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BICROSSED PRODUCTS OF GENERALIZED TAFT ALGEBRA AND GROUP ALGEBRAS

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Abstract. Let G be a group generated by a set of finite order elements. We prove that any bicrossed product $H_{m,d}(q) \bowtie k[G]$ between the generalized Taft algebra $H_{m,d}(q)$ and group algebra $k[G]$ is actually the smash product $H_{m,d}(q)\sharp k[G]$. Then we show that the classification of these smash products could be reduced to the description of the group automorphisms of G. As an application, the classification of $H_{m,d}(q) \bowtie k[C_{n_1} \times C_{n_2}]$ is completely presented by generators and relations, where C_n denotes the n-cyclic group.

Keywords: generalized Taft algebra; factorization problem; bicrossed product

MSC 2020: 16T05, 16S40

1. Introduction

The factorization problem was firstly considered by Maillet (see [17]) and was subsequently introduced to other mathematical objects for extensive research, such as algebra (see [9]), coalgebra (see [10]), Lie algebra (see [20]), groupoids (see [7]), Hopf algebra (see [19]), and so on. In the original setting, the factorization problem is to describe and classify all groups X which can factor through groups G and H, that is $X = GH$, and $G \cap H = \{1\}$. Although this problem seems very simple and natural, there is still no comprehensive and feasible way to solve it, and even describing and classifying a group which factors through two finite cycle groups is still an open question.

An important step in dealing with the factorization problem for groups was the bicrossed product construction introduced in the paper by Zappa (see [24]); later on, Takeuchi discovered the same construction in [23], where the terminology bicrossed

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product was firstly brought up. The main ingredients in this construction are the so-called *matched pairs of groups.* Subsequently, Majid in [18] generalized this notion to the context of Hopf algebras, and considered a more computational approach of the problem. The present paper is a contribution to the factorization problem for Hopf algebras.

In paper [3], the authors proposed a strategy for classifying the bicrossed product of Hopf algebras following the Majid's construction. The method proposed in [3] was followed in [8] to classify bicrossed products of two Sweedler's Hopf algebras, in [14], [15] to compute the automorphism of Drinfeld doubles of purely nonabelian finite group and quasitriangular structure of the doubles of finite group, respectively; then in [4] to classify bicrossed products of finite groups, in [5] to classify the complements for Hopf algebras and Lie algebras, in [1] to classify bicrossed products of two Taft algebras, and finally in [2], [6] to classify bicrossed products of Taft algebras and group algebras. In 2019, Lu, Ning and Wang in [16] described all Hopf algebras that could factor through the Sweedler's Hopf algebras and Kac-Paljutkin Hopf algebra by generators and relations.

Motivated by the above results, in this paper we describe the bicrossed products between the generalized Taft algebra $H_{m,d}(q)$ and the group algebra $k[G]$, where G is a group which admits a generating set of consisting of finite order elements.

This paper is organized as follows. In Section 2, we recall basic definitions and facts needed in this paper. In Section 3, we compute the bicrossed products $H_{m,d} \bowtie k[G]$. Theorem 3.1 shows that any bicrossed product $H_{m,d} \bowtie k[G]$ is in fact a smash product. In Section 4, as an application of the above theorem, we determine the sufficient and necessary conditions for two different bicrossed products $H_{m,d}\sharp k[G]$ and $H_{m,d}\sharp' k[G]$ to be isomorphic via quadruple (u,p,r,v). Finally, we apply our theorem to $H_{m,d}(q)$ and group algebra $k[C_{n_1} \times C_{n_2}]$, where C_n denotes the n-cyclic group.

Throughout this paper, k is an arbitrary field of characteristic zero. Unless otherwise specified, all algebras, coalgebras, bialgebra, Hopf algebra and homomorphisms are over k. We put $\otimes$ shorthand for $\otimes_k$. For a comultiplication of coalgebra C, we use Sweedler's notation $\Delta(c) = c_1 \otimes c_2$.

2. Preliminaries

In this section, we recall some basic definitions and results.

For any positive integer n, the cyclic group of all nth roots of unity in the field k is denoted by $U_n(k) = \{\omega \in k\colon\ \omega^n = 1\}$; if the order is n for $U_n(k)$, then any generator of $U_n(k)$ is called a *primitive nth root of unity.*

Let $m, d \geqslant 2$ be two fixed positive integers, $d \mid m$, and $q \in k$ the primitive dth root of unity. In [21] Radford considered the following Hopf algebra $H_{m,d}(q)$ (abbreviated as $H_{m,d}$) is generated by h, x as an algebra, subject to the following relations:

$$h^m = 1, \quad x^d = 0, \quad xh = qhx.$$

The coalgebra structure is given as follows:

$$\Delta(h) = h \otimes h, \quad \Delta(x) = x \otimes h + 1 \otimes x, \quad \varepsilon(h) = 1,$$
$$\varepsilon(x) = 0, \quad S(h) = h^{-1} = h^{m-1}, \quad S(x) = -xh^{-1}.$$

Obviously $\{h^i x^j \colon 0 \leqslant i \leqslant m-1, 0 \leqslant j \leqslant d-1\}$ is basis of $H_{m,d}$, and $\mathcal{G} = \{1, h, h^2, \ldots, h^{m-1}\}$ is the set of group-like elements. For any $j = \{0, 1, \ldots, d-1\}$, the set of $(h^j, 1)$-primitive elements is written as:

$$P_{h^j,1}(H_{m,d}) = \begin{cases} \alpha(h^j - 1), & j \neq 1, \\ \beta(h-1) + \gamma x, & j = 1, \end{cases} \quad \alpha, \beta, \gamma \in k. \tag{2.1}$$

Remark 2.1. It is worthwhile to note that if $d = m$, then $H_{m,d} = H_{m,m}$ is the m^2-dimensional Taft (Hopf) algebra (see [22]), the reason why $H_{m,d}$ is called a *generalized Taft algebra* in [11], [13]. Specifically, $H_{2,2}$ is the Sweedler's Hopf algebra H_4. The Hopf algebra $H_{m,d}$ can be also approached by quiver and relations, that is, $H_{m,d}$ is isomorphic to the quiver quantum group $KZ_m(q)/I_d$ constructed by Cibils in [12].

A matched pair of Hopf algebras is a quadruple $(A, H, \triangleleft, \triangleright)$, where A and H are Hopf algebras, and $\triangleleft\colon H \otimes A \to H$, $\triangleright\colon H \otimes A \to A$ are linear morphisms such that $(A, \triangleright)$ is a left H-module coalgebra, $(H, \triangleleft)$ is a right A-module coalgebra and the following compatible conditions hold:

$$y \triangleright 1 = \varepsilon_H(h) 1_A, \quad 1_H \triangleleft a = \varepsilon_A(a) 1_H, \tag{2.2}$$
$$y \triangleright (ab) = (y_{(1)} \triangleright a_{(1)})((y_{(2)} \triangleleft a_{(2)}) \triangleright b), \tag{2.3}$$
$$(yz) \triangleleft a = y \triangleleft (z_{(1)} \triangleright a_{(1)})(z_{(2)} \triangleleft a_{(2)}), \tag{2.4}$$
$$y_{(1)} \triangleleft a_{(1)} \otimes y_{(2)} \triangleright a_{(2)} = y_{(2)} \triangleleft a_{(2)} \otimes y_{(1)} \triangleright a_{(1)} \tag{2.5}$$

for all $a, b \in A$, $y, z \in H$.

If $(A, H, \triangleleft, \triangleright)$ is a matched pair, the associated bicrossed product $A \bowtie H$ of A and H is the vector space $A \otimes H$ endowed with the tensor product coalgebra and the multiplication:

$$(a \bowtie y)(b \bowtie z) = a(y_{(1)} \triangleright b_{(1)}) \bowtie (y_{(2)} \triangleleft b_{(2)})z,$$

the antipode is

$$S_{A \bowtie H}(a \bowtie y) = (1_A \bowtie S_H(y)) \cdot (S_A(a) \bowtie 1_H).$$

In particular, assume that $(A, \triangleright)$ is a left H-module coalgebra. Meanwhile consider H as a right A-module coalgebra with trivial action, namely $y \triangleleft a = \varepsilon_A(a) y$. Then $(A, H, \triangleleft, \triangleright)$ is a matched pair if and only if $(A, \triangleright)$ is left H-module algebra and the following condition holds:

$$y_{(1)} \otimes y_{(2)} \triangleright a = y_{(2)} \otimes y_{(1)} \triangleright a.$$

At this moment, the multiplication of $A \bowtie H$ becomes

$$(a \bowtie y) \cdot (b \bowtie z) = a(y_{(1)} \triangleright b) \bowtie y_{(2)}z$$

for all $a, b \in A$, $y, z \in H$. That is, $A \bowtie H$ is reduced to the smash product $A\sharp H$.

We say a Hopf algebra E factors through two Hopf algebras A and H if there exist injective Hopf algebra maps $i\colon A \to E$ and $j\colon H \to E$, such that the following map

$$A \otimes H \to E, \quad a \otimes y \mapsto i(a)j(y)$$

is bijective.

Theorem 2.2 ([3]). *Let A and H be two Hopf algebras. A Hopf algebra E factors through A and H if and only if there exists a matched pair $(A, H, \triangleleft, \triangleright)$ and an isomorphism $E \cong A \bowtie H$.*

Theorem 2.3 ([3]). *Let A, B and C be three coalgebras. Then there exists one to one correspondence between coalgebra map $\alpha\colon A \to B \otimes C$ and binary morphism (u, p), where $u\colon A \to B$ and $p\colon A \to C$ are both coalgebra morphisms such that*

$$p(a_1) \otimes u(a_2) = p(a_2) \otimes u(a_1)$$

for all $a \in A$. Furthermore, the correspondence relation is written as

$$\alpha(a) = u(a_1) \otimes p(a_2).$$

Theorem 2.4 ([3]). *Suppose $A\sharp H$ and $A'\sharp' H'$ are two smash products of Hopf algebra with the left actions $\triangleright\colon H \otimes A \to A$ and $\triangleright'\colon H' \otimes A' \to A'$ respectively, then there exists a bijective correspondence between the set of all morphisms of Hopf algebra $\psi\colon A\sharp H \to A'\sharp' H'$ and the set of all quadruples (u, p, r, v) , consisting of two unitary coalgebra maps $u\colon A \to A'$, and $r\colon H \to A'$, and two Hopf algebra maps $p\colon A \to H'$, $v\colon H \to H'$ subject to the following compatibilities:*

$$u(a_{(1)}) \otimes p(a_{(2)}) = u(a_{(2)}) \otimes p(a_{(1)}), \tag{2.6}$$

$$r(g_{(1)}) \otimes v(g_{(2)}) = r(g_{(2)}) \otimes v(g_{(1)}), \tag{2.7}$$

$$u(ab) = u(a_{(1)})(p(a_{(2)}) \triangleright' u(b)), \tag{2.8}$$

$$r(tg) = r(t_{(1)})(v(t_{(2)}) \triangleright' r(g)), \tag{2.9}$$

$$r(g_{(1)})(v(g_{(2)}) \triangleright' u(b)) = u(g_{(1)} \triangleright b_{(1)})(p(g_{(2)} \triangleright b) \triangleright' r(g_{(3)})), \tag{2.10}$$

$$v(g)p(b) = p(g_{(1)} \triangleright b)v(g_{(2)}) \tag{2.11}$$

for all $a, b \in A$, $g, t \in H$.

This correspondence is given by:

$$\psi(a\sharp g) = u(a_{(1)})(p(a_{(2)}) \triangleright' r(g_{(1)}))\sharp' p(a_{(3)})v(g_{(2)})$$

for all $a, b \in A$, $g, t \in H$.

Finally, the following theorem is useful for later calculations which is intensively used.

Theorem 2.5 ([1]). *Consider a matched pair of Hopf algebras* $(A, H, \triangleleft, \triangleright)$. *Let* $a, b \in \mathcal{G}(A)$, $g, h \in \mathcal{G}(H)$, *then*

(1) $g \triangleright a \in \mathcal{G}(A)$, $g \triangleleft a \in \mathcal{G}(H)$,

(2) *if* $x \in P_{a,1}(A)$ *then* $g \triangleleft x \in P_{g\triangleleft a,g}(A)$, $g \triangleright x \in P_{g\triangleright a,1}(A)$,

(3) *if* $y \in P_{g,1}(H)$ *then* $y \triangleleft a \in P_{g\triangleleft a,1}(A)$, $y \triangleright a \in P_{g\triangleright a,a}(A)$.

3. Bicrossed products of $H_{m,d}$ and $k[G]$

In this section we discuss the bicrossed products of a generalized Taft algebra $H_{m,d}$ and group algebra $k[G]$, where G admits a generating set consisting of finite order elements. At first we consider a special case: the bicrossed products of $H_{m,d}$ and $k[C_n]$, where $C_n = \langle g \rangle$ is the n-cyclic group.

Theorem 3.1. *Let* $(H_{m,d}, k[C_n], \triangleleft, \triangleright)$ *be a matched pair and* $U_n(k)$ *the set of* n*th roots of unity; then there exists a bijective between matched pairs* $(H_{m,d}, k[C_n], \triangleleft, \triangleright)$ *and* $U_n(k)$. *The actions are given by*

$$g^i \triangleright h^j x^k = \omega^{ik} h^j x^k, \quad g^i \triangleleft h^j x^k = g^i \varepsilon(x^k),$$

where $\omega \in U_n(k)$, $i = 0, 1, \ldots, n-1$, $j = 0, 1, \ldots, m-1$, *and* $k = 0, 1, \ldots, d-1$.

P r o o f. Let $(H_{m,d}, k[C_n], \triangleleft, \triangleright)$ be a matched pair. First of all, we have $g \triangleright h = h$, since by Theorem 2.5, $g \triangleright h \in \mathcal{G}(H_{m,d}) = \{1, h, \ldots, h^{m-1}\}$. If $g \triangleright h = 1$, by induction we obtain $1 = g^n \triangleright h = (g^{n-1}g) \triangleright h = h$, which is clearly a contradiction.

Then suppose $g \triangleright h = h^t$, $t \in \{2, 3, \ldots, m-1\}$, which implies $g \triangleright x \in P_{h^t,1}$. Because $t \neq 1$, we have $g \triangleright x = \alpha(1 - h^t)$, $\alpha \in k$, so $g^2 \triangleright x = g \triangleright (g \triangleright x) = \alpha - \alpha g \triangleright h^t$. By induction, we know that $x = g^n \triangleright x = \alpha - \alpha g^{n-1} \triangleright h^t$. But we also obtain $g^{n-1} \triangleright h^t \in G(H_{m,d}) = \{1, h, \ldots, h^{m-1}\}$, which leads to a contradiction. Hence, $g \triangleright h = h$, and $g \triangleright x = \alpha(1-h) + \beta x$, $\alpha, \beta \in k$. Thus, on one hand

$$g^2 \triangleright x = g \triangleright g \triangleright x = g \triangleright \alpha(1 - h^t) = \alpha(1+\beta)(1-h) + \beta^2 x.$$

Using induction

$$g^n \triangleright x = \alpha(1 + \beta + \ldots + \beta^{n-1})(1-h) + \beta^n x.$$

On the other hand, $g^n \triangleright x = 1 \triangleright x = x$. So we have

$$\beta^n = 1, \quad \alpha(1 + \beta + \ldots + \beta^{n-1}) = 0.$$

Now consider the right action $\triangleleft$. Again by Theorem 2.5, $g \triangleleft h \in k[C_n] = \{1, g, \ldots, g^{n-1}\}$, hence, $g \triangleleft h = g^t$, $t \in \{0, 1, \ldots, n-1\}$. If $g \triangleleft h = 1$, then $1 = g \triangleleft h^m = g \triangleleft 1 = g$, a contradiction. Therefore, $g \triangleleft h = g^t$, $t \in \{1, 2, \ldots, n-1\}$, and $g \triangleleft x \in P_{g^t,g}(k[C_n])$. Then by (2.1) we have $g \triangleleft x = \mu(g - g^t)$, $\mu \in k$. The condition (2.5) yields:

$$g_{(1)} \triangleleft x_{(1)} \otimes g_{(2)} \triangleright x_{(2)} = g_{(2)} \triangleleft x_{(2)} \otimes g_{(1)} \triangleright x_{(1)}.$$

The above equation can be converted to

$$\mu g \otimes h - \mu g^t \otimes h + \alpha g \otimes 1 - \alpha g \otimes h + \beta g \otimes x = \alpha g^t \otimes 1 - \alpha g^t \otimes h + \beta g^t \otimes x + \mu g^t \otimes 1.$$

We observed that if $t \neq 1$, then $\beta = 0$, which is a contradiction to $\beta^n = 1$, so we get $t = 1$.

By the compatibility (2.3)

$$g \triangleright hx = (g \triangleright h)((g \triangleright h) \triangleright x) = h(g \triangleright x) = \alpha h - \alpha h^2 + \beta hx,$$
$$g \triangleright xh = (g \triangleright x_1)((g \triangleleft x_2) \triangleright h) = (g \triangleright x)((g \triangleleft h) \triangleright h) = \alpha h - \alpha h^2 + \beta xh.$$

Since $xh = qhx$, combined with the above results, we have

$$q\alpha h - q\alpha h^2 + q\beta hx = \alpha h - \alpha h^2 + \beta xh.$$

Because $q \neq 1$, so we know $\alpha = 0$, thus $g \triangleright x = \beta x$. Again by induction for all $i \in \{0, 1, \ldots, n-1\}$, $j \in \{0, 1, \ldots, m-1\}$, $l \in \{0, 1, \ldots, d-1\}$, we have

$$g \triangleleft h^j = g, \quad g \triangleleft x^l = g\varepsilon(x^l), \quad g^i \triangleright h = h, \quad g^i \triangleright x = \beta^i x, \quad \beta \in U_n(k).$$

By induction, we also have $g^i \triangleleft h = g^i$, $i \in \{0, 1, \ldots, n-1\}$.

For any $i \in \{0, 1, \ldots, n-1\}$, $j \in \{0, 1, \ldots, m-1\}$, further calculations include

$$g^i \triangleleft h^j = g^i.$$

The above equation and the condition (2.3) imply

$$g^i \triangleright h^j = h^j.$$

By the condition (2.4)

$$g^2 \triangleleft x = (g \triangleleft (g \triangleright x_{(1)}))(g \triangleleft x_{(2)}) = (g \triangleleft (g \triangleright x))(g \triangleleft h) + (g \triangleleft (g \triangleright 1))(g \triangleleft x) = 0,$$
$$g^3 \triangleleft x = (g^2 \triangleleft (g \triangleright x_{(1)}))(g \triangleleft x_{(2)}) = (g^2 \triangleleft (g \triangleright x))(g \triangleleft h) + (g^2 \triangleleft (g \triangleright 1))(g \triangleleft x) = 0.$$

And more generally

$$g^i \triangleleft x^l = g^i \varepsilon(x^l).$$

By further computations

$$
\begin{aligned}
g^i \triangleright x^2 &= (g^i \triangleright x_{(1)})((g^i \triangleleft x_{(2)}) \triangleright x) \\
&= (g^i \triangleright x)((g^i \triangleleft h) \triangleright x) + (g^i \triangleright 1)((g^i \triangleleft x) \triangleright x) = \beta^{2i} x^2, \\
g^i \triangleright x^3 &= (g^i \triangleright x_{(1)})((g^i \triangleleft x_{(2)}) \triangleright x^2) \\
&= (g^i \triangleright x)((g^i \triangleleft h) \triangleright x^2) + (g^i \triangleright 1)((g^i \triangleleft x) \triangleright x^2) = \beta^{3i} x^3.
\end{aligned}
$$

Also by induction, we obtain

$$g^i \triangleright x^l = \beta^{il} x^l.$$

To sum up:

$$
\begin{aligned}
g^i \triangleright h^j x^l &= (g^i \triangleright h^j)((g^i \triangleleft h^j) \triangleright x^l) = h^j (g^i \triangleright x^l) = \beta^{il} h^j x^l, \\
g^i \triangleleft h^j x^l &= g^i \triangleleft h^j \triangleleft x^l = g^i \triangleleft x^l = g^i \varepsilon(x^l).
\end{aligned}
$$

The proof is completed. □

Corollary 3.2. *A Hopf algebra E could factorize through $H_{m,d}$ and $k[C_n]$ if and only if $E \cong T^{\omega}_{nmd}(q)$, where $\omega \in U_n(k)$, and $T^{\omega}_{nmd}(q)$ is a Hopf algebra generated by g, h, and x, subject to:*

$$g^n = h^m = 1, \quad x^d = 0, \quad xh = qhx, \quad hg = gh, \quad gx = \omega xg.$$

The coalgebra structure is

$$\Delta(g) = g \otimes g, \quad \Delta(h) = h \otimes h, \quad \Delta(x) = x \otimes h + 1 \otimes x, \quad \varepsilon(h) = \varepsilon(g) = 1, \quad \varepsilon(x) = 0.$$

and the antipode is

$$S(x) = -xh^{m-1}, \quad S(h) = h^{m-1}, \quad S(g) = g^{n-1}.$$

P r o o f. By Theorem 2.2, $E \cong H_{m,d} \bowtie k[C_n]$, and by Theorem 3.1, the action

$$\triangleright\colon\ H_{m,d}(q) \otimes k[C_n] \to k[C_n]$$

is trivial. Thus, E is isomorphic to the smash product $H_{m,d} \sharp k[C_n]$, which is generated by $h = h\sharp 1$, $x = x\sharp 1$, and $g = 1\sharp g$. Therefore,

$$
\begin{aligned}
gh &= (1\sharp g)(h\sharp 1) = g \triangleright h \sharp g \triangleleft h = h\sharp g = (h\sharp 1)(1\sharp g) = hg, \\
gx &= (1\sharp g)(x\sharp 1) = g \triangleright x_1 \sharp g \triangleleft x_2 = g \triangleright x \sharp g \triangleleft h + g \triangleright 1 \sharp g \triangleleft x = \omega x\sharp g = \omega xg.
\end{aligned}
$$

□

Remark 3.3. It is easy to see that

$$\{g^i h^j x^k \colon 0 \leqslant i \leqslant n-1,\ 0 \leqslant j \leqslant m-1,\ 0 \leqslant k \leqslant d-1\}$$

is the basis of H^{ω}_{nmd}. Thus, H^{ω}_{nmd} is nmd-dimensional.

The main theorem of this paper is given as follows:

Theorem 3.4. *Let G be a group generated by finite order elements. If a Hopf algebra E factorizes through $H_{m,d}$ and $k[G]$, then E is isomorphic to the smash product of $H_{m,d}$ and $k[G]$.*

Proof. Let G be a group generated by a set S of finite order elements. Assume $g \in S$ and $\operatorname{ord}(g) = u$, where $u \in N^*$. Assume $(H_{m,d}, k[G], \triangleleft, \triangleright)$ is a matched pair; by Theorem 3.1, we get $g \triangleright h = h$. Hence, $g \triangleright x = \alpha(1-h) + \beta x$, $\alpha, \beta \in k$, and we have

$$\alpha(1+\beta+\ldots+\beta^{u-1}) = 0, \quad \beta^u = 1.$$

Since $g \triangleleft h \in \mathcal{G}(k[G])$ and $g \triangleleft h \neq 1$, thus $g \triangleleft h = g_{i_1}^{a_1} g_{i_2}^{a_2} \ldots g_{i_s}^{a_s}$, where $s, a_1, \ldots, a_s \in N^*$, $g_{i_1}, \ldots, g_{i_s} \in S$, so we get

$$g \triangleleft x \in P_{g \triangleleft h, g \triangleleft 1}(k[G]) = P_{g_{i_1}^{a_1} g_{i_2}^{a_2} \ldots g_{i_s}^{a_s}, g}(k[G]),$$

that is

$$g \triangleleft x = \gamma(g - g_{i_1}^{a_1} g_{i_2}^{a_2} \ldots g_{i_s}^{a_s}), \quad \gamma \in k.$$

Applying compatibility (2.5) we obtain

$$g \triangleleft x \otimes h + g \otimes g \triangleright x = g \triangleleft h \otimes g \triangleright x + g \triangleleft x \otimes 1.$$

By further calculation

$$\begin{aligned}
&\gamma g \otimes h - \gamma g_{i_1}^{a_1} g_{i_2}^{a_2} \ldots g_{i_s}^{a_s} \otimes h + \alpha g \otimes 1 - \alpha g \otimes h + \beta g \otimes x \\
&\quad = \alpha g_{i_1}^{a_1} g_{i_2}^{a_2} \ldots g_{i_s}^{a_s} \otimes 1 - \alpha g_{i_1}^{a_1} g_{i_2}^{a_2} \ldots g_{i_s}^{a_s} \otimes h + \beta g_{i_1}^{a_1} g_{i_2}^{a_2} \ldots g_{i_s}^{a_s} \otimes x \\
&\qquad + \gamma g \otimes 1 - \gamma g_{i_1}^{a_1} g_{i_2}^{a_2} \ldots g_{i_s}^{a_s} \otimes 1.
\end{aligned}$$

So

$$\beta g \otimes x = \beta g_{i_1}^{a_1} g_{i_2}^{a_2} \ldots g_{i_s}^{a_s} \otimes x.$$

Since $\beta^u = 1$ implies $\beta \neq 0$, $g_{i_1}^{a_1} g_{i_2}^{a_2} \ldots g_{i_s}^{a_s} = g$; therefore, $g \triangleleft h = g$ and $g \triangleleft x = 0$.

According to condition (2.3)

$$\begin{aligned}
&g \triangleright hx = (g \triangleright h)((g \triangleleft h) \triangleright x) = h(g \triangleright x) = \alpha(h - h^2) + \beta hx, \\
&g \triangleright xh = (g \triangleright x_{(1)})((g \triangleleft x_{(2)}) \triangleright h) = \alpha(h - h^2) + \beta xh = \alpha(h - h^2) + \beta qhx.
\end{aligned}$$

By the relation $xh = qhx$,

$$q\alpha h - q\alpha h^2 + q\beta hx = \alpha h - \alpha h^2 + \beta xh.$$

Since $q \neq 1$, we obtain $\alpha = 0$; therefore, $g \triangleright x = \beta x$.

Applying the similar method for any $g_i \in S$ with order u_i, we have

$$g_i \triangleright h = h, \quad g_i \triangleright x = \beta_i x, \quad g_i \triangleleft h = g_i, \quad g_i \triangleleft x = 0, \quad \beta_i \in U_{u_i}(k).$$

For all $g_i \in S$,

$$g_i^2 \triangleleft h = (g_i(g_i \triangleright h))(g_i \triangleleft h) = (g_i \triangleleft h)(g_i \triangleleft h) = g_i^2.$$

By induction one can see that for all $v \in \{0, 1, \ldots, u_i - 1\}$,

$$g_i^v \triangleleft h = g_i^v,$$

and for any $j \in \{0, 1, \ldots, m-1\}$,

$$g_i^v \triangleleft h^j = g_i^v.$$

Similarly we have

$$g_i^v \triangleright h^j = h^j.$$

By the condition (2.4),

$$g_{i_1}^{v_1} g_{i_2}^{v_2} \triangleleft h^j = (g_{i_1}^{v_1} \triangleleft (g_{i_2}^{v_2} \triangleright h^j))(g_{i_2}^{v_2} \triangleleft h^j) = (g_{i_1}^{v_1} \triangleleft h^j)(g_{i_2}^{v_2} \triangleleft h^j) = g_{i_1}^{v_1} g_{i_2}^{v_2}$$

for all $g_{i_1}^{v_1}$, $g_{i_2}^{v_2} \in S$, $v_1 \in \{0, 1, \ldots, u_{i_1} - 1\}$, $v_2 \in \{0, 1, \ldots, u_{i_2} - 1\}$. By further computation we obtain

$$g_{i_1}^{v_1} g_{i_2}^{v_2} \ldots g_{i_t}^{v_t} \triangleleft h^j = g_{i_1}^{v_1} g_{i_2}^{v_2} \ldots g_{i_t}^{v_t},$$

where $g_{i_l}^{v_l} \in S$, $v_l \in \{0, 1, \ldots, u_{i_l} - 1\}$, $l \in \{1, 2, \ldots, t\}$.

Now let us calculate the right action of x on S.

$$g_i^2 \triangleleft x \overset{(2.4)}{=} (g_i \triangleleft (g_i \triangleright x_{(1)}))(g_i \triangleleft x_{(2)}) = (g_i \triangleleft (g_i \triangleright x))(g_i \triangleleft h) + (g_i \triangleleft (g_i \triangleright 1))(g_i \triangleleft x) = 0,$$
$$g_i^3 \triangleleft x = (g_i^2 \cdot g_i) \triangleleft x = 0.$$

Again by induction we get

$$g_i^v \triangleleft x^j = 0$$

for all $v \in \{0, 1, \ldots, u_i - 1\}$, $j \in \{1, 2, \ldots, d-1\}$. In fact, we have

$$g_i^v \triangleleft x^j = g_i^v \varepsilon(x^j).$$

Similarly to the previous calculation

$$g_{i_1}^{v_1} g_{i_2}^{v_2} \dots g_{i_t}^{v_t} \triangleleft x^j = g_{i_1}^{v_1} g_{i_2}^{v_2} \dots g_{i_t}^{v_t} \varepsilon(x^j)$$

for all $g_{i_l} \in S$, $v_l \in \{0, 1, \dots, u_{i_l} - 1\}$, $l \in \{0, 1, \dots, t\}$. And

$$\begin{aligned} g_{i_1}^{v_1} g_{i_2}^{v_2} \dots g_{i_t}^{v_t} \triangleleft h^j x^{j'} &= (g_{i_1}^{v_1} g_{i_2}^{v_2} \dots g_{i_t}^{v_t} \triangleleft h^j) \triangleleft x^{j'} = g_{i_1}^{v_1} g_{i_2}^{v_2} \dots g_{i_t}^{v_t} \varepsilon(x^{j'}) \\ &= g_{i_1}^{v_1} g_{i_2}^{v_2} \dots g_{i_t}^{v_t} \varepsilon(h^j x^{j'}), \end{aligned}$$

which shows that the right action is indeed trivial. The proof is completed. □

As an application, we consider a special case when $G = C_{n_1} \times C_{n_2}$.

Corollary 3.5. *Let g_1 and g_2 be the generating element of C_{n_1} and C_{n_2}, respectively. There is a bijective correspondence between the matched pairs $(H_{m,d}, k[C_{n_1} \times C_{n_2}], \triangleleft, \triangleright)$ and $U_{n_1}(k) \times U_{n_2}(k)$. Explicitly for any $\beta_1 \in U_{n_1}(k)$, $\beta_2 \in U_{n_2}(k)$, the correspondence is given by*

$$(g_1^{t_1}, g_2^{t_2}) \triangleright h^j x^k = \beta_1^{t_1 k} \beta_2^{t_2 k} h^j x^k, \quad (g_1^{t_1}, g_2^{t_2}) \triangleleft h^j x^k = g_1^{t_1} g_2^{t_2} \varepsilon(h^j x^k)$$

for all $t_1 \in \{0, 1, \dots, n_1 - 1\}$, $t_2 \in \{0, 1, \dots, n_2 - 1\}$, $j \in \{0, 1, \dots, m - 1\}$, $k \in \{0, 1, \dots d - 1\}$. Moreover, the bicrossed product $H_{m,d} \bowtie k[C_{n_1} \times C_{n_2}])$ is isomorphic to $H_{n_1 n_2 md}^{\beta_1, \beta_2}$, which is a Hopf algebra generated by g_1, g_2, h and x subject to the following relations

$$g_1^{n_1} = g_2^{n_2} = h^m = 1, \quad x^d = 0, \quad g_1 h = h g_1, \quad g_2 h = h g_2,$$
$$g_1 x = \beta_1 x g_1, \quad g_2 x = \beta_2 x g_2, \quad xh = qhx.$$

Its coalgebra structure and the antipode are given as follows:

$$\Delta(g_1) = g_1 \otimes g_1, \;\; \Delta(g_2) = g_2 \otimes g_2, \;\; \Delta(h) = h \otimes h, \;\; \Delta(x) = x \otimes h + 1 \otimes x, \;\; \varepsilon(x) = 0,$$
$$\varepsilon(g_1) = \varepsilon(g_2) = \varepsilon(h) = 1, \;\; S(g_1) = g_1^{n_1 - 1}, \;\; S(g_2) = g_2^{n_2 - 1}, \;\; S(x) = -x h^{m-1}.$$

P r o o f. Since $(H_{m,d}, k[C_{n_1} \times C_{n_2}], \triangleleft, \triangleright)$ is a matched pair, from Theorem 3.4 we know the right action is trivial, and

$$g_1^{t_1} \triangleright h = h, \quad g_1^{t_1} \triangleright x = \beta_1^{t_1} x, \quad g_2^{t_2} \triangleright h = h, \quad g_2^{t_2} \triangleright x = \beta_2^{t_2} x,$$

where $t_1 \in \{0, 1, \dots, n_1 - 1\}$, $t_2 \in \{0, 1, \dots, n_2 - 1\}$. By the relation (2.3) we get

$$g_1^{t_1} \triangleright h^j = h^j,$$

as $g_1^{t_1} \triangleright x = \beta_1^{t_1} x$ by induction we know

$$g_1^{t_1} \triangleright x^j = \beta_1^{t_1 j} x^j.$$

Therefore,

$$g_1^{t_1} \triangleright h^j x^k \overset{(2.3)}{=} (g_1^{t_1} \triangleright h^j)((g_1^{t_1} \triangleleft h^j) \triangleright x^k) = h^j(g_1^{t_1} \triangleright x^k) = \beta_1^{t_1 k} h^j x^k.$$

For g_2 we have the similar results

$$g_2^{t_2} \triangleright h^j = h^j, \quad g_2^{t_2} \triangleright x^j = \beta_2^{t_2 j} x^j,$$

where $j \in \{0, 1, \ldots, d-1\}$. Hence, $(g_1^{t_1}, g_2^{t_2}) \triangleright h^j x^k = \beta_1^{ik} \beta_2^{ik} h^j x^k$. The proof is completed. $\square$

Remark 3.6. In Corollary 3.5, if $n_2 = 1$, then we could obtain Theorem 3.1.

4. The isomorphism between the bicrossed products of $H_{m,d}$ and $k[G]$

In this section we describe the isomorphism of $H_{m,d} \bowtie k[G]$, where G is a group generated by finite order elements. By Theorem 3.4, $H_{m,d} \bowtie k[G]$ is actually the smash product between them.

Theorem 4.1. *Let $H_{m,d} \sharp k[G]$ and $H_{m,d} \sharp' k[G]$ be two smash products, then there exists one to one correspondence between Hopf algebra isomorphism φ: $H_{m,d} \sharp k[G] \to H_{m,d} \sharp' k[G]$ and triple (u, r, v), where $u\colon H_{m,d} \to H_{m,d}$, $r\colon k[G] \to H_{m,d}$ are unital coalgebra maps, and $v\colon k[G] \to k[G]$ is an automorphism of Hopf algebra, such that*

$$\varphi(a \sharp t) = u(a) r(t_{(1)}) \sharp' v(t_{(2)})$$

for all $a \in H_{m,d}$, $t \in k[G]$.

Proof. Let $S = \{g_i \in G\colon i \in I\}$ be the generating set of G, where I is the index set and $\operatorname{ord}(g_i) = u_i$ for all $i \in I$. Denote the left actions in the smash products $H_{m,d} \sharp k[G]$ and $H_{m,d} \sharp' k[G]$ by $\triangleright\colon k[G] \otimes H_{m,d} \to H_{m,d}$, $\triangleright'\colon k[G] \otimes H_{m,d} \to H_{m,d}$ respectively. By Theorem 3.3 for all $i \in I$,

$$g_i \triangleright h = h, \quad g_i \triangleright x = \alpha_i x, \quad g_i \triangleright' h = h, \quad g_i \triangleright' x = \overline{\alpha_i} x,$$

where $\alpha_i, \overline{\alpha_i} \in U_{u_i}(k)$.

From Theorem 2.3 we know there is a one to one correspondence between a Hopf morphism $\varphi\colon H_{m,d} \sharp k[G] \to H_{m,d} \sharp' k[G]$ and quadruple (u, p, r, v), where $u\colon H_{m,d} \to H_{m,d}$, $r\colon k[G] \to H_{m,d}$ are coalgebra maps, and $p\colon H_{m,d} \to k[G]$, $v\colon k[G] \to k[G]$ are Hopf algebra maps.

We claim that $p\colon H_{m,d} \to k[G]$ is trivial, i.e., $p = \varepsilon$. Indeed since $p(h) \in \mathcal{G}(k[G])$, we get $p(h) = g_{i_1}^{a_1} g_{i_2}^{a_2} \dots g_{i_s}^{a_s}$ for all $s, a_1, \dots a_s \in N$, $g_{i_1}^{a_1}, \dots g_{i_s}^{a_s} \in S$, and $p(x) \in P_{p(h),1}(k[G])$. By relation (2.1) we have $p(x) = \beta(g_{i_1}^{a_1} g_{i_2}^{a_2} \dots g_{i_s}^{a_s} - 1)$ for all $\beta \in k$. Hence,

$$\begin{aligned} p(hx) = p(h)p(x) &= \beta g_{i_1}^{a_1} g_{i_2}^{a_2} \dots g_{i_s}^{a_s} (g_{i_1}^{a_1} g_{i_2}^{a_2} \dots g_{i_s}^{a_s} - 1) \\ &= \beta(g_{i_1}^{a_1} g_{i_2}^{a_2} \dots g_{i_s}^{a_s} - 1) g_{i_1}^{a_1} g_{i_2}^{a_2} \dots g_{i_s}^{a_s} = p(xh). \end{aligned}$$

Due to $p(xh) = qp(hx)$, $\beta(q-1) g_{i_1}^{a_1} g_{i_2}^{a_2} \dots g_{i_s}^{a_s} (g_{i_1}^{a_1} g_{i_2}^{a_2} \dots g_{i_s}^{a_s} - 1) = 0$. Since $q \neq 1$, we have $\beta(g_{i_1}^{a_1} g_{i_2}^{a_2} \dots g_{i_s}^{a_s} - 1) = 0$, i.e., $p(x) = 0$.

Because $u\colon H_{m,d} \to H_{m,d}$ is a coalgebra map, we have $u(1) = 1$, $u(h) = h^c$, $c \in \{0, 1, \dots m-1\}$. So

$$u(h^2) = u(h \cdot h) = u(h)(p(h) \triangleright' u(h)) = h^c (g_{i_1}^{a_1} g_{i_2}^{a_2} \dots g_{i_s}^{a_s} - 1 \triangleright' h^d) = h^{2c}.$$

It is easy to get $u(h^j) = h^{jc}$, so $u(x) \in P_{h^c,1}(H_{m,d})$. In other words, when $c \neq 1$, $u(x) = \mu(h^c - 1)$; when $c = 1$, $u(x) = \mu(h-1) + \gamma x$ for $\mu, \gamma \in k$.

Firstly, suppose $c \neq 1$, $u(x) = \mu(h^c - 1)$, then we have

$$\begin{aligned} &u(hx) = u(h)(p(h) \triangleright' u(x)) = h^d (g_{i_1}^{a_1} g_{i_2}^{a_2} \dots g_{i_s}^{a_s} - 1 \triangleright' \mu(h^c - 1)) = \mu h^c (h^c - 1), \\ &u(xh) = u(x_1)(p(x_2) \triangleright' u(h)) = u(x)(p(h) \triangleright' u(h)) + p(x) \triangleright' u(h) = \mu h^c (h^c - 1). \end{aligned}$$

As $u(xh) = qu(hx)$, from the above identity, we get $\mu(q-1)h^d(h^c - 1) = 0$. But $q \neq 1$, so $\mu(h^c - 1) = 0$, $u(x) = 0$. And

$$\begin{aligned} \psi(x\sharp 1) = u(x_{(2)})(p(x_{(2)}) \triangleright' r(1)) \sharp p_(x_{(3)}) v(1) &= u(x_{(1)}) \sharp p(x_{(2)}) \\ &= u(x) \sharp p(h) + 1 \sharp p(x) = 0, \end{aligned}$$

which is a contradiction to the assumption that ψ is an bijective. Hence, we have

$$c = 1, \quad u(x) = \mu(h-1) + \gamma x.$$

Similarly, we have

$$\begin{aligned} u(hx) &= u(h)(p(h) \triangleright' u(x)) = h[g_{i_1}^{a_1} g_{i_2}^{a_2} \dots g_{i_s}^{a_s} \triangleright' (\mu(h-1) + \gamma x)] \\ &= \mu h(h-1) + \gamma \alpha_{i_1}^{a_1} \alpha_{i_2}^{a_2} \dots \alpha_{i_s}^{a_s}, \\ u(xh) &= \mu h(h-1) + \gamma q h x. \end{aligned}$$

Since $u(xh) = qu(hx)$, we can obtain $\mu = q\mu$ and $q\gamma \alpha_{i_1}^{a_1} \alpha_{i_2}^{a_2} \dots \alpha_{i_s}^{a_s} = q\gamma$. Due to $q \neq 1$, we have $\mu = 0$, $\gamma \neq 0$ and $\alpha_{i_1}^{a_1} \alpha_{i_2}^{a_2} \dots \alpha_{i_s}^{a_s} = 1$. In fact, if $\gamma = 0$, then $u(x) = 0$, which is a contradiction. Therefore,

$$u(h^i) = h^i, \quad u(x^j) = \gamma^j x^j,$$

where $\gamma \in k^*$, $i \in \{0, 1, \dots m-1\}$, $j \in \{0, 1, \dots, d-1\}$.

Set $a = x$. From condition $u(a_{(1)}) \otimes p(a_{(2)}) = u(a_{(2)}) \otimes p(a_{(1)})$, we have

$$\begin{aligned} u(x_{(1)}) \otimes p(x_{(2)}) = u(x_{(2)}) \otimes p(x_{(1)}) \\ \Leftrightarrow u(x) \otimes p(h) + u(1) \otimes p(x) = u(h) \otimes p(x) + u(x) \otimes p(1) \\ \Leftrightarrow \gamma x \otimes g_{i_1}^{a_1} g_{i_2}^{a_2} \ldots g_{i_s}^{a_s} = \gamma x \otimes 1. \end{aligned}$$

From the above we get $\gamma \neq 0$, so $g_{i_1}^{a_1} g_{i_2}^{a_2} \ldots g_{i_s}^{a_s} = 1$, therefore, $p(x) = 0$, $p(h) = 1$, which implies that p is trivial. Hence, the mentioned quadruple becomes (u, ε, r, v) and for all $a \sharp t \in H_{m,d} \sharp k[G]$,

$$\psi(a \sharp t) = u(a) r(t_{(1)}) \sharp' v(t_{(2)}).$$

Next we show that v is an automorphism of Hopf algebra. Consider the inverse $\varphi\colon H_{m,d} \sharp' k[G] \to H_{m,d} \sharp k[G]$ of ψ given by

$$\varphi(a \sharp' t) = \bar{u}(a) \bar{r}(t_{(1)}) \sharp \overline{v}(t_{(2)}).$$

Then for any $t \in k[G]$,

$$1 \sharp' t = \psi \circ \varphi(1 \sharp' t) = \psi(\bar{r}(t_{(1)}) \sharp \overline{v}(t_{(2)})) = u(\bar{r}(t_{(1)})) r(\overline{v}(t_{(2)})) \sharp' v(\overline{v}(t_{(3)})).$$

Applying $\varepsilon \otimes \mathrm{Id}$ role in the above equation, we can get $v \circ \overline{v}(t) = t$. On the other hand, we can obtain $\overline{v} \circ v(t) = t$. Thus, v is an automorphism of a Hopf algebra. The proof is completed. $\square$

Theorem 4.2. *With the Hopf algebras $H_{n_1 n_2 md}^{\beta_1, \beta_2}$ and $H_{n_1 n_2 md}^{\overline{\beta_1}, \overline{\beta_2}}$ defined in Corollary 3.5, then $H_{n_1 n_2 md}^{\beta_1, \beta_2}$ and $H_{n_1 n_2 md}^{\overline{\beta_1}, \overline{\beta_2}}$ are isomorphic if and only if the following conditions hold:*

$$m \mid n_1 l_1, \quad m \mid n_2 l_2, \quad (t_1, n_1) = 1, \quad (t_2, n_2) = 1, \quad \overline{\beta_1}^{t_1} = \beta_1 q^{l_1}, \quad \overline{\beta_2}^{t_2} = \beta_2 q^{l_2},$$

where $t_1 \in \{0, 1, \ldots n_1 - 1\}$, $t_2 \in \{0, 1, \ldots n_2 - 1\}$, $l_1, l_2 \in \{0, 1, \ldots m - 1\}$.

Proof. From Theorem 4.1 we know that any isomorphism $\psi\colon H_{n_1 n_2 md}^{\beta_1, \beta_2} \to H_{n_1 n_2 md}^{\overline{\beta_1}, \overline{\beta_2}}$ corresponds to a triple (u, r, v), where $v \in \mathrm{Aut}_{\mathrm{Hopf}}(k[C_{n_1} \times C_{n_2}])$, and since p is trivial, the equation (2.8) implies that $u\colon H_{m,d} \to H_{m,d}$ is a Hopf algebra map satisfying

$$u(h^i x^j) = u(h^i) u(x^j) = \gamma^j h^i x^j, \quad \gamma \in k^*.$$

Since $k[C_{n_1} \times C_{n_2}]$ is cocommutative, some compatibility conditions are auto satisfied. The morphims v and r are determined completely by integers $t_1 \in \{0, 1, \ldots n_1 - 1\}$, $t_2 \in \{0, 1, \ldots n_2 - 1\}$, $l_1, l_2 \in \{0, 1, \ldots m - 1\}$ such that

$$v(g_1) = g_1^{t_1}, \quad v(g_2) = g_2^{t_2}, \quad r(g_1) = h^{l_1}, \quad r(g_2) = h^{l_2},$$

where $(t_1, n_1) = 1$, $(t_2, n_2) = 1$.

By the condition (2.9) and induction, we obtain $r(g_1^i) = h^{il_1}$. In particular, we have $1 = r(g_1^{n_1}) = h^{n_1 l_1}$ and $m \mid n_1 l_1$. Consider $b = x$, the compatibility

$$r(g_{(1)})(v(g_{(2)}) \triangleright' u(b)) = u(g_{(1)} \triangleright b_{(1)})(p(g_{(2)} \triangleright b_{(2)}) \triangleright' r(g_{(3)} \triangleleft b_{(3)}))$$

becomes

$$\begin{aligned} r(g_1)(v(g_1) \triangleright' u(x)) = u(g_1 \triangleright x) r(g_1) &\Leftrightarrow h^{l_1}(g^{t_1} \triangleright' \gamma x) = u(\beta_1 x) h^{l_1} \\ &\Leftrightarrow \gamma \overline{\beta_1}^{t_1} h^{l_1} x = \gamma^j \beta_1 x h^{l_1} \\ &\Leftrightarrow \gamma \overline{\beta_1}^{t_1} h^{l_1} x = \gamma^j \beta_1 q^{l_1} h^{l_1} x. \end{aligned}$$

So we get $\overline{\beta_1}^{t_1} = \beta_1 q^{l_1}$. Similarly for g_2 we have $m \mid n_2 l$, $\overline{\beta_2}^{t_2} = \beta_2 q^{l_2}$. This proves the necessity.

Now we prove sufficiency. Construct a coalgebra map $r_{l_1,l_2} \colon k[C_{n_1} \times C_{n_2}] \to H_{m,d}$ and a Hopf algebra automorphism $v_{t_1,t_2} \colon k[C_{n_1} \times C_{n_2}] \to k[C_{n_1} \times C_{n_2}]$ for all $i, j \in N$ such that

$$r_{l_1,l_2}(g_1^i) = h^{il_1}, \quad r_{l_1,l_2}(g_2^j) = h^{jl_2}, \quad v_{t_1,t_2}(g_1) = g_1^{t_1}, \quad v_{t_1,t_2}(g_2) = g_2^{t_2}.$$

We prove that the following map is an automorphism of a Hopf algebra:

$$\varphi \colon H_{n_1 n_2 md}^{\beta_1,\beta_2} \to H_{n_1 n_2 md}^{\overline{\beta_1},\overline{\beta_2}}, \quad \varphi(a \sharp y) = a r_{l_1,l_2}(y_1) \sharp v_{t_1,t_2}(y_2).$$

At first, it is straightforward to check that φ is a Hopf algebra map. Now we are going to show φ is bijective.

As $(t_1, n_1) = 1$, so $\tau t_1 + n_1 \tau' = 1$, where $\tau, \tau' \in Z$, and there are unique integers $\alpha, \beta, \eta \in Z$, $\overline{l_1} \in \{0, 1, \ldots, m-1\}$ such that

$$\tau = \eta n_1 + \tau', \quad -l_1 \tau_1 = \beta m + \overline{l_1}.$$

Define a coalgebra map $r_{\overline{l_1},\overline{l_2}} \colon k[C_{n_1} \times C_{n_2}] \to H_{m,d}$, and a Hopf algebra map $v_{\tau_1,\tau_2} \colon k[C_{n_1} \times C_{n_2}] \to k[C_{n_1} \times C_{n_2}]$ by

$$r_{\overline{l_1},\overline{l_2}}(g_1^k) = h^{k\overline{l_1}}, \quad r_{\overline{l_1},\overline{l_2}}(g_2^k) = h^{k\overline{l_2}}, \quad v_{\tau_1,\tau_2}(g_1) = g^{\tau_1}, \quad v_{\tau_1,\tau_2}(g_2) = g^{\tau_2}.$$

Define $\overline{\varphi} \colon H_{n_1 n_2 md}^{\overline{\beta_1},\overline{\beta_2}} \to H_{n_1 n_2 md}^{\beta_1,\beta_2}$, $\overline{\varphi}(a \sharp y) = a r_{\overline{l_1},\overline{l_2}}(y_1) \sharp v_{\tau_1,\tau_2}(y_2)$. Then we have

$$\begin{aligned} \varphi \circ \overline{\varphi}((h^i x^j \sharp g_1^{k_1})) &= \varphi(h^i x^j r_{l_1,l_2}(g_1^k) \sharp v_{\tau_1,\tau_2}(g_1^k)) = \varphi(h^i x^j h^{k\overline{l_1}} \sharp g_1^{k\tau_1}) \\ &= h^i x^j h^{k\overline{l_1}} r_{l_1,l_2}(g_1^{k\tau_1}) \sharp v_{t_1,t_2}(g_1^{k\tau_1}) = h^i x^j h^{k\overline{l_1}} h^{k\tau_1 l_1} \sharp g_1^{k\tau_1 t_1} \\ &= h^i x^j h^{k\overline{l_1} + k\tau_1 l_1} \sharp g_1^{k(\tau - \eta n_1) t_1} = h^i x^j h^{k(\overline{l_1} + \tau_1 l)} \sharp g_1^{k\tau t_1} \\ &= h^i x^j h^{k\beta m} \sharp g_1^{k(1 - n_1 \tau')} = h^i x^j \sharp g_1^k, \end{aligned}$$

for all $i \in \{0, 1, \ldots, m-1\}$, $j \in \{0, 1, \ldots, d-1\}$ and $k \in \{0, 1, \ldots n_1 - 1\}$. Thus, $\varphi \circ \overline{\varphi} = \mathrm{Id}$. Similarly, we have $\overline{\varphi} \circ \varphi = \mathrm{Id}$. The proof is completed. $\square$

Example 4.3. Consider the following data:

$$m = 6,\ d = 2,\ q = -1; \quad n_1 = 4,\ t_1 = 3,\ l_1 = 3; \quad n_2 = 18,\ t_2 = 5,\ l_2 = 2;$$
$$\beta_1 = \mathrm{i}, \quad \beta_2 = \mathrm{e}^{\mathrm{i}10\pi/9}, \quad \overline{\beta_1} = \mathrm{i}, \quad \overline{\beta_2} = \mathrm{e}^{\mathrm{i}2\pi/9},$$

where i denotes the imaginary unit. It is straightforward to check that the above data satisfy the conditions in Theorem 4.2.

Define a coalgebra map $r\colon k[C_4 \times C_{18}] \to H_{6,2}$ and a Hopf algebra automorphism $v\colon k[C_4 \times C_{18}] \to k[C_4 \times C_{18}]$ by

$$r(g_1^i) = h^{3i}, \quad r(g_2^j) = h^{2j}, \quad v(g_1) = g_1^3, \quad v(g_2) = g_2^5.$$

Then we have the an automorphism of a Hopf algebra $\varphi\colon H_{n_1n_2md}^{\beta_1,\beta_2} \to H_{n_1n_2md}^{\overline{\beta_1},\overline{\beta_2}}$ as follows:

$$\begin{aligned}\varphi(h^i x^j g_1^{k_1} g_2^{k_2}) &= h^i x^j r(g_1^{k_1} g_2^{k_2}) v(g_1^{k_1} g_2^{k_2}) \overset{(2.9)}{=} h^i x^j r(g_1^{k_1})(v(g_1^{k_1}) \triangleright' (g_2^{k_2})) v(g_1^{k_1}) v(g_2^{k_2})\\ &= h^i x^j h^{3k_1}(g_1^{3k_1} \triangleright' h^{2k_2}) g_1^{3k_1} g_2^{5k_2} = h^i x^j h^{3k_1+2k_2} g_1^{3k_1} g_2^{5k_2}\\ &= (-1)^{jk_1} h^{i+3k_1+2k_2} x^j g_1^{3k_1} g_2^{5k_2}.\end{aligned}$$

Remark 4.4. In Theorem 4.2, assume $(n_1, n_2) = 1$. Since the group $C_{n_1} \times C_{n_2}$ is commutative, we deduce that $C_{n_1} \times C_{n_2}$ is a cycle group generated by (g_1, g_2) of order $n_1 n_2$. In this case, Theorem 3.5 of [2] could be applied.

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