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WEAK n-INJECTIVE AND WEAK n-FAT MODULES

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Abstract. We introduce and study the concepts of weak n-injective and weak n-flat modules in terms of super finitely presented modules whose projective dimension is at most n, which generalize the n-FP-injective and n-flat modules. We show that the class of all weak n-injective R-modules is injectively resolving, whereas that of weak n-flat right R-modules is projectively resolving and the class of weak n-injective (or weak n-flat) modules together with its left (or right) orthogonal class forms a hereditary (or perfect hereditary) cotorsion theory.

Keywords: weak injective module; weak flat module; weak n-injective module; weak n-flat module; cotorsion theory

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1. Introduction

Throughout this paper, R denotes an associative ring with identity and an R-module will always mean a left R-module unless stated otherwise. For any R-module N, $\mathrm{pd}_R N$ denotes the projective dimension of N. Denote by R-Mod (or Mod-R), the category of left (or right) R-modules and we use M^+ to denote the character module $\mathrm{Hom}_{\mathbb{Z}}(M, \mathbb{Q}/\mathbb{Z})$.

In 1970s, Stenström in [9] extended the injective modules to FP-injective modules and characterized coherent rings in terms of FP-injective modules. In 2002, Lee in [7] introduced and studied the notions of n-flat and n-FP-injective (or n-absolutely pure) modules. Further, Yang and Liu [10] studied these modules well and gave some nice

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characterizations of n-coherent rings in terms of n-flat and n-FP-injective modules. Recently in [6], Gao et al. introduced and studied the notions of weak injective and weak flat modules, which are generalizations of FP-injective and flat modules, respectively. Inspired by the above works, we introduce and study the notions of weak n-injective and weak n-flat modules, which are generalizations of n-FP-injective and n-flat modules, respectively. This paper is organized as follows.

In Section 2, we present some known definitions and terminologies in order to use them in the following sections.

In Section 3, we investigate some homological properties of weak injective and weak flat modules. Further, we prove that the class of all weak injective R-modules is injectively resolving, whereas that of weak flat right R-modules is projectively resolving, and we show that the class of all weak injective (or weak flat) modules together with its left (or right) orthogonal class forms a hereditary (or perfect hereditary) cotorsion theory.

In the last section, we introduce and study the notions of weak n-injective and weak n-flat modules, which are the generalizations of n-FP-injective and n-flat modules, respectively. We see that the principal results on weak injective and weak flat modules remain true for weak n-injective and weak n-flat modules. Further, we show that the class of all weak n-injective (or weak n-flat) left (or right) R-modules is closed under direct sums, direct products, direct limits, pure submodules and it is an injectively (or a projectively) resolving class.

2. Preliminaries

In this section, we first recall some known definitions and terminologies which will be needed in the sequel.

Let $\mathcal{C}$ be a class of R-modules in R-Mod. By a left orthogonal class of $\mathcal{C}$ (denoted by ${}^{\perp}\mathcal{C}$) we mean the class of all R-modules F such that $\mathrm{Ext}^1(F, C) = 0$ for every $C \in \mathcal{C}$ and by a right orthogonal class of $\mathcal{C}$ (denoted by $\mathcal{C}^{\perp}$) we mean the class of all R-modules F such that $\mathrm{Ext}^1(C, F) = 0$ for every $C \in \mathcal{C}$. A pair of classes of R-modules $(\mathcal{F}, \mathcal{C})$ is called a *cotorsion theory* if $\mathcal{F}^{\perp} = \mathcal{C}$ and ${}^{\perp}\mathcal{C} = \mathcal{F}$. A cotorsion theory $(\mathcal{F}, \mathcal{C})$ is said to be complete if for every R-module A there is an exact sequence $0 \to C \to F \to A \to 0$ such that $F \in \mathcal{F}$ and $C \in \mathcal{C}$. A cotorsion theory $(\mathcal{F}, \mathcal{C})$ is said to be hereditary if the following statement holds: if $0 \to F' \to F \to F'' \to 0$ is exact with $F, F'' \in \mathcal{F}$, then $F' \in \mathcal{F}$.

Let $\mathcal{C}$ be a class of R-modules in R-Mod. We say that $\mathcal{C}$ is projectively resolving if it contains all projective modules and for any short exact sequence $0 \to A \to B \to C \to 0$ with $C \in \mathcal{C}$, the conditions $A \in \mathcal{C}$ and $B \in \mathcal{C}$ are equivalent. Also we say that $\mathcal{C}$ is injectively resolving if it contains all injective R-modules and for any short exact sequence $0 \to A \to B \to C \to 0$ with $A \in \mathcal{C}$, the conditions $B \in \mathcal{C}$ and $C \in \mathcal{C}$ are equivalent.

Recall that an R-module M is called *FP-injective* if $\mathrm{Ext}^1_R(N, M) = 0$ for every finitely presented R-module N.

Definition 2.1 ([7]). Let n be a nonnegative integer. An R-module M is called *n-FP-injective* if $\mathrm{Ext}^1_R(N, M) = 0$ for every finitely presented R-module N with $\mathrm{pd}_R N \leqslant n$. A right R-module M is called *n-flat* if $\mathrm{Tor}^R_1(M, N) = 0$ for every finitely presented left R-module N with $\mathrm{pd}_R N \leqslant n$.

Definition 2.2 ([5]). An R-module M is said to be *super finitely* presented if there exists an exact sequence of R-modules

$$\ldots \to P_n \to \ldots \to P_1 \to P_0 \to M \to 0,$$

where each P_i is finitely generated and projective.

Remark 2.1. (See [1].) Note that every super finitely presented R-module is finitely presented but the converse is not true in general. Also, every finitely presented R-module is super finitely presented if and only if R is a coherent ring.

Definition 2.3 ([8], Definition 6.3.1). A cotorsion pair $(\mathcal{A}, \mathcal{B})$ in an Abelian category $\mathcal{C}$ is said to be *cogenerated* by a set of objects $\mathcal{S} \subseteq \mathcal{A}$ if $\mathcal{B} = \mathcal{S}^{\perp}$.

Theorem 2.1 ([8], Theorem 6.3.8, (Eklof and Trlifaj Theorem)). *Let $(\mathcal{A}, \mathcal{B})$ be a cotorsion pair in a Grothendieck categoty $\mathcal{C}$ with (functorially) enough projectives. If $(\mathcal{A}, \mathcal{B})$ is cogenerated by a set $\mathcal{S} \subseteq \mathcal{A}$, then $(\mathcal{A}, \mathcal{B})$ is (functorially) right complete.*

3. Weak injective and weak flat modules

In this section, we investigate some homological properties of weak injective and weak flat modules, and we show that the class of all weak injective (or weak flat) modules together with its left (or right) orthogonal class forms a hereditary (or perfect hereditary) cotorsion theory.

Definition 3.1 ([6]). An R-module M is called *weak injective* if $\mathrm{Ext}^1_R(N, M) = 0$ for every super finitely presented R-module N. A right R-module M is called *weak flat* if $\mathrm{Tor}^R_1(M, N) = 0$ for every super finitely presented R-module N.

Remark 3.1 ([6]).

(1) It is clear that every FP-injective left R-module is weak injective and every flat right R-module is weak flat. Further, over a left coherent ring R, the class of all

weak injective modules coincides with that of FP-injective modules as well as the class of all weak flat right R-modules coincides with that of flat right R-modules.

(2) Note that the class of all weak injective R-modules is closed under direct limits by [6], Proposition 2.6. The fact that the class of all weak flat right R-modules is closed under direct limits follows from the standard isomorphism:

$$\mathrm{Tor}_1^R(\varinjlim M_i, N) \cong \varinjlim \mathrm{Tor}_1^R(M_i, N)$$

for any left R-module N.

The following results have been obtained from [4], Proposition 2.6.

Proposition 3.1. *Let $0 \to M_1 \to M_2 \to M_3 \to 0$ be an exact sequence of R-modules. Then the following hold.*

(1) *If M_1 and M_2 are weak injective, then M_3 is weak injective.*

(2) *If M_1 and M_3 are weak injective, then M_2 is weak injective.*

Proposition 3.2. *The class of all weak injective R-modules is injectively resolving and closed under direct summands.*

P r o o f. Since every FP-injective R-module is weak injective by Remark 3.1, the class of all weak injective modules contains all injective R-modules. Hence, the class of all weak injective R-modules is injectively resolving by Proposition 3.1 and it is closed under direct summands by the additivity of the bifunctor $\mathrm{Ext}_R^1(\cdot,\cdot)$. $\square$

The following results have been obtained from [4], Proposition 2.6.

Proposition 3.3. *Let $0 \to M_1 \to M_2 \to M_3 \to 0$ be an exact sequence of right R-modules. Then the following hold.*

(1) *If M_2 and M_3 are weak flat, then so is M_1.*

(2) *If M_1 and M_3 are weak flat, then so is M_2.*

Proposition 3.4. *The class of all weak flat right R-modules is projectively resolving and closed under direct summands.*

P r o o f. Since every flat right R-module is weak flat by Remark 3.1, the class of all weak flat right R-modules is projectively resolving by Proposition 3.3. On the other hand, it is closed under direct summands by the additivity of the bifunctor $\mathrm{Tor}_1^R(\cdot,\cdot)$. $\square$

We denote by $\mathcal{WF}$ (or $\mathcal{WI}$) the class of all weak flat (or weak injective) right (or left) R-modules.

Proposition 3.5. *The following statements are equivalent for any ring R.*

(1) *Every weak flat right R-module is flat.*

(2) *Every cotorsion right R-module belongs to $\mathcal{WF}^{\perp}$.*

(3) *Every weak injective left R-module is FP-injective.*

(4) *Every finitely presented left R-module belongs to ${}^{\perp}\mathcal{WI}$.*

Proof. (1) $\Leftrightarrow$ (2) Suppose every weak flat right R-module is flat. Let C be a cotorsion right R-module. Then $\mathrm{Ext}^1(F,C)=0$ for all $F\in\mathcal{F}$, where $\mathcal{F}$ is the class of all flat right R-modules, and hence $\mathrm{Ext}^1(F,C)=0$ for all $F\in\mathcal{WF}$. Thus, $C\in\mathcal{WF}^{\perp}$. Conversely, we suppose that $\mathcal{F}^{\perp}\subset\mathcal{WF}^{\perp}$. Notice that the last inclusion is an equality since $\mathcal{F}\subset\mathcal{WF}$.

(1) $\Rightarrow$ (3) Let M be any weak injective left R-module. Then M^{+} is weak flat by [6], Proposition 2.10, and so M^{+} is flat by (1). On the other hand, for any finitely presented R-module N, there is an exact sequence

$$\mathrm{Tor}_1^R(M^{+},N)\to\mathrm{Ext}^1_R(N,M)^{+}\to 0$$

by [2], Lemma 2.7 (1). Thus, $\mathrm{Ext}^1_R(N,M)=0$ and hence M is FP-injective.

(3) $\Rightarrow$ (1) is clear from the notion of orthogonal classes.

(3) $\Leftrightarrow$ (4) We denote the class of all FP-injective left R-modules by $\mathcal{FP}$. If $\mathcal{WI}\subset\mathcal{FP}$, then we obtain that every finitely presented R-module is in ${}^{\perp}\mathcal{WI}$. Conversely, if every finitely presented R-module belongs to ${}^{\perp}\mathcal{WI}$, then by the definition of FP-injective R-module, we deduce that $\mathcal{WI}\subset\mathcal{FP}$. $\square$

Proposition 3.6. *The following statements hold for any ring R.*

(1) *If M is a weak injective R-module and A is any pure submodule of M, then M/A is weak injective.*

(2) *If M is a weak flat right R-module and A is any pure submodule of M, then M/A is weak flat.*

Proof. (1) Since the class of weak injective modules is closed under pure quotients, the result can be obtained from [6], Proposition 2.9 (2) and [4], Proposition 2.6 (1).

(2) is trivial. $\square$

Theorem 3.1. *The following statements are true for any ring R.*

(1) *$({}^{\perp}\mathcal{WI},\mathcal{WI})$ is a hereditary cotorsion theory.*

(2) *$(\mathcal{WF},\mathcal{WF}^{\perp})$ is a perfect hereditary cotorsion theory.*

P r o o f. (1) It follows directly from the definition of cotorsion theory and the fact that $\mathcal{WI}$ is the right orthogonal class of the super finitely presented modules.

(2) Let $\mathrm{Card}(R) \leqslant \aleph_\beta$ and $F \in \mathcal{WF}$. Then we can write F as a union of a continuous chain $(F_\alpha)_{\alpha<\lambda}$ of pure submodules of F such that $\mathrm{Card}(F_0) \leqslant \aleph_\beta$ and $\mathrm{Card}(F_{\alpha+1}/F_\alpha) \leqslant \aleph_\beta$ whenever $\alpha + 1 < \lambda$. If N is an R-module such that $\mathrm{Ext}^1(F_0, N) = 0$ and $\mathrm{Ext}^1(F_{\alpha+1}/F_\alpha, N) = 0$ whenever $\alpha + 1 < \lambda$, then $\mathrm{Ext}^1(F, N) = 0$ by [3], Theorem 7.3.4. Since F_α is a pure submodule of F for any $\alpha < \lambda$, we have $F_\alpha \in \mathcal{WF}$ by [6], Proposition 2.9. Further, F_α is a pure submodule of $F_{\alpha+1}$ whenever $\alpha + 1 < \lambda$, and so $F_{\alpha+1}/F_\alpha \in \mathcal{WF}$ by Proposition 3.6. Let X be a set of representatives of all modules $C \in \mathcal{WF}$ with $\mathrm{Card}(C) \leqslant \aleph_\beta$. Then $X^\perp = \mathcal{WF}^\perp$. Hence, $(\mathcal{WF}, \mathcal{WF}^\perp)$ is a cotorsion theory and it is cogenerated by the set X; see Definition 2.3. Therefore, by Theorem 2.1, $(\mathcal{WF}, \mathcal{WF}^\perp)$ is a complete cotorsion theory. Since $\mathcal{WF}$ is closed under direct limits by Remark 3.1, the cotorsion pair $(\mathcal{WF}, \mathcal{WF}^\perp)$ is perfect by [3], Theorem 7.2.6. On the other hand, $(\mathcal{WF}, \mathcal{WF}^\perp)$ is hereditary by Proposition 3.3. $\square$

4. Weak n-injective and weak n-flat modules

In this section, we introduce the notions of weak n-injective and weak n-flat modules, which are generalizations of n-FP-injective and n-flat modules. We see that the principal results on weak injective and weak flat modules remain true for weak n-injective and weak n-flat modules.

Definition 4.1. Let n be a nonnegative integer. An R-module M is called *weak n-injective* if $\mathrm{Ext}^1_R(N, M) = 0$ for any super finitely presented R-module N with $\mathrm{pd}_R N \leqslant n$. A right R-module M is called *weak n-flat* if $\mathrm{Tor}_1^R(M, N) = 0$ for any super finitely presented R-module N with $\mathrm{pd}_R N \leqslant n$.

Remark 4.1.

(1) Because every super finitely presented R-module whose projective dimension at most n is a finitely presented R-module with projective dimension at most n, every n-FP-injective (or n-flat right) R-module is weak n-injective (weak n-flat).
(2) Over a left coherent ring R, the class of all super finitely presented R-modules coincides with that of finitely presented R-modules and so the class of all weak n-injective (or weak n-flat right) R-modules coincides with that of n-FP-injective (or n-flat right) R-modules.

From the definitions of weak n-injective and weak n-flat modules, we immediately get the following proposition.

Proposition 4.1. *Let R be any ring and $\{M_i\}_{i\in I}$ be any family of R-modules. Then the following statements hold:*

(1) *$\prod M_i$ is weak n-injective if and only if each M_i is weak n-injective.*

(2) *$\bigoplus M_i$ is a weak n-flat right R-module if and only if each M_i is a weak n-flat right R-module.*

(3) *$\bigoplus M_i$ is weak n-injective if and only if each M_i is weak n-injective.*

Proof. (1) and (2) follows from the standard isomorphisms

$$\mathrm{Ext}_R^1\Big(N, \prod M_i\Big) \cong \prod \mathrm{Ext}_R^1(N, M_i) \quad \text{and} \quad \mathrm{Tor}_1^R\Big(\bigoplus M_i, N\Big) \cong \bigoplus \mathrm{Tor}_1^R(M_i, N),$$

respectively.

(3) Since the isomorphism

$$\mathrm{Ext}_R^1\Big(N, \bigoplus M_i\Big) \cong \bigoplus \mathrm{Ext}_R^1(N, M_i)$$

is true for every finitely presented R-module N, the result follows. □

Proposition 4.2. *Let n be a nonnegative integer and R be any ring. Then:*

(1) *The class of all weak n-injective R-modules is closed under direct limits.*

(2) *The class of all weak n-flat R-modules is closed under direct limits.*

Proof. (1) Let $\{M_i\}_{i\in I}$ be a family of weak n-injective R-modules and N be any super finitely presented R-module. Then by [6], Lemma 2.4, we have the following isomorphism:

$$\mathrm{Ext}_R^1(N, \varinjlim M_i) \cong \varinjlim \mathrm{Ext}_R^1(N, M_i).$$

Since M_i is weak n-injective for every i, we have $\mathrm{Ext}_R^1(N, M_i) = 0$ for all super finitely presented R-modules N with $\mathrm{pd}_R N \leqslant n$. Hence, $\{M_i\}_{i\in I}$ is closed under direct limits.

(2) It follows from the isomorphism

$$\mathrm{Tor}_1^R(\varinjlim M_i, N) \cong \varinjlim \mathrm{Tor}_1^R(M_i, N)$$

for any family of right R-modules M_i and any left R-module N. □

The relationship between weak n-injective and weak n-flat modules is the following:

Proposition 4.3. *A right R-module M is weak n-flat if and only if M^+ is a weak n-injective left R-module.*

Proof. It follows from the standard isomorphism $\mathrm{Ext}_R^1(N, M^+) \cong \mathrm{Tor}_1^R(M, N)^+$ for any left R-module N. □

Proposition 4.4. *Let R be any ring. Then the following statements hold:*

(1) *The class of all weak n-injective R-modules is closed under pure submodules.*

(2) *The class of all weak n-flat right R-modules is closed under pure submodules.*

Proof. (1) Let M be a weak n-injective R-module and M_1 be any pure submodule of M. Then there exists a short exact sequence

$$0 \to M_1 \to M \to M/M_1 \to 0,$$

which induces the exact sequence:

$$\mathrm{Hom}_R(N, M) \to \mathrm{Hom}_R(N, M/M_1) \to \mathrm{Ext}^1_R(N, M_1) \to 0$$

for any super finitely presented R-module N with $\mathrm{pd}_R N \leqslant n$. Since M_1 is a pure submodule of M, we have the following exact sequence:

$$\mathrm{Hom}_R(N, M) \to \mathrm{Hom}_R(N, M/M_1) \to 0.$$

Hence, $\mathrm{Ext}^1_R(N, M_1) = 0$ and hence M_1 is weak n-injective.

(2) Let A be any pure submodule of a weak n-flat right R-module M. Then there exists a pure exact sequence

$$0 \to A \to M \to M/A \to 0,$$

which induces a split exact sequence

$$0 \to (M/A)^+ \to M^+ \to A^+ \to 0.$$

By Proposition 4.3, M^+ is a weak n-injective R-module. Since A^+ is isomorphic to a direct summand of M^+, A^+ is weak n-injective by Proposition 4.1 (2). Therefore, A is weak n-flat by Proposition 4.3. $\square$

Proposition 4.5. *An R-module M is weak n-injective if and only if M^+ is a weak n-flat right R-module.*

Proof. The proof is similar to that of [6], Theorem 2.10. $\square$

Proposition 4.6.

(1) *An R-module M is weak n-injective if and only if M^{++} is weak n-injective.*

(2) *A right R-module M is weak n-flat if and only if M^{++} is weak n-flat.*

Proof. It follows from Propositions 4.3 and 4.5. $\square$

In [7], Theorem 5, it is proved that the direct product of n-flat right R-modules is n-flat if and only if R is a left n-coherent ring. Over an arbitrary ring R, we have the following proposition for weak n-flat right R-modules.

Proposition 4.7. *Let R be any ring. Then the following statements hold:*

(1) *Any direct product of weak n-flat right R-modules is weak n-flat.*

(2) *Any direct product of copies of R is a weak n-flat right R-module.*

P r o o f. The proof is similar to that of [6], Theorem 2.13. $\square$

Proposition 4.8. *Let R be any ring. Then every right R-module has a weak n-flat preenvelope.*

P r o o f. Let M be any right R-module. By [3], Lemma 5.3.12, there is an infinite cardinal $\aleph_\alpha$ such that for any R-homomorphism $f\colon M \to L$ with L weak n-flat, there is a pure submodule K of L such that $\mathrm{Card}(K) \leqslant \aleph_\alpha$ and $f(M) \subseteq K$. Since K is weak n-flat by Proposition 4.4, M has a weak n-flat preenvelope by Proposition 6.2.1 of [3] and Proposition 4.7. $\square$

Proposition 4.9. *The following statements are equivalent for a ring R.*

(1) *R is weak n-injective as a left R-module.*

(2) *Every right R-module has a monic weak n-flat preenvelope.*

(3) *Every injective right R-module is weak n-flat.*

(4) *Every flat left R-module is weak n-injective.*

P r o o f. (1) $\Rightarrow$ (2) Let M be a right R-module. By Proposition 4.8, M has a weak n-flat preenvelope $f\colon M \to L$. Since $({}_RR)^+$ is a cogenerator in the category of right R-modules, there is an exact sequence $0 \to M \to \prod({}_RR)^+$. Since ${}_RR$ is weak n-injective left R-module by assumption, we have that $({}_RR)^+$ is a weak n-flat right R-module by Proposition 4.5. It follows that $\prod({}_RR)^+$ is weak n-flat by Proposition 4.6. Thus, f is monic and hence M has a monic weak n-flat preenvelope.

(2) $\Rightarrow$ (3) Let I be any injective right R-module. By (2), there exists an exact sequence $0 \to I \to F \to N \to 0$, where $I \to F$ is a weak n-flat preenvelope with F weak n-flat. Since I is injective, the short exact sequence splits and hence I is a direct summand of F. Therefore, I is weak n-flat by Proposition 4.1.

(3) $\Rightarrow$ (4) Let M be a flat left R-module. Then M^+ is an injective right R-module. Thus M^+ is weak n-flat by hypothesis. This implies that M is weak n-injective by Proposition 4.5.

(4) $\Rightarrow$ (1) Since ${}_RR$ is flat, ${}_RR$ is weak n-injective by hypothesis. $\square$

Proposition 4.10.

(1) *An R-module M is weak n-injective if and only if $\mathrm{Ext}_R^i(N, M) = 0$ for all $i \geqslant 1$ and for any super finitely presented R-module N with $\mathrm{pd}_R N \leqslant n$.*

(2) *A right R-module M is weak n-flat if and only if* $\operatorname{Tor}_i^R(M,N)=0$ *for all* $i\geqslant 1$ *and for any super finitely presented R-module N with* $\operatorname{pd}_R N\leqslant n$.

Proof. The proof is similar to that of [6], Proposition 3.1. □

The following proposition is a direct consequence of Proposition 4.10.

Proposition 4.11. *Let $0\to M_1\to M_2\to M_3\to 0$ be an exact sequence of R-modules. Then the following statements hold.*

(1) *If M_1 and M_2 are weak n-injective, then M_3 is weak n-injective.*

(2) *If M_1 and M_3 are weak n-injective, then M_2 is weak n-injective.*

Proposition 4.12. *The class of all weak n-injective R-modules is injectively resolving and closed under direct summands.*

Proof. If we denote by $\mathcal{A}$ the class of super finitely presented modules with projective dimension at most n, then we have that $\mathcal{WI}_n$, the class of weak n-injective R-modules, is equal to

$$\mathcal{A}^{\perp_\infty}:=\{N\in R\text{-Mod}\colon \operatorname{Ext}^i(A,N)=0 \text{ for every } A\in\mathcal{A} \text{ and every } i>0\}.$$

Such orthogonal classes are always injectively resolving and closed under direct summands. □

Proposition 4.13. *Let $0\to M_1\to M_2\to M_3\to 0$ be an exact sequence of right R-modules. Then the following statements hold.*

(1) *If M_2 and M_3 are weak n-flat, then so is M_1.*

(2) *If M_1 and M_3 are weak n-flat, then so is M_2.*

Proof. (1) Suppose $0\to M_1\to M_2\to M_3\to 0$ is an exact sequence of right R-modules. Then for any left R-module N we have the following long exact sequence:

$$\ldots\to\operatorname{Tor}_2^R(M_3,N)\to\operatorname{Tor}_1^R(M_1,N)\to\operatorname{Tor}_1^R(M_2,N)\to\operatorname{Tor}_1^R(M_3,N)\to\ldots.$$

Since M_2 and M_3 are weak n-flat right R-modules, by Proposition 4.10 (2), we have that $\operatorname{Tor}_i^R(M_2,N)=0=\operatorname{Tor}_i^R(M_3,N)$ for any $i\geqslant 1$ and any super finitely presented R-module N with $\operatorname{pd}_R N\leqslant n$. This implies that $\operatorname{Tor}_1^R(M_1,N)=0$ for any super finitely presented R-module N with $\operatorname{pd}_R N\leqslant n$. Therefore, M_1 is a weak n-flat R-module.

(2) is trivial. □

Proposition 4.14. *The class of all weak n-flat right R-modules is projectively resolving and closed under direct summands.*

P r o o f. Since every flat right R-module is weak n-flat by Remark 4.1, the class of weak n-flat right R-modules is projectively resolving by Proposition 4.13. On the other hand, it is closed under direct summands by the additivity of the bifunctor $\operatorname{Tor}_1^R(\cdot,\cdot)$. □

We denote by $\mathcal{WF}_n$ (or $\mathcal{WI}_n$) the class of all weak n-flat (or weak n-injective) right (or left) R-modules. Recall from [3] that an R-module M is called *cotorsion* if $\operatorname{Ext}^1(F,M)=0$ for all flat R-modules F.

Proposition 4.15. *The following statements are equivalent for any ring R.*

(1) *Every weak n-flat right R-module is flat.*

(2) *Every cotorsion right R-module belongs to $\mathcal{WF}_n^{\perp}$.*

(3) *Every weak n-injective left R-module is FP-injective.*

(4) *Every finitely presented left R-module belongs to ${}^{\perp}\mathcal{WI}_n$.*

P r o o f. The proof is similar to that of Proposition 3.5. □

Proposition 4.16. *The following statements hold for any ring R.*

(1) *If M is a weak n-injective R-module and A is any pure submodule of M, then M/A is weak n-injective.*

(2) *If M is a weak n-flat right R-module and A is any pure submodule of M, then M/A is weak n-flat.*

P r o o f. The proof is similar to that of Proposition 3.6. □

In [10], Theorem 2.1, it is proved that the class of all n-FP-injective modules (or n-flat modules) together with its left (or right) orthogonal class forms a complete cotorsion (or perfect cotorsion) theory. For the classes of weak n-injective and weak n-flat modules we have the following theorem.

Theorem 4.1. *The following statements hold for any ring R.*

(1) $({}^{\perp}\mathcal{WI}_n,\mathcal{WI}_n)$ *is a hereditary cotorsion theory.*

(2) $(\mathcal{WF}_n,\mathcal{WF}_n^{\perp})$ *is a perfect hereditary cotorsion theory.*

P r o o f. (1) It is clear that $\mathcal{WI}_n\subseteq({}^{\perp}\mathcal{WI}_n)^{\perp}$. Now let $X\in({}^{\perp}\mathcal{WI}_n)^{\perp}$ and N be any super finitely presented R-module with $\operatorname{pd}_R N\leqslant n$. Then $N\in{}^{\perp}\mathcal{WI}_n$. This gives that $\operatorname{Ext}_R^1(N,X)=0$, and hence $X\in\mathcal{WI}_n$. Thus, $({}^{\perp}\mathcal{WI}_n,\mathcal{WI}_n)$ is a cotorsion theory. On the other hand, consider the short exact sequence

$$0\to A\to B\to C\to 0$$

with B and C in ${}^{\perp}\mathcal{WI}_n$. Then for any R-module M we have the following long exact sequence:

$$\ldots\to\operatorname{Ext}_R^1(C,M)\to\operatorname{Ext}_R^1(B,M)\to\operatorname{Ext}_R^1(A,M)\to\operatorname{Ext}_R^2(C,M)\to\ldots.$$

Since $\mathrm{Ext}_R^i(C,M) = 0 = \mathrm{Ext}_R^i(B,M)$ for all $i \geqslant 1$ and for all weak n-injective R-modules M by Proposition 4.10, we have $\mathrm{Ext}_R^1(A,M) = 0$ for all weak n-injective R-modules M. This gives that $A \in {}^{\perp}\mathcal{WI}_n$. Therefore, the cotorsion pair $({}^{\perp}\mathcal{WI}_n, \mathcal{WI}_n)$ is hereditary.

(2) The proof is similar to that of Theorem 3.1 (2). □

Proposition 4.17. *The following conditions are equivalent:*

(1) *every left R-module is weak n-injective;*

(2) *every cotorsion left R-module is weak n-injective;*

(3) *every right R-module is weak n-flat;*

(4) *every cotorsion right R-module is weak n-flat;*

(5) *every right R-module in $\mathcal{WF}_n^{\perp}$ is injective;*

(6) *every left R-module in ${}^{\perp}\mathcal{WI}_n$ is projective.*

Proof. (1) $\Rightarrow$ (2) and (3) $\Rightarrow$ (4) are obvious.

(2) $\Rightarrow$ (3) Let M be any right R-module. Then M^+ is a cotorsion left R-module by [3], Lemma 5.3.23. Thus, M^+ is weak n-injective by (2). Therefore, M is weak n-flat by Proposition 4.3.

(4) $\Rightarrow$ (1) Let M be a left R-module. Then M^+ is a cotorsion right R-module by [3], Lemma 5.3.23 and so M^+ is weak n-flat by (4). Hence, M is weak n-injective by Proposition 4.5.

(3) $\Leftrightarrow$ (5) and (1) $\Leftrightarrow$ (6) follow from Theorem 4.1 □

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References

[1] *D. Bravo, J. Gillespie, M. Hovey*: The stable module category of a general ring. Available at `https://arxiv.org/abs/1405.5768` (2014), 38 pages.

[2] *J. Chen, N. Ding*: On n-coherent rings. Commun. Algebra *24* (1996), 3211–3216. zbl MR doi

[3] *E. E. Enochs, O. M. G. Jenda*: Relative Homological Algebra. De Gruyter Expositions in Mathematics 30. Walter De Gruyter, Berlin, 2000. zbl MR doi

[4] *Z. Gao, Z. Huang*: Weak injective covers and dimension of modules. Acta Math. Hung. *147* (2015), 135–157. zbl MR doi

[5] *Z. Gao, F. Wang*: All Gorenstein hereditary rings are coherent. J. Algebra Appl. *13* (2014), Article ID 1350140, 5 pages. zbl MR doi

[6] *Z. Gao, F. Wang*: Weak injective and weak flat modules. Commun. Algebra *43* (2015), 3857–3868. zbl MR doi

[7] *S. B. Lee*: n-coherent rings. Commun. Algebra *30* (2002), 1119–1126. zbl MR doi

[8] *M. A. Pérez*: Introduction to Abelian Model Structures and Gorenstein Homological Dimensions. Monographs and Research Notes in Mathematics. CRC Press, Boca Raton, 2016. zbl MR doi

[9] *B. Stenström*: Coherent rings and FP-injective modules. J. Lond. Math. Soc., II. Ser. *2* (1970), 323–329. zbl MR doi
[10] *X. Yang, Z. Liu*: n-flat and n-FP-injective modules. Czech. Math. J. *61* (2011), 359–369. zbl MR doi

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