x^2(t)+y^2(t)) - \lambda_4\Delta(t),$$

where $\lambda_4 = \min\{\alpha_1, \alpha_2, \frac{1}{2}\alpha_5\}$. Therefore, from (2.1) and (2.2), we have

$$\begin{aligned}
W'_{(1.3)} &= \Big(V' - \frac{1}{\eta}\omega(t)V\Big)\mathrm{e}^{-\eta^{-1}\int_{t_1}^t \omega(s)\,\mathrm{d}s}\\
&\leqslant \Big(\lambda_3\omega(t)(x^2(t)+y^2(t)) - \lambda_4\Delta(t) - \frac{\lambda_0}{\eta}\omega(t)\Delta(t)\Big)\mathrm{e}^{-\eta^{-1}\int_{t_1}^t \omega(s)\,\mathrm{d}s}.
\end{aligned}$$

By taking $\eta = \lambda_0/\lambda_3 = 2\lambda_0/(d+1)$, we obtain

$$W'_{(1.3)} \leqslant -\lambda_5\Delta(t),$$

where $\lambda_5 = \lambda_4 \mathrm{e}^{-n/\eta}$.

The uniform asymptotic stability of the zero solution of (1.1) follows immediately. □

3. Boundedness

Our main theorem in this section is for the forced equation (1.2). The corresponding system becomes

$$(3.1)\qquad \begin{cases} x'(t) = y(t), \\ y'(t) = z(t), \\ (z(t) + \Omega(z(t-r)))' = -\Psi(x(t))z(t) - \Phi(x(t))y(t) - h(x(t)) \\ \qquad\qquad + \displaystyle\int_{t-\sigma}^{t} h'(x(s))y(s)\,\mathrm{d}s + e(t). \end{cases}$$

Theorem 3.1. *In addition to the conditions of Theorem* 2.1, *assume there exists a positive constant* e_1 *such that*

(ix) $\int_{t_0}^{t} |e(s)|\,\mathrm{d}s < e_1$ *for all* $t \geqslant t_0$.

Then there exists a positive constant N *such that any solution of* (3.1) *satisfies*

$$(3.2)\qquad |x(t)| \leqslant N, \quad |y(t)| \leqslant N, \quad \text{and} \quad |Z(t)| \leqslant N.$$

Proof. On differentiating (2.1) along the solutions of system (3.1), we obtain

$$W'_{(3.1)} \leqslant -\lambda_5\Delta(t) + |e(t)|(d|x(t)| + d|y(t)| + |Z(t)|)\mathrm{e}^{-\eta^{-1}\int_{t_1}^{t}\omega(s)\,\mathrm{d}s}.$$

Now, from (2.5) and applying the inequality $|u| \leqslant u^2 + 1$, we find that

$$W'_{(3.1)} \leqslant -\lambda_5\Delta(t) + \lambda_6|e(t)|(\Delta(t) + 3),$$

where $\lambda_6 = \max\{d, 1\}$. In view of (2.3), the above estimates imply that

$$(3.3)\qquad W'_{(3.1)} \leqslant -\lambda_5\Delta(t) + \frac{\lambda_6}{\lambda_1}|e(t)|W(t) + 3\lambda_6|e(t)|.$$

An integration of (3.3) from t_1 to t gives

$$W(t) \leqslant e_2 + \frac{\lambda_6}{\lambda_1}\int_{t_1}^{t} W(s)|e(s)|\,\mathrm{d}s$$

for some positive constant e_2. An application of Gronwall's inequality shows that $W(t)$ is bounded, and the conclusion of the theorem follows immediately. □

4. Square integrability

Our next result concerns the square integrability of the solutions of equation (1.2).

Theorem 4.1. *If all the conditions of Theorem* 3.1 *are satisfied, then for a solution x of* (1.2)

$$\int_{t_0}^{\infty} \Gamma(s)\,\mathrm{d}s < \infty.$$

P r o o f. From (2.8), we have

$$\begin{aligned} V'_{(1.3)} &\leqslant \lambda_3\omega(t)(x^2(t)+y^2(t)) - \alpha_1 x^2(t) - \alpha_2 y^2(t) - \alpha_3 z^2(t) \\ &\leqslant \lambda_3\omega(t)(x^2(t)+y^2(t)) - \beta_1\Gamma(t), \end{aligned} \tag{4.1}$$

where $\beta_1 = \min\{\alpha_1, \alpha_2, \alpha_3\}$. Therefore, from (2.1), (2.2), and (4.1), we have

$$\begin{aligned} W'_{(1.3)} &= \Big(V'_{(1.3)} - \frac{1}{\eta}\omega(t)V\Big)\mathrm{e}^{-\eta^{-1}\int_{t_1}^{t}\omega(s)\,\mathrm{d}s} \\ &\leqslant \Big(\lambda_3\omega(t)(x^2(t)+y^2(t)) - \beta_1\Gamma(t) - \frac{\lambda_0}{\eta}\omega(t)\Delta(t)\Big)\mathrm{e}^{-\eta^{-1}\int_{t_1}^{t}\omega(s)\,\mathrm{d}s}. \end{aligned}$$

Now

$$W'_{(3.1)} = W'_{(1.3)} + e(t)(dx(t) + dy(t) + Z(t))\mathrm{e}^{-\eta^{-1}\int_{t_1}^{t}\omega(s)\,\mathrm{d}s},$$

and since $\lambda_0/\eta = \lambda_3$,

$$\begin{aligned} W'_{(3.1)} &\leqslant -\beta_2\Gamma(t) + |e(t)|(d|x(t)| + d|y(t)| + |Z(t)|)\mathrm{e}^{-\eta^{-1}\int_{t_1}^{t}\omega(s)\,\mathrm{d}s} \\ &\leqslant -\beta_2\Gamma(t) + \Big(\frac{\lambda_6}{\lambda_1}W(t) + 3\lambda_6\Big)|e(t)|, \end{aligned} \tag{4.2}$$

where $\beta_2 = \beta_1\mathrm{e}^{-n\lambda_3/\lambda_0}$. Define $H(t)$ by

$$H(t) = W(t) + \tau\int_{t_1}^{t}\Gamma(s)\,\mathrm{d}s \quad \forall\, t \geqslant t_1, \tag{4.3}$$

where $\tau > 0$ is a constant to be specified later. Differentiating H and using (4.2), we obtain

$$H'(t) \leqslant (\tau - \beta_2)\Gamma(t) + \Big(\frac{\lambda_6}{\lambda_1}W(t) + 3\lambda_6\Big)|e(t)|.$$

Choosing $\tau - \beta_2 < 0$, then from the boundedness of $W(t)$,

$$H'(t) \leqslant \lambda_7|e(t)| \tag{4.4}$$

for some $\lambda_7 > 0$. Integrating (4.4) from t_1 to t, and using condition (ix), we see that $H(t)$ is bounded. In view of (4.3), this implies that

$$\int_{t_1}^{\infty} \Gamma(s)\,\mathrm{d}s$$

is bounded, which is what we wished to show. $\square$

R e m a r k 4.2. Notice that by Theorem 4.1,

$$\int_{t_1}^{\infty} (x^2(s) + y^2(s) + z^2(s))\,\mathrm{d}s < \infty$$

and consequently

$$\int_{t_1}^{\infty} Z^2(s)\,\mathrm{d}s < \infty,$$

i.e., the solutions of system (3.1) are square integrable.

We conclude this paper with an example to illustrate our results.

E x a m p l e 4.3. Consider the third order neutral delay differential equation

$$\begin{aligned}(4.5)\qquad \Big(x''(t) + \frac{1}{100}\frac{x''(t-r)}{1+\mathrm{e}^{x''(t-r)}}\Big)' + \Big(\frac{1}{10+x^2} + 3.3\Big)x'' + \Big(\frac{\cos x}{4+x^2} + 6.25\Big)x' \\ + \Big(2x(t-\tfrac{1}{10}) + \frac{x(t-\frac{1}{10})}{1+|x(t-\frac{1}{10})|}\Big) = \frac{1}{1+t^2}.\end{aligned}$$

Taking $d = 0.7$, we see that

$$\psi_0 = 2.6 = 3.3 - 0.7 \leqslant \Psi(x) - d = \frac{1}{10+x^2} + 3.3 - 0.7 \leqslant \frac{1}{10} + 3.3 - 0.7 = 2.7 = \psi_1.$$

We also have

$$\varphi_0 = 6 \leqslant \Phi(x) = \frac{\cos x}{4+x^2} + 6.25 \leqslant 6.5 = \varphi_1$$

and

$$|\Omega(x)| = \frac{1}{100}\Big|\frac{x}{1+\mathrm{e}^x}\Big| < \frac{1}{100}|x| = K|x|.$$

Now

$$h(x) = 2x + \frac{x}{1+|x|},$$

so $h(0) = 0$,

$$\frac{h(x)}{x} \geqslant 2 = \delta_1 \quad \text{for } x \neq 0, \quad \text{and} \quad |h'(x)| = \Big|2 + \frac{1}{(1+|x|)^2}\Big| \leqslant 3 = \delta_0.$$

Simple calculations show that

$$\int_{-\infty}^{\infty} |\Psi'(u)|\,\mathrm{d}u = \int_{-\infty}^{\infty} \Big|\frac{-2u}{(10+u^2)^2}\Big|\,\mathrm{d}u = 2\int_0^{\infty} \frac{2u}{(10+u^2)^2}\,\mathrm{d}u = \frac{1}{5}$$

and

$$\int_{-\infty}^{\infty} |\Phi'(u)|\,\mathrm{d}u \leqslant \int_{-\infty}^{\infty} \Big(\Big|\frac{\sin u}{4+u^2}\Big| + \Big|\frac{2u\cos u}{(4+u^2)^2}\Big|\Big)\,\mathrm{d}u \leqslant \pi.$$

Hence, conditions (i)–(iv) hold.

If we take $\varepsilon = K = \frac{1}{100}$, it is easy to see that

$$\begin{aligned}
&-d\delta_1 + \frac{1}{2}(\delta_0 K + d\psi_1) = -0.44 = -A,\\
&\delta_0 - d\varphi_0 + \frac{1}{2}(d + K(d+\varphi_1)) = -0.814 = -B,\\
&(d-\psi_0) + \frac{d}{2}(\psi_1+1) + \frac{K}{2}(3d + \varphi_1 + 2\delta_0 + 2\psi_1) + \varepsilon = -0.495 = -C,\\
&\frac{\delta_0}{\varphi_0} = \frac{3}{6} = 0.5 < 0.7 = d < \min\Big\{\frac{\psi_0}{3}, \frac{\varphi_0}{2}\Big\} = \min\Big\{\frac{2.6}{3}, 3\Big\} \approx 0.867,
\end{aligned}$$

and so (v)–(viii) hold. Clearly, $e(t) = 1/(1+t^2)$ satisfies

$$\int_0^t |e(s)|\,\mathrm{d}s < \infty \quad \forall\, t \geqslant t_0,$$

so (ix) holds. Finally, if

$$\sigma = 0.1 < \frac{2}{\delta_0}\min\Big\{\frac{A}{d}, \frac{B}{(3d+1+K)}, C, \frac{\varepsilon}{K}\Big\} \approx 0.1745,$$

then all the conditions of Theorems 2.1, 3.1, and 4.1 hold, so all solutions of equation (4.5) are bounded, x, x', and x'' are square integrable, and if $e(t) \equiv 0$, then the zero solution of (4.5) is uniformly asymptotically stable.

Acknowledgment. The authors would like to thank the referees for carefully reading the manuscript and making several suggestions for improving the paper.

References

[1] *A. T. Ademola, P. O. Arawomo*: Uniform stability and boundedness of solutions of nonlinear delay differential equations of the third order. Math. J. Okayama Univ. *55* (2013), 157–166. zbl MR

[2] *B. Baculíková, J. Džurina*: On the asymptotic behavior of a class of third order nonlinear neutral differential equations. Cent. Eur. J. Math. *8* (2010), 1091–1103. zbl MR doi

[3] *D. Beldjerd, M. Remili*: Boundedness and square integrability of solutions of certain third-order differential equations. Math. Bohem. *143* (2018), 377–389. zbl MR doi

[4] *P. Das, N. Misra*: A necessary and sufficient condition for the solutions of a functional differential equation to be oscillatory or tend to zero. J. Math. Anal. Appl. *205* (1997), 78–87. zbl MR doi

[5] *B. Dorociaková*: Some nonoscillatory properties of third order differential equations of neutral type. Tatra Mt. Math. Publ. *38* (2007), 71–76. zbl MR

[6] *Z. Došlá, P. Liška*: Comparison theorems for third-order neutral differential equations. Electron. J. Differ. Equ. *2016* (2016), Article ID 38, 13 pages. zbl MR

[7] *Z. Došlá, P. Liška*: Oscillation of third-order nonlinear neutral differential equations. Appl. Math. Lett. *56* (2016), 42–48. zbl MR doi

[8] *J. O. C. Ezeilo*: On the stability of solutions of certain differential equations of the third order. Q. J. Math., Oxf. II. Ser. *11* (1960), 64–69. zbl MR doi

[9] *R. M. Goldwyn, K. S. Narendra*: Stability of Certain Nonlinear Differential Equations Using the Second Method of Liapunov. Technical Report No. 403. Harvard University, Cambridge, 1963.

[10] *J. R. Graef, D. Beldjerd, M. Remili*: On stability, ultimate boundedness, and existence of periodic solutions of certain third order differential equations with delay. Panam. Math. J. *25* (2015), 82–94. zbl MR

[11] *J. R. Graef, L. D. Oudjedi, M. Remili*: Stability and square integrability of solutions to third order neutral delay differential equations. Tatra Mt. Math. Publ. *71* (2018), 81–97. zbl MR doi

[12] *J. K. Hale*: Theory of Functional Differential Equations. Applied Mathematical Sciences 3. Springer, New York, 1977. zbl MR

[13] *J. K. Hale, S. M. Verduyn Lunel*: Introduction to Functional Differential Equations. Applied Mathematical Sciences 99. Springer, New York, 1993. zbl MR doi

[14] *T. Hara*: Remarks on the asymptotic behavior of the solutions of certain non-autonomous differential equations. Proc. Japan Acad. *48* (1972), 549–552. zbl MR doi

[15] *T. Hara*: On the asymptotic behavior of the solutions of some third and fourth order non-autonomous differential equations. Publ. Res. Inst. Math. Sci., Kyoto Univ. *9* (1974), 649–673. zbl MR doi

[16] *M. R. S. Kulenović, G. Ladas, A. Meimaridou*: Stability of solutions of linear delay differential equations. Proc. Am. Math. Soc. *100* (1987), 433–441. zbl MR doi

[17] *T. Li, C. Zhang, G. Xing*: Oscillation of third-order neutral delay differential equations. Abstr. Appl. Anal. *2012* (2012), Article ID 569201, 11 pages. zbl MR doi

[18] *A. M. Liapounoff*: Problème général de la stabilité du mouvement. Annals of Mathematics Studies 17. Princeton University Press, Princeton, 1947. (In French.) zbl MR doi

[19] *B. Mihalíková, E. Kostiková*: Boundedness and oscillation of third order neutral differential equations. Tatra Mt. Math. Publ. *43* (2009), 137–144. zbl MR doi

[20] *M. Nakashima*: Asymptotic behavior of the solutions of some third order differential equations. Rep. Fac. Sci., Kagoshima Univ. *4* (1971), 7–15. MR

[21] *M. O. Omeike*: New results on the asymptotic behavior of a third-order nonlinear differential equation. Differ. Equ. Appl. *2* (2010), 39–51. zbl MR doi

[22] *M. O. Omeike*: New results on the stability of solution of some non-autonomous delay differential equations of the third order. Differ. Uravn. Protsessy Upr. *1* (2010), 18–29. zbl MR

[23] *C. Qian*: On global stability of third-order nonlinear differential equations. Nonlinear Anal., Theory Methods Appl., Ser. A *42* (2000), 651–661. zbl MR doi

[24] *C. Qian*: Asymptotic behavior of a third-order nonlinear differential equation. J. Math. Anal. Appl. *284* (2003), 191–205. zbl MR doi

[25] *M. Remili, D. Beldjerd*: On the asymptotic behavior of the solutions of third order delay differential equations. Rend. Circ. Mat. Palermo (2) *63* (2014), 447–455. zbl MR doi

[26] *M. Remili, D. Beldjerd*: A boundedness and stability results for a kind of third order delay differential equations. Appl. Appl. Math. *10* (2015), 772–782. zbl MR

[27] *M. Remili, D. Beldjerd*: On ultimate boundedness and existence of periodic solutions of kind of third order delay differential equations. Acta Univ. M. Belii, Ser. Math. *24* (2016), 43–57. zbl MR

[28] *M. Remili, D. Beldjerd*: Stability and ultimate boundedness of solutions of some third order differential equations with delay. J. Assoc. Arab Univers. Basic Appl. Sci. *23* (2017), 90–95. doi

[29] *S. N. Šimanov*: On stability of solution of a nonlinear equation of the third order. Prikl. Mat. Mekh. *17* (1953), 369–372. (In Russian.) zbl MR

[30] *K. E. Swick*: Asymptotic behavior of the solutions of certain third order differential equations. SIAM J. Appl. Math. *19* (1970), 96–102. zbl MR doi

[31] *Y. Tian, Y. Cai, Y. Fu, T. Li*: Oscillation and asymptotic behavior of third-order neutral differential equations with distributed deviating arguments. Adv. Difference Equ. *2015* (2015), Article ID 267, 14 pages. zbl MR doi

[32] *C. Tunç*: On the stability and boundedness of solutions of nonlinear third order differential equations with delay. Filomat *24* (2010), 1–10. zbl MR doi

[33] *C. Tunç*: Some stability and boundedness conditions for non-autonomous differential equations with deviating arguments. Electron. J. Qual. Theory Differ. Equ. *2010* (2010), Article ID 1, 12 pages. zbl MR doi

[34] *J. Yu*: Asymptotic stability for a class of nonautonomous neutral differential equations. Chin. Ann. Math., Ser. B *18* (1997), 449–456. zbl MR

[35] *J. Yu, Z. Wang, C. Qian*: Oscillation of neutral delay differential equation. Bull. Aust. Math. Soc. *45* (1992), 195–200. zbl MR doi

[36] *L. Zhang, L. Si*: Global asymptotic stability of a class of third order nonlinear system. Acta Math. Appl. Sin. *30* (2007), 99–103. (In Chinese.) zbl MR

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