Czechoslovak Mathematical Journal, 72 (147) (2022), 709–734

WEIGHTED MULTI-PARAMETER MIXED HARDY SPACES AND THEIR APPLICATIONS

Wei Ding, Yun Xu, Yueping Zhu, Nantong

Received March 27, 2021. Published online January 26, 2022.

Abstract. Applying discrete Calderón's identity, we study weighted multi-parameter mixed Hardy space $H^p_{\rm mix}(\omega, \mathbb{R}^{n_1} \times \mathbb{R}^{n_2})$. Different from classical multi-parameter Hardy space, this space has characteristics of local Hardy space and Hardy space in different directions, respectively. As applications, we discuss the boundedness on $H^p_{\rm mix}(\omega, \mathbb{R}^{n_1} \times \mathbb{R}^{n_2})$ of operators in mixed Journé's class.

Keywords: weight; multi-parameter; mixed Hardy spaces; singular integral operator

MSC 2020: 42B35, 42B30, 42B25, 42B20

1. Introduction

The study of Hardy spaces has a long history. In 1915, to study the behavior of holomorphic functions on the unit disk, Hardy in [26] considered the averages

$$M_p(r, F) = \left(\frac{1}{2\pi}\int_0^{2\pi} |F(re^{i\theta})|^p \, d\theta\right)^{1/p}, \quad 1 < p < \infty$$

and

$$M_\infty(r, F) = \max_{0 \leqslant \theta \leqslant 2\pi} |F(re^{i\theta})|$$

for $0 < r < 1$. This paper can be regarded as the starting point to study Hardy spaces. After that, the theory of Hardy spaces has been extensively studied. Precisely, Stein and Weiss in [32] generalized the definitions of Hardy spaces to higher dimensional, Fefferman and Stein in [8] defined Hardy spaces via a purely real method, Coifman in [1] and Latter in [29] gave atomic characterizations, respectively. Garcia-Cuerva in [15] studies weighted Hardy spaces $H^p_\omega(\mathbb{R}^n)$ for Muckenhoupt's weights ω,

Supported by NNSF of China grants (11771223).

DOI: 10.21136/CMJ.2022.0115-21

Gundy and Wheeden in [20] gave a characterization of $H^p_\omega(\mathbb{R}^n)$ by Lusin area integral. We also refer the readers to the works about Musielak-Orlicz Hardy spaces in [33], [34].

In recent decades, multi-parameter Hardy spaces have been developed greatly. In the fundamental work (see [19]), Gundy and Stein developed the product Hardy space $H^p(\mathbb{R}\times\mathbb{R})$. Using discrete Calderón's identity, Han, Lu and Zhao in [25] proved that product Hardy space can also be characterised by a discrete Littlewood-Paley square function. After that, authors in [4], [30], [31] studied multi-parameter weighted Hardy spaces. Following the footprints, multi-parameter local Hardy spaces and multi-parameter mixed Hardy spaces have been introduced recently in [5], [7], respectively.

Let $\psi_0^{(1)}, \psi^{(1)} \in \mathcal{S}(\mathbb{R}^{n_1})$ with

$$\operatorname{supp}\widehat{\psi_0^{(1)}} \subseteq \{\xi\in\mathbb{R}^{n_1}\colon\ |\xi|\leqslant 2\};\quad \widehat{\psi_0^{(1)}}(\xi)=1\quad \text{if } |\xi|\leqslant 1, \tag{1.1}$$

$$\operatorname{supp}\widehat{\psi^{(1)}} \subseteq \Big\{\xi\in\mathbb{R}^{n_1}\colon\ \frac{1}{2}\leqslant|\xi|\leqslant 2\Big\} \tag{1.2}$$

and

$$|\widehat{\psi_0^{(1)}}(\xi)|^2+\sum_{j=1}^{\infty}|\widehat{\psi^{(1)}}(2^{-j}\xi)|^2=1\quad \forall\,\xi\in\mathbb{R}^{n_1}. \tag{1.3}$$

Let $\psi^{(2)}\in\mathcal{S}(\mathbb{R}^{n_2})$ with

$$\operatorname{supp}\widehat{\psi^{(2)}} \subseteq \Big\{\xi\in\mathbb{R}^{n_2}\colon\ \frac{1}{2}\leqslant|\xi|\leqslant 2\Big\} \tag{1.4}$$

and

$$\sum_{j\in\mathbb{Z}}|\widehat{\psi^{(2)}}(2^{-j}\xi)|^2=1\quad \forall\,\xi\in\mathbb{R}^{n_2}\setminus\{0\}. \tag{1.5}$$

Then for $j,k\in\mathbb{Z}$, $j\geqslant 1$, set $\psi_j^{(1)}(x)=2^{jn_1}\psi^{(1)}(2^jx)$, $\psi_k^{(2)}(x)=2^{kn_2}\psi^{(2)}(2^kx)$ and $\psi_{j,k}(x,y)=\psi_j^{(1)}(x)\psi_k^{(2)}(y)$, $\psi_{0,k}(x,y)=\psi_0^{(1)}(x)\psi_k^{(2)}(y)$.

Denote $\mathcal{S}_0(\mathbb{R}^{n_1+n_2})=\{f\in\mathcal{S}(\mathbb{R}^{n_1+n_2})\colon\ \int_{\mathbb{R}^{n_2}}f(x_1,x_2)x_2^\alpha\,\mathrm{d}x_2=0$ for all $|\alpha|\geqslant 0$ for all $x_1\in\mathbb{R}^{n_1}\}$, $\mathcal{S}_M(\mathbb{R}^n)=\{\phi\in\mathcal{S}(\mathbb{R}^n)\colon\ \int_{\mathbb{R}^n}\phi(x)x^\alpha\,\mathrm{d}x=0$ for all $0\leqslant|\alpha|\leqslant M\}$, and $\mathcal{S}_\infty(\mathbb{R}^n)=\{\phi\in\mathcal{S}(\mathbb{R}^n)\colon\ \int_{\mathbb{R}^n}\phi(x)x^\alpha\,\mathrm{d}x=0$ for all $|\alpha|\geqslant 0\}$. For $i=1,2$ and any $j\in\mathbb{Z}$, denote $\Pi_j^{n_i}=\{I\colon\ I$ are dyadic cubes in $\mathbb{R}^{n_i}$ with the side length $l(I)=2^{-j}$, and the left lower corners of I are $x_I=2^{-j}\ell$, $\ell\in\mathbb{Z}^{n_i}\}$, and $\Pi=\bigcup\limits_{\substack{j\in\mathbb{N}\\ k\in\mathbb{Z}}}\Pi_j^{n_1}\times\Pi_k^{n_2}$.

By taking the Fourier transform, we have the following continuous Calderón's identity:

$$f(x) = \sum_{j=0}^{\infty} \sum_{k=-\infty}^{\infty} \psi_{j,k} * \psi_{j,k} * f(x), \tag{1.6}$$

where the series converges in $L^2(\mathbb{R}^{n_1+n_2})$, $\mathcal{S}_0(\mathbb{R}^{n_1+n_2})$ and its dual space $\mathcal{S}_0'(\mathbb{R}^{n_1+n_2})$. Moreover, (1.6) can be discretized as follows.

Theorem A. *Suppose that* $\psi_0^{(1)}, \psi^{(1)} \in \mathcal{S}(\mathbb{R}^{n_1})$, $\psi^{(2)} \in \mathcal{S}(\mathbb{R}^{n_2})$ *are functions satisfying conditions* (1.1)–(1.5), *see* [7]. *Then*
(1.7)

$$\begin{aligned} f(x_1,x_2) = & \sum_{j=0}^{\infty} \sum_{k=-\infty}^{\infty} \sum_{(\ell_1,\ell_2)\in\mathbb{Z}^{n_1}\times\mathbb{Z}^{n_2}} 2^{-jn_1-kn_2}(\psi_{j,k} * f)(2^{-j}\ell_1, 2^{-k}\ell_2) \\ & \times \psi_{j,k}(x_1 - 2^{-j}\ell_1, x_2 - 2^{-k}\ell_2) \\ = & \sum_{j=0}^{\infty} \sum_{k=-\infty}^{\infty} \sum_{I\times J\in\Pi_j^{n_1}\times\Pi_k^{n_2}} |I||J|(\psi_{j,k} * f)(x_I, x_J)\psi_{j,k}(x_1 - x_I, x_2 - x_J), \end{aligned}$$

where the series converges in $L^2(\mathbb{R}^{n_1+n_2})$, $\mathcal{S}_0(\mathbb{R}^{n_1+n_2})$ *and* $\mathcal{S}_0'(\mathbb{R}^{n_1+n_2})$.

According to discrete Calderón's identity (1.7), mixed Hardy spaces are introduced in [7]. In the present paper, on the basis of [7], we consider the weighted multi-parameter mixed Hardy spaces and study the boundedness of some operators on them. For this purpose, we now recall some definitions of product weights in two parameter setting, see [16]. For $1 < p < \infty$, a weight function, a nonnegative locally integrable function $\omega \in A_p(\mathbb{R}^{n_1} \times \mathbb{R}^{n_2})$ if there exists a constant $C > 0$ such that

$$\left(\frac{1}{|R|}\int_R \omega(x)\,\mathrm{d}x\right)\left(\frac{1}{|R|}\int_R \omega(x)^{-1/(p-1)}\,\mathrm{d}x\right)^{p-1} < C$$

for any dyadic cuboid $R = I \times J$ on $\mathbb{R}^{n_1} \times \mathbb{R}^{n_2}$, that is, I and J are cubes on $\mathbb{R}^{n_1}$, $\mathbb{R}^{n_2}$, respectively. We say $\omega \in A_1(\mathbb{R}^{n_1} \times \mathbb{R}^{n_2})$ if there exists a constant $C > 0$ such that

$$M_s\omega(x) \leqslant C\omega(x)$$

for almost every $x \in \mathbb{R}^{n_1+n_2}$, where M_s is the strong maximal function defined by

$$M_s f(x) = \sup_{x\in R} \frac{1}{|R|}\int_R |f(v)|\,\mathrm{d}v$$

for any cuboid R on $\mathbb{R}^{n_1} \times \mathbb{R}^{n_2}$.

At last we define $\omega \in A_\infty(\mathbb{R}^{n_1} \times \mathbb{R}^{n_2})$ by

$$A_\infty(\mathbb{R}^{n_1} \times \mathbb{R}^{n_2}) = \bigcup_{1\leqslant p<\infty} A_p(\mathbb{R}^{n_1} \times \mathbb{R}^{n_2}).$$

If $\omega \in A_\infty(\mathbb{R}^{n_1} \times \mathbb{R}^{n_2})$, then $q_\omega = \inf\{q\colon \omega \in A_q\}$ is called the *critical index* of $\omega(\mathbb{R}^{n_1} \times \mathbb{R}^{n_2})$. Notice that if $\omega \in A_\infty$, then $q_\omega < \infty$. In this paper, we use $A_p(\mathbb{R}^n)$ to denote the classical Muckenhoupt's weights on $\mathbb{R}^n$.

Given a weight function ω on $\mathbb{R}^n$ for $0 < r < \infty$ we define $L^r_\omega(\mathbb{R}^n)$ as

$$L^r_\omega(\mathbb{R}^n) = \left\{f\colon \int_{\mathbb{R}^n} |f(x)|^r \omega(x)\,\mathrm{d}x < \infty\right\}.$$

With discrete Calderón's identity, we define the following weighted multi-parameter mixed Hardy spaces.

Definition 1.1. Let $0 < p < \infty$ and $\omega \in A_\infty(\mathbb{R}^{n_1} \times \mathbb{R}^{n_2})$. Suppose that $\psi^{(1)}_0, \psi^{(1)} \in \mathcal{S}(\mathbb{R}^{n_1})$, $\psi^{(2)} \in \mathcal{S}(\mathbb{R}^{n_2})$ are functions satisfying conditions (1.1)–(1.5). The *weighted multiparameter mixed Hardy space* $H^p_{\rm mix}(\omega, \mathbb{R}^{n_1} \times \mathbb{R}^{n_2})$ is defined by

$$\begin{aligned} &H^p_{\rm mix}(\omega, \mathbb{R}^{n_1} \times \mathbb{R}^{n_2}) \\ &\qquad = \{f \in \mathcal{S}'_0(\mathbb{R}^{n_1+n_2})\colon \|f\|_{H^p_{\rm mix}(\omega,\mathbb{R}^{n_1}\times\mathbb{R}^{n_2})} = \|S(f)(x)\|_{L^p_\omega(\mathbb{R}^{n_1+n_2})} < \infty\}, \end{aligned}$$

where

$$S(f)(x) = \biggl(\sum_{\substack{j\in\mathbb{N}\\ k\in\mathbb{Z}}} \sum_{I\times J\in\Pi^{n_1}_j\times\Pi^{n_2}_k} |\psi_{j,k} * f(x_I, x_J)|^2 \chi_I(x_1)\chi_J(x_2)\biggr)^{1/2}. \tag{1.8}$$

Here the symbols $\mathbb{N}$, $\mathbb{Z}$ denote the sets of natural numbers, integers, respectively. By discrete Calderón's identity (1.7), we can prove that the multi-parameter mixed Hardy spaces are well defined, that is, for $0 < p < \infty$, the space $H^p_{\rm mix}(\omega, \mathbb{R}^{n_1} \times \mathbb{R}^{n_2})$ is independent of the choice of functions $\psi^{(1)}_0$, $\psi^{(1)}$ and $\psi^{(2)}$, due to the following result.

Theorem 1.1. *Let $0 < p < \infty$ and $\omega \in A_\infty(\mathbb{R}^{n_1} \times \mathbb{R}^{n_2})$. Suppose that $\psi^{(1)}_0, \psi^{(1)} \in \mathcal{S}(\mathbb{R}^{n_1})$, $\psi^{(2)} \in \mathcal{S}(\mathbb{R}^{n_2})$ are functions satisfying conditions* (1.1)–(1.5)*, and $\phi^{(1)}_0, \phi^{(1)} \in \mathcal{S}(\mathbb{R}^{n_1})$, $\phi^{(2)} \in \mathcal{S}(\mathbb{R}^{n_2})$ is another group satisfying conditions* (1.1)–(1.5). *Then for any $f \in \mathcal{S}'_0(\mathbb{R}^{n_1} \times \mathbb{R}^{n_2})$, one has*

$$\begin{aligned} &\biggl\|\biggl(\sum_{\substack{j\in\mathbb{N}\\ k\in\mathbb{Z}}} \sum_{I\times J\in\Pi^{n_1}_j\times\Pi^{n_2}_k} |\psi_{j,k} * f(x_I, x_J)|^2 \chi_I(x)\chi_J(y)\biggr)^{1/2}\biggr\|_{L^p_\omega(\mathbb{R}^{n_1+n_2})} \\ &\qquad \approx \biggl\|\biggl(\sum_{\substack{j\in\mathbb{N}\\ k\in\mathbb{Z}}} \sum_{I\times J\in\Pi^{n_1}_j\times\Pi^{n_2}_k} |\phi_{j,k} * f(x_I, x_J)|^2 \chi_I(x)\chi_J(y)\biggr)^{1/2}\biggr\|_{L^p_\omega(\mathbb{R}^{n_1+n_2})}. \end{aligned}$$

We refer the reader to Theorem 1.3 in [7] for the proof of Theorem 1.1. The only difference is in the last step, where $L^{p/r}(\ell^{2/r})$ is replaced by $L^{p/r}_\omega(\ell^{2/r})$.

A locally integrable function defined away from the diagonal $x = y$ in $\mathbb{R}^n \times \mathbb{R}^n$ is called a *one-parameter Calderón-Zygmund kernel* with regularity exponent $\varepsilon > 0$ if there exists a constant $C > 0$ such that

$$|\mathcal{K}(x,y)| \leqslant C\frac{1}{|x-y|^n} \quad \text{for } x \neq y, \quad \text{and} \quad |\mathcal{K}(x,y) - \mathcal{K}(x,y')| \leqslant C\frac{|y-y'|^\varepsilon}{|x-y|^{n+\varepsilon}}$$

whenever $|y - y'| \leqslant \frac{1}{2}|x-y|$, and

$$|\mathcal{K}(x,y) - \mathcal{K}(x',y)| \leqslant C\frac{|x-x'|^\varepsilon}{|x-y|^{n+\varepsilon}}$$

whenever $|x - x'| \leqslant \frac{1}{2}|x-y|$. The smallest such constant C is denoted by $\|\mathcal{K}\|_{\mathrm{CZ}}$. We call an operator T *one-parameter Calderón-Zygmund operator* if T is a singular integral operator associated with a one-parameter Calderón-Zygmund kernel $\mathcal{K}(x,y)$ given by $\langle T(f), g\rangle = \int \mathcal{K}(x,y)f(y)g(x)\,\mathrm{d}x\,\mathrm{d}y$ for all $f, g \in \mathcal{S}(\mathbb{R}^n)$ with disjoint supports. Define $\|T\|_{\mathrm{CZ}}$ by $\|T\|_{\mathrm{CZ}} = \|\mathcal{K}\|_{\mathrm{CZ}} + \|T\|_{L^2\to L^2}$. Then T is bounded on $L^p(\mathbb{R}^n)$ $(1 < p < \infty)$ if $\|T\|_{\mathrm{CZ}} < \infty$. However, by a classical result, the H^p $(0 < p \leqslant 1)$ boundedness of T implies that the operator must satisfy additional condition: $\int T(\varphi)(x)\,\mathrm{d}x = 0$ for any $\varphi \in \mathcal{S}_\infty(\mathbb{R}^n)$, namely $T^*(1) = 0$. Moreover, to ensure the boundedness of T on local Hardy spaces h^p $(0 < p \leqslant 1)$, additional condition $|\mathcal{K}(x,y)| \leqslant C1/|x-y|^{n+\delta}$ for some $\delta > 0$ should be included, see [17].

In the two-parameter setting, instead of the classical one-parameter dilation, mathematicians usually consider the dilations $\delta\colon x \to (\delta_1 x_1, \delta_2 x_2)$, $x = (x_1, x_2) \in \mathbb{R}^{n_1} \times \mathbb{R}^{n_2}$, $\delta = (\delta_1, \delta_2)$, $\delta_1, \delta_2 > 0$, see [11], [12], [13], [21], [27], [28]. For instance, Fefferman and Stein studied the product convolution singular integral operators which satisfy analogous conditions enjoyed by the double Hilbert transform defined on $\mathbb{R} \times \mathbb{R}$, see [13]. Journé introduced non-convolution product singular integral operators in [27]. After that, more and more new results about operators in Journé class were obtained, see [11], [12], [21], [28]. Inspired by those, the authors in [6] introduced the following mixed Journé class to discuss the boundedness of singular integral operators on $H^p_{\mathrm{mix}}(\mathbb{R}^{n_1} \times \mathbb{R}^{n_2})$.

Definition 1.2. A singular integral operator T is said to be in *mixed Journé class* on $\mathbb{R}^{n_1} \times \mathbb{R}^{n_2}$ with regularity exponent $\varepsilon \in (0,1]$ and $\delta > 0$ if

$$T(f)(x_1,x_2) = \int_{\mathbb{R}^{n_1}\times\mathbb{R}^{n_2}} \mathcal{K}(x_1,y_1,x_2,y_2)f(y_1,y_2)\,\mathrm{d}y_1\,\mathrm{d}y_2,$$

where the kernel $\mathcal{K}$ satisfies the following conditions.

For each $x_1, y_1 \in \mathbb{R}^{n_1}$, set $\widetilde{\mathcal{K}}^1(x_1,y_1)$ to be the singular integral operator acting on functions on $\mathbb{R}^{n_2}$ with the kernel

$$\widetilde{\mathcal{K}}^1(x_1,y_1)(x_2,y_2) = \mathcal{K}(x_1,y_1,x_2,y_2)$$

and similarly, $\widetilde{\mathcal{K}}^2(x_2,y_2)(x_1,y_1) = \mathcal{K}(x_1,y_1,x_2,y_2)$. Then there exists a constant $C>0$ such that

$$(1)\quad \|\widetilde{\mathcal{K}}^1(x_1,y_1)\|_{\mathrm{CZ}} \leqslant C\min\Big\{\frac{1}{|x_1-y_1|^{n_1}}, \frac{1}{|x_1-y_1|^{n_1+\delta}}\Big\} \quad \text{if } |x_1-y_1|>0;$$

$$\|\widetilde{\mathcal{K}}^1(x_1,y_1)-\widetilde{\mathcal{K}}^1(x_1,y_1')\|_{\mathrm{CZ}} \leqslant C\frac{|y_1-y_1'|^{\varepsilon}}{|x_1-y_1|^{n_1+\varepsilon}} \quad \text{if } |y_1-y_1'|\leqslant \frac{1}{2}|x_1-y_1|;$$

$$(2)\quad \|\widetilde{\mathcal{K}}^1(x_1,y_1)-\widetilde{\mathcal{K}}^1(x_1',y_1)\|_{\mathrm{CZ}} \leqslant C\frac{|x_1-x_1'|^{\varepsilon}}{|x_1-y_1|^{n_1+\varepsilon}} \quad \text{if } |x_1-x_1'|\leqslant \frac{1}{2}|x_1-y_1|;$$

$$\|\widetilde{\mathcal{K}}^2(x_2,y_2)\|_{\mathrm{CZ}} \leqslant C\frac{1}{|x_2-y_2|^{n_2}} \quad \text{if } |x_2-y_2|>0;$$

$$\|\widetilde{\mathcal{K}}^2(x_2,y_2)-\widetilde{\mathcal{K}}^2(x_2,y_2')\|_{\mathrm{CZ}} \leqslant C\frac{|y_2-y_2'|^{\varepsilon}}{|x_2-y_2|^{n_2+\varepsilon}} \quad \text{if } |y_2-y_2'|\leqslant \frac{1}{2}|x_2-y_2|;$$

$$\|\widetilde{\mathcal{K}}^2(x_2,y_2)-\widetilde{\mathcal{K}}^2(x_2',y_2)\|_{\mathrm{CZ}} \leqslant C\frac{|x_2-x_2'|^{\varepsilon}}{|x_2-y_2|^{n_2+\varepsilon}} \quad \text{if } |x_2-x_2'|\leqslant \frac{1}{2}|x_2-y_2|.$$

Theorem 1.2. *Let $\omega \in A_2(\mathbb{R}^{n_1}\times\mathbb{R}^{n_2})$. Suppose that T is a singular integral operator in mixed Journé class on $\mathbb{R}^{n_1}\times\mathbb{R}^{n_2}$ with regularity exponent $\varepsilon\in(0,1]$, $\delta>0$ and bounded on $L^2_\omega(\mathbb{R}^{n_1+n_2})$. Then T is bounded on $H^p_{\mathrm{mix}}(\omega,\mathbb{R}^{n_1}\times\mathbb{R}^{n_2})$ for $\max\{n_1/(n_1+\delta), n_1/(n_1+\varepsilon), n_2/(n_2+\varepsilon)\} < p \leqslant 1$ if $T_1(1)=T_2(1)=T_1^*(1)=T_2^*(1)=0$.*

The meaning of $T_1(1)=0$ is as follows: for any fixed $y_2\in\mathbb{R}^{n_2}$,

$$\int \mathcal{K}(x_1,y_1,x_2,y_2)\psi_1(x_1)\psi_2(x_2)\,\mathrm{d}x_1\,\mathrm{d}x_2\,\mathrm{d}y_1 = 0 \quad \text{for any } \psi_i\in\mathcal{S}_M(\mathbb{R}^{n_i}).$$

Similarly for $T_2(1)=0$, where $M=M_{p,n_1,n_2}$ is a large positive integer dependent only on p, n_1, n_2. $T_1^*(1)=0$ is known for any fixed $x_2\in\mathbb{R}^{n_2}$ as

$$\int \mathcal{K}(x_1,y_1,x_2,y_2)\psi_1(y_1)\psi_2(y_2)\,\mathrm{d}y_1\,\mathrm{d}y_2\,\mathrm{d}x_1 = 0 \quad \text{for any } \psi_i\in\mathcal{S}_M(\mathbb{R}^{n_i}),\ i=1,2,$$

similarly for $T_2^*(1)=0$.

As a corollary of Theorem 1.2, we have the following result.

Theorem 1.3. *Let $\omega\in A_2(\mathbb{R}^{n_1}\times\mathbb{R}^{n_2})$. Suppose that T is a singular integral operator satisfying the conditions of Theorem* 1.2. *Then T can be extended to a bounded operator from $H^p_{\mathrm{mix}}(\omega,\mathbb{R}^{n_1}\times\mathbb{R}^{n_2})$ to $L^p_\omega(\mathbb{R}^{n_1+n_2})$ if $\max\{n_1/(n_1+\delta), n_1/(n_1+\varepsilon), n_2/(n_2+\varepsilon)\}<p\leqslant 1$.*

Finally, we establish some conventions. Throughout the article, C denotes a positive constant that is independent of the main parameters involved, but whose value may vary from line to line. Constants in subscript, such as C_1, do not change in different occurrences. We denote $f\leqslant Cg$ by $f\lesssim g$. If $f\lesssim g\lesssim f$, we write $f\approx g$.

2. Almost orthogonality estimates

In this section, we give some almost orthogonality estimates which are critical in the discrete Littlewood-Paley theory. The following almost orthogonality estimates can be seen in [18].

Lemma 2.1. *Let $\psi, \varphi \in \mathcal{S}(\mathbb{R}^n)$ and $j, k \in \mathbb{Z}$, $j \leqslant k$. Suppose that $\int \varphi(x) x^\alpha \, \mathrm{d}x = 0$ for all $|\alpha| \leqslant N-1$ for a positive integer N. Then for any given positive integer L there exists a constant C dependent only on ψ, φ, n and L such that*

$$|\psi_j * \varphi_k(x)| \leqslant C \frac{2^{jn} 2^{-(k-j)N}}{(1+2^j|x|)^L}.$$

It is convenient to set $j \wedge k = \min\{j,k\}$. Then by Lemma 2.1 it is easy to have some corollaries.

Lemma 2.2. *Let $\varphi, \psi \in \mathcal{S}_\infty(\mathbb{R}^n)$ and $\phi \in \mathcal{S}(\mathbb{R}^n)$. Then for any given positive integer N, L there exist constants C_1 and C_2 independent of what such that*

$$|\varphi_j * \psi_k(x)| \leqslant C_1 \frac{2^{(j\wedge k)n} 2^{-|j-k|N}}{(1+2^{(j\wedge k)}|x|)^L} \quad \forall\, j,k \in \mathbb{Z} \quad \text{and} \quad |\phi * \psi_i(x)| \leqslant C_2 \frac{2^{-iN}}{(1+|x|)^L} \quad \forall\, i \in \mathbb{N}.$$

Furthermore, one can deduce the following results.

Lemma 2.3. *Suppose that $\psi^{(i)} \in \mathcal{S}_\infty(\mathbb{R}^{n_i})$, $i = 1,2$, $f \in \mathcal{S}_0(\mathbb{R}^{n_1+n_2})$. Then for any given positive integers N_1, N_2, L_1, L_2,*

$$|\psi_{j,k} * f(x,y)| \leqslant C \frac{2^{-jN_1}}{(1+|x|)^{L_1}} \frac{2^{-|k|N_2}}{(1+|y|)^{L_2}} \quad \forall\, j,k \in \mathbb{Z},\ j>0, \tag{2.1}$$

where C is constant independent of j, k.

Lemma 2.4. *Suppose that $\psi_0^{(1)} \in \mathcal{S}(\mathbb{R}^{n_1})$, $\psi^{(2)} \in \mathcal{S}_\infty(\mathbb{R}^{n_2})$, $f \in \mathcal{S}_0(\mathbb{R}^{n_1+n_2})$. Then for any given positive integers N_2, L_1, L_2,*

$$|\psi_{0,k} * f(x,y)| \leqslant C \frac{1}{(1+|x|)^{L_1}} \frac{2^{-|k|N_2}}{(1+|y|)^{L_2}} \quad \forall\, k \in \mathbb{Z}, \tag{2.2}$$

where C is a constant independent of k.

With a similar proof of Lemma 4.1 in [7], one can obtain the following estimates of the discrete version about strong maximal function.

Lemma 2.5. *Let $j, j' \in \mathbb{N}$, $k, k' \in \mathbb{Z}$, $I \times J \in \Pi_j^{n_1} \times \Pi_k^{n_2}$ and $\Lambda \subseteq \Pi_{j'}^{n_1} \times \Pi_{k'}^{n_2}$. Suppose that $\phi_0^{(1)}, \phi^{(1)} \in \mathcal{S}(\mathbb{R}^{n_1}), \phi^{(2)} \in \mathcal{S}(\mathbb{R}^{n_2})$ are functions satisfying conditions (1.1)–(1.5). Then for any $f \in \mathcal{S}_0'(\mathbb{R}^{n_1} \times \mathbb{R}^{n_2})$, $u, u^* \in I$, $v, v^* \in J$, we have*

$$\sum_{(I',J')\in\Lambda} \frac{2^{(j\wedge j')n_1} 2^{(k\wedge k')n_2} |I'||J'|}{(1+2^{j\wedge j'}|u-x_{I'}|)^L (1+2^{k\wedge k'}|v-x_{J'}|)^L} |\phi_{j',k'} * f(x_{I'}, y_{J'})|$$
$$\leqslant C_1 \bigg\{ M_s \bigg(\sum_{(I',J')\in\Lambda} |\phi_{j',k'} * f(x_{I'}, y_{J'})|^2 \chi_{I'} \chi_{J'} \bigg)^{\sigma/2} (u^*, v^*) \bigg\}^{1/\sigma},$$

where $C_1 = C2^{(1-1/\sigma)[n_1(j\wedge j'-j')+n_2(k\wedge k'-k')]}$ and $\max\{n_1/L, n_2/L\} < \sigma \leqslant 1$, which can be arbitrarily small if L is big enough.

The following result is about the boundedness of maximal function on vector-value spaces.

Lemma 2.6 ([3], Theorem 1.12)**.** *Suppose that $1 < p, q < \infty$, $\omega \in A_p(\mathbb{R}^n)$. Then*

$$\|\{M(f_i)\}_i\|_{L^p_\omega(l^q)} \leqslant C_{n,p,q,\omega} \|\{f_i\}_i\|_{L^p_\omega(l^q)},$$

where M denotes the Hardy-Littlewood maximal operator and

$$L^p_\omega(l^q) = \bigg\{ f = \{f_v\} \colon \|f\|_{L^p(l^q)} = \bigg\| \bigg(\sum_v |f_v|^q \bigg)^{1/q} \bigg\|_{L^p(\omega)} < \infty \bigg\}.$$

Remark 2.1. Since product weighted $\omega \in A_p(\mathbb{R}^{n_1} \times \mathbb{R}^{n_2})$ for a.e. $x \in \mathbb{R}^{n_1}$, $y \in \mathbb{R}^{n_2}$, $\omega(\cdot, y) \in A_p(\mathbb{R}^{n_1})$, $\omega(x, \cdot) \in A_p(\mathbb{R}^{n_2})$, uniformly. Moreover, the strong maximal operator $M_s \leqslant M \circ M$. By iteration, Lemma 2.6 also holds for M_s when $\omega \in A_p(\mathbb{R}^{n_1} \times \mathbb{R}^{n_2})$.

3. Some Propositions of $H^p_{\rm mix}(\omega, \mathbb{R}^{n_1} \times \mathbb{R}^{n_2})$

Proposition 3.1. *Let $\omega \in A_\infty(\mathbb{R}^{n_1} \times \mathbb{R}^{n_2})$. Then $\mathcal{S}_0(\mathbb{R}^{n_1+n_2})$ is dense in $H^p_{\rm mix}(\omega, \mathbb{R}^{n_1} \times \mathbb{R}^{n_2})$.*

P r o o f. We first prove $\mathcal{S}_0(\mathbb{R}^{n_1+n_2}) \subseteq H^p_{\rm mix}(\omega, \mathbb{R}^{n_1} \times \mathbb{R}^{n_2})$. Let $f \in \mathcal{S}_0(\mathbb{R}^{n_1+n_2})$, then by definition,

$$\|f\|^p_{H^p_{\rm mix}(\omega, \mathbb{R}^{n_1}\times\mathbb{R}^{n_2})} = \bigg\| \bigg(\sum_{\substack{j\in\mathbb{N}\\ k\in\mathbb{Z}}} \sum_{I\times J\in\Pi_j^{n_1}\times\Pi_k^{n_2}} |\psi_{j,k} * f(x_I, x_J)|^2 \chi_I(x_1)\chi_J(x_2) \bigg)^{1/2} \bigg\|^p_{L^p_\omega(\mathbb{R}^{n_1+n_2})}.$$

By (2.1) and (2.2) for any given positive integers N_1, N_2, L_1, L_2 one has

$$|\psi_{j,k} * f(x,y)| \leqslant C \frac{2^{-jN_1}}{(1+|x|)^{L_1}} \frac{2^{-|k|N_2}}{(1+|y|)^{L_2}} \quad \forall\, j,k \in \mathbb{Z},\ j \geqslant 0.$$

Then

$$|\psi_{j,k} * f(x_I, x_J)|^2 \chi_I(x_1)\chi_J(x_2) \lesssim \frac{2^{-2jN_1}}{(1+|x_I|)^{2L_1}} \frac{2^{-2|k|N_2}}{(1+|x_J|)^{2L_2}} \chi_I(x_1)\chi_J(x_2).$$

If $a, b \in J \in \Pi_i^n$, one has $1+|a| \lesssim 2^{|i|} + |b| \leqslant 2^{|i|}(1+|b|)$. Hence,

$$|\psi_{j,k} * f(x_I, x_J)|^2 \chi_I(x_1)\chi_J(x_2) \lesssim \frac{2^{-2jN_1}}{(1+|x_1|)^{2L_1}} \frac{2^{-2|k|(N_2-L_2)}}{(1+|x_2|)^{2L_2}} \chi_I(x_1)\chi_J(x_2),$$

choosing N_2, L_2 such that $N_2 - L_2 > 0$, which implies that

$$\begin{aligned}
&\|f\|^p_{H^p_{\mathrm{mix}}(\omega, \mathbb{R}^{n_1}\times\mathbb{R}^{n_2})} \\
&\quad\lesssim \Bigg\| \sum_{\substack{j\in\mathbb{N}\\ k\in\mathbb{Z}}} \sum_{I\times J \in \Pi_j^{n_1}\times\Pi_k^{n_2}} \frac{2^{-jN_1}}{(1+|x_1|)^{L_1}} \frac{2^{-|k|(N_2-L_2)}}{(1+|x_2|)^{L_2}} \chi_I(x_1)\chi_J(x_2) \Bigg\|^p_{L^p_\omega(\mathbb{R}^{n_1+n_2})} \\
&\quad\lesssim \int \frac{1}{(1+|x_1|)^{pL_1}} \frac{1}{(1+|x_2|)^{pL_2}} \omega(x_1,x_2)\,\mathrm{d}x_1\,\mathrm{d}x_2 \\
&\quad= \sum_{j,k\in\mathbb{N}} \int_{2^j\leqslant|x_1|<2^{j+1}} \int_{2^k\leqslant|x_2|<2^{k+1}} \frac{1}{(1+|x_1|)^{pL_1}} \frac{1}{(1+|x_2|)^{pL_2}} \omega(x_1,x_2)\,\mathrm{d}x_1\,\mathrm{d}x_2 \\
&\quad\lesssim \sum_{j,k\in\mathbb{N}} 2^{-jpL_1} 2^{-kpL_2} \int_{|x_1|<2^{j+1}} \int_{|x_2|<2^{k+1}} \omega(x_1,x_2)\,\mathrm{d}x_1\,\mathrm{d}x_2.
\end{aligned}$$

By Lebesgue differential theorem, it is easy to see that if $\omega \in A_q(\mathbb{R}^{n_1}\times\mathbb{R}^{n_2})$, $\omega(\cdot, x_2) \in A_q(\mathbb{R}^{n_1})$ for a.e. $x_2 \in \mathbb{R}^{n_2}$ uniformly. It implies that $\omega(x_1,x_2)\,\mathrm{d}x_1$ is a doubling measure on $\mathbb{R}^{n_1}$ for a.e. $x_2 \in \mathbb{R}^{n_2}$ uniformly, precisely, for all $\lambda > 1$ and all balls $B \subseteq \mathbb{R}^{n_1}$, we have

$$\int_{\lambda B} \omega(x_1,x_2)\,\mathrm{d}x_1 \leqslant C\lambda^{n_1 q} \int_B \omega(x_1,x_2)\,\mathrm{d}x_1,$$

where C is a constant independent of x_2, see [18]. This result also holds for $\omega(x_1,x_2)\,\mathrm{d}x_2$. Therefore,

$$\|f\|^p_{H^p_{\mathrm{mix}}(\omega, \mathbb{R}^{n_1}\times\mathbb{R}^{n_2})} \lesssim \sum_{j,k\in\mathbb{N}} 2^{-jpL_1} 2^{-kpL_2} 2^{2jn_1} 2^{2kn_2} \int_{|x_1|<1} \int_{|x_2|<1} \omega(x_1,x_2)\,\mathrm{d}x_1\,\mathrm{d}x_2,$$

which yields that $f \in H^p_{\mathrm{mix}}(\omega, \mathbb{R}^{n_1}\times\mathbb{R}^{n_2})$ since $\omega \in A_2(\mathbb{R}^{n_1}\times\mathbb{R}^{n_2})$ and L_1, L_2 can be big enough.

We now prove the density of $\mathcal{S}_0(\mathbb{R}^{n_1+n_2})$ in $H^p_{\mathrm{mix}}(\omega, \mathbb{R}^{n_1} \times \mathbb{R}^{n_2})$. Let $f \in H^p_{\mathrm{mix}}(\omega, \mathbb{R}^{n_1} \times \mathbb{R}^{n_2})$. Then by (1.7),

$$f(x_1, x_2) = \sum_{j=0}^{\infty} \sum_{k=-\infty}^{\infty} \sum_{I \times J \in \Pi_j^{n_1} \times \Pi_k^{n_2}} |I||J|(\psi_{j,k} * f)(x_I, x_J)\psi_{j,k}(x_1 - x_I, x_2 - x_J).$$

For any fixed $N > 0$, denote

$$\begin{aligned} E_N = \{(j,k,I,J)\colon\ & j \in \mathbb{N},\ k \in \mathbb{Z},\ I \in \Pi_j^{n_1},\ J \in \Pi_k^{n_2}, \\ & j \leqslant N,\ |k| \leqslant N,\ |x_I| \leqslant N,\ |x_J| \leqslant N\}, \end{aligned}$$

and

$$f_N(x_1, x_2)\colon = \sum_{(j,k,I,J) \in E_N} |I||J|(\psi_{j,k} * f)(x_I, x_J)\psi_{j,k}(x_1 - x_I, x_2 - x_J).$$

Obviously, $f_N \in \mathcal{S}_0(\mathbb{R}^{n_1+n_2})$. Moreover, for fixed $j \in \mathbb{N}$, $k \in \mathbb{Z}$, let $(E_N^{j,k})^c$ be the orthogonal projection of $(E_N)^c$ on $\Pi_j^{n_1} \times \Pi_k^{n_2}$, where $(E_N)^c$ is the complementary of E_N on $\{(j,k,I,J)\colon\ j \in \mathbb{N},\ k \in \mathbb{Z},\ I \in \Pi_j^{n_1},\ J \in \Pi_k^{n_2}\}$. Then

$$f(x_1, x_2) - f_N(x_1, x_2) := \sum_{\substack{j \in \mathbb{N} \\ k \in \mathbb{Z}}} \sum_{I \times J \in (E_N^{j,k})^c} |I||J|(\psi_{j,k} * f)(x_I, x_J)\psi_{j,k}(x_1 - x_I, x_2 - x_J)$$

in $\mathcal{S}_0'(\mathbb{R}^{n_1+n_2})$. To complete the proof, we only need to prove that

$$\|f - f_N\|_{H^p_{\mathrm{mix}}(\omega, \mathbb{R}^{n_1} \times \mathbb{R}^{n_2})} \to 0 \quad \text{as } N \to \infty.$$

By definition,

$$\begin{aligned} &\|f - f_N\|_{H^p_{\mathrm{mix}}(\omega, \mathbb{R}^{n_1} \times \mathbb{R}^{n_2})} \\ &= \bigg\| \bigg(\sum_{\substack{j' \in \mathbb{N} \\ k' \in \mathbb{Z}}} \sum_{I' \times J' \in \Pi_{j'}^{n_1} \times \Pi_{k'}^{n_2}} |\varphi_{j',k'} * (f - f_N)(x_{I'}, x_{J'})|^2 \chi_{I'}(x_1)\chi_{J'}(x_2) \bigg)^{1/2} \bigg\|_{L^p_\omega(\mathbb{R}^{n_1+n_2})}, \end{aligned}$$

where $\varphi_{j',k'}$ satisfies the conditions of Definition 1.1. It is easy to have that

$$\begin{aligned} &\varphi_{j',k'} * (f - f_N)(x_{I'}, x_{J'}) \\ &\quad = \sum_{\substack{j \in \mathbb{N} \\ k \in \mathbb{Z}}} \sum_{I \times J \in (E_N^{j,k})^c} |I||J|(\psi_{j,k} * f)(x_I, x_J)\varphi_{j',k'} * \psi_{j,k}(x_{I'} - x_I, x_{J'} - x_J). \end{aligned}$$

By Lemma 2.2,

$$\begin{aligned} (3.1) \qquad & |\varphi_{j',k'} * \psi_{j,k}(x_{I'} - x_I, x_{J'} - x_J)| \\ & \leqslant C \frac{2^{(j \wedge j')n_1} 2^{-|j-j'|N}}{(1 + 2^{(j \wedge j')}|x_{I'} - x_I|)^L} \frac{2^{(k \wedge k')n_2} 2^{-|k-k'|N}}{(1 + 2^{(k \wedge k')}|x_{J'} - x_J|)^L} \end{aligned}$$

for all j, j', k, k', which implies that

$$\begin{aligned}
&|\varphi_{j',k'} * (f - f_N)(x_{I'}, x_{J'})| \\
&\qquad \lesssim \sum_{\substack{j\in\mathbb{N}\\ k\in\mathbb{Z}}} \sum_{I\times J\in(E_N^{j,k})^c} |I||J||(\psi_{j,k} * f)(x_I, x_J)| \\
&\qquad\quad \times \frac{2^{(j\wedge j')n_1} 2^{-|j-j'|N}}{(1+2^{(j\wedge j')}|x_{I'} - x_I|)^L} \frac{2^{(k\wedge k')n_2} 2^{-|k-k'|N}}{(1+2^{(k\wedge k')}|x_{J'} - x_J|)^L} \\
&\qquad \lesssim \sum_{\substack{j\in\mathbb{N}\\ k\in\mathbb{Z}}} 2^{-|j-j'|N} 2^{-|k-k'|N} \\
&\qquad\quad \times C_1 \bigg\{ M_s \bigg(\sum_{I\times J\in(E_N^{j,k})^c} |\psi_{j,k} * f(x_I, x_J)|^2 \chi_I \chi_J \bigg)^{r/2} (u', v') \bigg\}^{1/r}
\end{aligned}$$

for any $u' \in I'$, $v' \in J'$ by Lemma 2.5, where $C_1 = 2^{(1-1/r)[n_1(j'\wedge j - j) + n_2(k'\wedge k - k)]}$ and $\max\{n_1/L, n_2/L\} < r \leqslant 1$.

Therefore, by Cauchy's inequality,

$$\begin{aligned}
&\sum_{\substack{j'\in\mathbb{N}\\ k'\in\mathbb{Z}}} \sum_{I'\times J'\in\Pi_{j'}^{n_1}\times\Pi_{k'}^{n_2}} |\varphi_{j',k'} * (f - f_N)(x_{I'}, x_{J'})|^2 \chi_{I'}(x_1)\chi_{J'}(x_2) \\
&\quad \leqslant C \sum_{\substack{j',k'\in\mathbb{Z}\\ j'\geqslant 0}} \bigg[\sum_{\substack{j\in\mathbb{N}\\ k\in\mathbb{Z}}} 2^{-|j-j'|N} 2^{-|k-k'|N} \\
&\qquad \times C_1 \bigg\{ M_s \bigg(\sum_{I\times J\in(E_N^{j,k})^c} |\psi_{j,k} * f(x_I, x_J)|^2 \chi_I \chi_J \bigg)^{r/2} (u', v') \bigg\}^{1/r} \bigg]^2 \\
&\quad \leqslant C \sum_{\substack{j',k'\in\mathbb{Z}\\ j'\geqslant 0}} \bigg\{ \bigg[\sum_{\substack{j\in\mathbb{N}\\ k\in\mathbb{Z}}} 2^{-|j-j'|N} 2^{-|k-k'|N} C_1 \bigg] \bigg[\sum_{\substack{j\in\mathbb{N}\\ k\in\mathbb{Z}}} 2^{-|j-j'|N} 2^{-|k-k'|N} \\
&\qquad \times C_1 \bigg\{ M_s \bigg(\sum_{I\times J\in(E_N^{j,k})^c} |\psi_{j,k} * f(x_I, x_J)|^2 \chi_I \chi_J \bigg)^{r/2} (u', v') \bigg\}^{2/r} \bigg] \bigg\}.
\end{aligned}$$

It can be seen that

$$2^{-|j-j'|N} 2^{-|k-k'|N} C_1 \leqslant C 2^{-|j-j'|[N-(1/r-1)n_1]} 2^{-|k-k'|[N-(1/r-1)n_2]}, \tag{3.2}$$

which yields that

$$\sum_{\substack{j,k\in\mathbb{Z}\\ j\geqslant 0}} 2^{-|j-j'|N} 2^{-|k-k'|N} C_1 \quad \text{and} \quad \sum_{\substack{j',k'\in\mathbb{Z}\\ j'\geqslant 0}} 2^{-|j-j'|N} 2^{-|k-k'|N} C_1$$

are all bounded by a constant independent of j, k, j', k' if N is big enough. Hence,

$$\sum_{\substack{j'\in\mathbb{N}\\k'\in\mathbb{Z}}}\sum_{I'\times J'\in\Pi_{j'}^{n_1}\times\Pi_{k'}^{n_2}}|\varphi_{j',k'}*(f-f_N)(x_{I'},x_{J'})|^2\chi_{I'}(x_1)\chi_{J'}(x_2)$$
$$\leqslant C\sum_{\substack{j\in\mathbb{N}\\k\in\mathbb{Z}}}\left\{M_s\bigg(\sum_{I\times J\in(E_N^{j,k})^c}|\psi_{j,k}*f(x_I,x_J)|^2\chi_I\chi_J\bigg)^{r/2}(u',v')\right\}^{2/r}.$$

Note that for any $p>0$, one can choose r small enough such that $p/r>q_\omega$. Hence, by Remark 2.1 on $L_\omega^{p/r}(\ell^{2/r})$, we have

(3.3)

$$\left\|\bigg(\sum_{\substack{j'\in\mathbb{N}\\k'\in\mathbb{Z}}}\sum_{I'\times J'\in\Pi_{j'}^{n_1}\times\Pi_{k'}^{n_2}}|\varphi_{j',k'}*(f-f_N)(x_{I'},x_{J'})|^2\chi_{I'}(x_1)\chi_{J'}(x_2)\bigg)^{1/2}\right\|_{L_\omega^p(\mathbb{R}^{n_1+n_2})}$$
$$\leqslant C\left\|\Bigg(\sum_{\substack{j\in\mathbb{N}\\k\in\mathbb{Z}}}\left\{M_s\bigg(\sum_{I\times J\in(E_N^{j,k})^c}|\psi_{j,k}*f(x_I,x_J)|^2\chi_I\chi_J\bigg)^{r/2}(u',v')\right\}^{2/r}\Bigg)^{1/2}\right\|_{L_\omega^p(\mathbb{R}^{n_1+n_2})}$$
$$=C\left\|\Bigg(\sum_{\substack{j\in\mathbb{N}\\k\in\mathbb{Z}}}\left\{M_s\bigg(\sum_{I\times J\in(E_N^{j,k})^c}|\psi_{j,k}*f(x_I,x_J)|^2\chi_I\chi_J\bigg)^{r/2}(u',v')\right\}^{2/r}\Bigg)^{r/2}\right\|_{L_\omega^{p/r}(\mathbb{R}^{n_1+n_2})}^{1/r}$$
$$\leqslant C\left\|\bigg(\sum_{\substack{j\in\mathbb{N}\\k\in\mathbb{Z}}}\sum_{I\times J\in(E_N^{j,k})^c}|\psi_{j,k}*f(x_I,x_J)|^2\chi_I(x)\chi_J(y)\bigg)^{1/2}\right\|_{L_\omega^p(\mathbb{R}^{n_1+n_2})},$$

which implies that $\|f-f_N\|_{H^p_{\mathrm{mix}}(\omega,\mathbb{R}^{n_1}\times\mathbb{R}^{n_2})}\to 0$ as $N\to\infty$.

Thus, we complete the proof. □

As a direct corollary of Proposition 3.1, we have the following result.

Corollary 3.1. *Let* $1\leqslant q<\infty$ *and* $\omega\in A_\infty(\mathbb{R}^{n_1}\times\mathbb{R}^{n_2})$. *Then* $L^q(\mathbb{R}^{n_1+n_2})\cap H^p_{\mathrm{mix}}(\omega,\mathbb{R}^{n_1}\times\mathbb{R}^{n_2})$ *is dense in* $H^p_{\mathrm{mix}}(\omega,\mathbb{R}^{n_1}\times\mathbb{R}^{n_2})$ *for all* $0<p<\infty$.

Note that $\omega\in A_q(\mathbb{R}^{n_1+n_2})$ if $\omega\in A_q(\mathbb{R}^{n_1}\times\mathbb{R}^{n_2})$ for some $1\leqslant q<\infty$ and $\mathcal{S}_0(\mathbb{R}^{n_1+n_2})\subseteq\mathcal{S}(\mathbb{R}^{n_1+n_2})\subseteq L^q_\omega(\mathbb{R}^{n_1+n_2})$ by classical result. The above result also holds for weighted L^q spaces.

Corollary 3.2. *Suppose that* $\omega\in A_q(\mathbb{R}^{n_1}\times\mathbb{R}^{n_2})$ *for some* $1\leqslant q<\infty$. *Then* $L^q_\omega(\mathbb{R}^{n_1+n_2})\cap H^p_{\mathrm{mix}}(\omega,\mathbb{R}^{n_1}\times\mathbb{R}^{n_2})$ *is dense in* $H^p_{\mathrm{mix}}(\omega,\mathbb{R}^{n_1}\times\mathbb{R}^{n_2})$ *for all* $0<p<\infty$.

Proposition 3.2. *For* $i=1,2$, *let* $\psi_0^{(i)}$ *and* $\psi^{(i)}\in\mathcal{S}(\mathbb{R}^{n_i})$ *be functions satisfying conditions* (1.1)–(1.3). *Suppose that* $\omega\in A_\infty(\mathbb{R}^{n_1}\times\mathbb{R}^{n_2})$. *Then for* $p\in(0,\infty)$ *if*

$f \in L^2(\mathbb{R}^{n_1+n_2}) \cap H^p_{\text{mix}}(\omega, \mathbb{R}^{n_1} \times \mathbb{R}^{n_2})$, *one has*

$$\|f\|_{H^p_{mix}(\omega,\mathbb{R}^{n_1}\times\mathbb{R}^{n_2})} \approx \left\| \left(\sum_{\substack{j\in\mathbb{N}\\ k\in\mathbb{Z}}} |\psi_{j,k} * f(x)|^2 \right)^{1/2} \right\|_{L^p_\omega(\mathbb{R}^{n_1+n_2})}. \tag{3.4}$$

P r o o f. Given any positive integer N, by (1.6), one can split f as

$$\begin{aligned} f(x_1,x_2) &= \sum_{j=0}^{\infty} \sum_{k=-\infty}^{\infty} \int \psi_{j,k}(x-y)\psi_{j,k} * f(y)\,\mathrm{d}y \\ &= \sum_{j=0}^{\infty} \sum_{k=-\infty}^{\infty} \sum_{I\times J\in\Pi^{n_1}_{j+N}\times\Pi^{n_2}_{k+N}} \int_{I\times J} \psi_{j,k}(x-y)\psi_{j,k} * f(y)\,\mathrm{d}y, \end{aligned}$$

where the series converges in $L^2(\mathbb{R}^{n_1+n_2})$. Hence, for any $u \in I$, $v \in J$,

$$\begin{aligned} f(x_1,x_2) = &\sum_{\substack{j\in\mathbb{N}\\ k\in\mathbb{Z}}} \sum_{I\times J\in\Pi^{n_1}_{j+N}\times\Pi^{n_2}_{k+N}} |I||J|[(\psi_{j,k} * f)(u,v)]\psi_{j,k}(x_1-u, x_2-v) \\ &+ \Bigg\{ \sum_{\substack{j\in\mathbb{N}\\ k\in\mathbb{Z}}} \sum_{I\times J\in\Pi^{n_1}_{j+N}\times\Pi^{n_2}_{k+N}} \int_{I\times J} \psi_{j,k}(x-y)\psi_{j,k} * f(y)\,\mathrm{d}y \\ &- \sum_{\substack{j\in\mathbb{N}\\ k\in\mathbb{Z}}} \sum_{I\times J\in\Pi^{n_1}_{j+N}\times\Pi^{n_2}_{k+N}} |I||J|(\psi_{j,k} * f)(u,v)\psi_{j,k}(x_1-u, x_2-v) \Bigg\} \\ := &\, T_N(f)(x) + \mathcal{R}_N(f)(x). \end{aligned}$$

It is easy to check that $\|T_N(f)\|_{L^2(\mathbb{R}^{n_1+n_2})} \lesssim \|f\|_{L^2(\mathbb{R}^{n_1+n_2})}$. Furthermore, one can divide $\mathcal{R}_N(f)(x)$ as follows:

$$\begin{aligned} &\mathcal{R}_N(f)(x) \\ &= \sum_{\substack{j\in\mathbb{N}\\ k\in\mathbb{Z}}} \sum_{I\times J\in\Pi^{n_1}_{j+N}\times\Pi^{n_2}_{k+N}} \int_{I\times J} \psi_{j,k}(x-y)\psi_{j,k} * f(y)\,\mathrm{d}y \\ &\quad - \sum_{\substack{j\in\mathbb{N}\\ k\in\mathbb{Z}}} \sum_{I\times J\in\Pi^{n_1}_{j+N}\times\Pi^{n_2}_{k+N}} \int_{I\times J} [(\psi_{j,k} * f)(u,v)]\psi_{j,k}(x_1-u, x_2-v)\,\mathrm{d}y \\ &= \sum_{\substack{j\in\mathbb{N}\\ k\in\mathbb{Z}}} \sum_{I\times J\in\Pi^{n_1}_{j+N}\times\Pi^{n_2}_{k+N}} \int_{I\times J} [\psi_{j,k}(x-y) - \psi_{j,k}(x_1-u, x_2-v)]\psi_{j,k} * f(y)\,\mathrm{d}y \\ &\quad + \sum_{\substack{j\in\mathbb{N}\\ k\in\mathbb{Z}}} \sum_{I\times J\in\Pi^{n_1}_{j+N}\times\Pi^{n_2}_{k+N}} \int_{I\times J} [(\psi_{j,k} * f)(y) - \psi_{j,k} * f)(u,v)]\psi_{j,k}(x_1-u, x_2-v)\,\mathrm{d}y \\ &= \mathcal{R}^1_N(f)(x) + \mathcal{R}^2_N(f)(x). \end{aligned}$$

We claim that for $i=1,2$,

$$\|\mathcal{R}_N^i(f)\|_{H^p_{\rm mix}(\omega,\mathbb{R}^{n_1}\times\mathbb{R}^{n_2})}\leqslant C2^{-N}\|f\|_{H^p_{\rm mix}(\omega,\mathbb{R}^{n_1}\times\mathbb{R}^{n_2})} \tag{3.5}$$

and

$$\|\mathcal{R}_N^i(f)\|_{L^2_\omega(\mathbb{R}^{n_1+n_2})}\leqslant C2^{-N}\|f\|_{L^2_\omega(\mathbb{R}^{n_1+n_2})}.$$

We only give the proof of the boundedness of $\mathcal{R}_N^1(f)$ on $H^p_{\rm mix}(\omega,\mathbb{R}^{n_1}\times\mathbb{R}^{n_2})$. For this, let $\varphi_{j',k'}$ satisfy Definition 1.1. Applying discrete Calderón's identity (1.7), one has

$$\begin{aligned}
&\varphi_{j',k'}*\mathcal{R}_N^1(f)(x)\\
&\quad=\sum_{\substack{j\in\mathbb{N}\\k\in\mathbb{Z}}}\sum_{I\times J\in\Pi^{n_1}_{j+N}\times\Pi^{n_2}_{k+N}}\int_{I\times J}\varphi_{j',k'}*[\psi_{j,k}(\cdot-y)-\psi_{j,k}(\cdot-u,\cdot-v)(x)]\\
&\qquad\times\psi_{j,k}*f(y)\,\mathrm{d}y\\
&\quad=\sum_{\substack{j\in\mathbb{N}\\k\in\mathbb{Z}}}\sum_{I\times J\in\Pi^{n_1}_{j+N}\times\Pi^{n_2}_{k+N}}\int_{I\times J}\varphi_{j',k'}*[\psi_{j,k}(\cdot-y)-\psi_{j,k}(\cdot-u,\cdot-v)(x)]\\
&\qquad\times\psi_{j,k}*\Bigg\{\sum_{\substack{j''\in\mathbb{N}\\k''\in\mathbb{Z}}}\sum_{I''\times J''\in\Pi^{n_1}_{j''}\times\Pi^{n_2}_{k''}}|I''||J''|(\psi_{j'',k''}*f)(x_{I''},x_{J''})\\
&\qquad\times\psi_{j'',k''}(\cdot-x_{I''},\cdot-x_{J''})\Bigg\}(y)\,\mathrm{d}y.
\end{aligned}$$

Denote $\widetilde{\psi}_{j,k}(z_1,z_2)=\psi_{j,k}(z-y)-\psi_{j,k}(z_1-u,z_2-v)$, then it is easy to see that $\widetilde{\psi}_{j,k}\in\mathcal{S}(\mathbb{R}^{n_1+n_2})$ and its behavior is similar as $2^{-N}\psi_{j,k}(z_1-y_1,z_2-y_2)$. Hence, by Lemma 2.2, one has

$$|\varphi_{j',k'}*\widetilde{\psi}_{j,k}(x_1,x_2)|\lesssim2^{-N}\frac{2^{(j\wedge j')n_1}2^{-|j-j'|M_1}}{(1+2^{(j\wedge j')}|x_1-y_1|)^{L_1}}\frac{2^{(k\wedge k')n_2}2^{-|k-k'|M_2}}{(1+2^{(k\wedge k')}|x_2-y_2|)^{L_2}}.$$

It implies that

$$|\varphi_{j',k'}*\widetilde{\psi}_{j,k}(x_1,x_2)|\lesssim2^{-N}\frac{2^{j'n_1}2^{-|j-j'|(M_1-L_1)}}{(1+2^{j'}|x_1-y_1|)^{L_1}}\frac{2^{k'n_2}2^{-|k-k'|(M_2-L_2)}}{(1+2^{k'}|x_2-y_2|)^{L_2}}$$

for all $j,j'\in\mathbb{N},k,k'\in\mathbb{Z}$. Similarly,

$$\begin{aligned}
&|(\psi_{j,k}*\psi_{j'',k''}(\cdot-x_{I''},\cdot-x_{J''}))(y_1,y_2)|\\
&\quad=|\psi_{j,k}*\psi_{j'',k''}(y_1-x_{I''},y_2-x_{J''})|\\
&\quad\lesssim\frac{2^{(j\wedge j'')n_1}2^{-|j-j''|M_1}}{(1+2^{(j\wedge j'')}|x_{I''}-y_1|)^{L_1}}\frac{2^{(k\wedge k'')n_2}2^{-|k-k''|M_2}}{(1+2^{(k\wedge k'')}|x_{J''}-y_2|)^{L_2}}\\
&\quad\lesssim\frac{2^{j''n_1}2^{-|j-j''|(M_1-L_1)}}{(1+2^{j''}|x_{I''}-y_1|)^{L_1}}\frac{2^{k''n_2}2^{-|k-k''|(M_2-L_2)}}{(1+2^{k''}|x_{J''}-y_2|)^{L_2}}.
\end{aligned}$$

Therefore,

$$
\begin{aligned}
&|\varphi_{j',k'} * \mathcal{R}_N^1(f)(x)| \\
&\quad \lesssim 2^{-N} \sum_{\substack{j\in\mathbb{N}\\ k\in\mathbb{Z}}} \sum_{I\times J\in\Pi_{j+N}^{n_1}\times\Pi_{k+N}^{n_2}} \sum_{j'',k''\in\mathbb{N}} \sum_{I''\times J''\in\Pi_{j''}^{n_1}\times\Pi_{k''}^{n_2}} |I''||J''|(\psi_{j'',k''} * f)(x_{I''}, x_{J''}) \\
&\qquad \times \int_{I\times J} \bigg\{ \frac{2^{j'n_1} 2^{-|j-j'|(M_1-L_1)}}{(1+2^{j'}|x_1-y_1|)^{L_1}} \frac{2^{k'n_2} 2^{-|k-k'|(M_2-L_2)}}{(1+2^{k'}|x_2-y_2|)^{L_2}} \\
&\qquad \times \frac{2^{j''n_1} 2^{-|j-j''|(M_1-L_1)}}{(1+2^{j''}|x_{I''}-y_1|)^{L_1}} \frac{2^{k''n_2} 2^{-|k-k''|(M_2-L_2)}}{(1+2^{k''}|x_{J''}-y_2|)^{L_2}} \bigg\} \, \mathrm{d}y \\
&\quad = 2^{-N} \sum_{\substack{j\in\mathbb{N}\\ k\in\mathbb{Z}}} \sum_{j'',k''\in\mathbb{N}} \sum_{I''\times J''\in\Pi_{j''}^{n_1}\times\Pi_{k''}^{n_2}} |I''||J''|(\psi_{j'',k''} * f)(x_{I''}, x_{J''}) \\
&\qquad \times \int_{\mathbb{R}^{n_1}\times\mathbb{R}^{n_2}} \bigg\{ \frac{2^{j'n_1} 2^{-|j-j'|(M_1-L_1)}}{(1+2^{j'}|x_1-y_1|)^{L_1}} \frac{2^{k'n_2} 2^{-|k-k'|(M_2-L_2)}}{(1+2^{k'}|x_2-y_2|)^{L_2}} \\
&\qquad \times \frac{2^{j''n_1} 2^{-|j-j''|(M_1-L_1)}}{(1+2^{j''}|x_{I''}-y_1|)^{L_1}} \frac{2^{k''n_2} 2^{-|k-k''|(M_2-L_2)}}{(1+2^{k''}|x_{J''}-y_2|)^{L_2}} \bigg\} \, \mathrm{d}y \\
&\quad \lesssim 2^{-N} \sum_{\substack{j\in\mathbb{N}\\ k\in\mathbb{Z}}} \sum_{j'',k''\in\mathbb{N}} \sum_{I''\times J''\in\Pi_{j''}^{n_1}\times\Pi_{k''}^{n_2}} |I''||J''| 2^{-|j-j''|(M_1-L_1)} 2^{-|k-k''|(M_2-L_2)} \\
&\qquad \times \frac{2^{(j'\wedge j'')n_1} 2^{-|j-j'|(M_1-L_1)}}{(1+2^{(j'\wedge j'')}|x_1-x_{I''}|)^{L_1}} \frac{2^{(k'\wedge k'')n_2} 2^{-|k-k'|(M_2-L_2)}}{(1+2^{(k'\wedge k'')}|x_2-x_{J''}|)^{L_2}} (\psi_{j'',k''} * f)(x_{I''}, x_{J''}).
\end{aligned}
$$

Hence, by Lemma 2.5,

$$
\begin{aligned}
&|\varphi_{j',k'} * \mathcal{R}_N^1(f)(x_{I'}, x_{J'})| \\
&\quad \lesssim 2^{-N} \sum_{\substack{j\in\mathbb{N}\\ k\in\mathbb{Z}}} \sum_{j'',k''\in\mathbb{N}} 2^{-|j-j''|(M_1-L_1)} 2^{-|k-k''|(M_2-L_2)} 2^{-|j-j'|(M_1-L_1)} 2^{-|k-k'|(M_2-L_2)} \\
&\qquad \times C_1 \bigg\{ M_s \bigg(\sum_{I''\times J''\in\Pi_{j''}^{n_1}\times\Pi_{k''}^{n_2}} |\psi_{j'',k''} * f(x_{I''}, y_{J''})|^2 \chi_{I''}\chi_{J''} \bigg)^{r/2} (u^*, v^*) \bigg\}^{1/r} \\
&\quad \leqslant 2^{-N} \sum_{j'',k''\in\mathbb{N}} 2^{-|j'-j''|(M_1-L_1)} 2^{-|k'-k''|(M_2-L_2)} \\
&\qquad \times C_1 \bigg\{ M_s \bigg(\sum_{I''\times J''\in\Pi_{j''}^{n_1}\times\Pi_{k''}^{n_2}} |\psi_{j'',k''} * f(x_{I''}, y_{J''})|^2 \chi_{I''}\chi_{J''} \bigg)^{r/2} (u^*, v^*) \bigg\}^{1/r}
\end{aligned}
$$

for any $(u^*, v^*) \in I' \times J'$, where $C_1 = 2^{(1-1/r)[n_1(j'\wedge j''-j'')+n_2(k'\wedge k''-k'')]}$ and $\max\{n_1/L_1, n_2/L_2\} < r \leqslant 1$.

Repeating the process to obtain (3.3), we have

$$\begin{aligned}&\Big\|\Big(\sum_{j',k'\in\mathbb{N}}\sum_{R'\in\Pi^{n_1}_{j'}\times\Pi^{n_2}_{k'}}|\varphi_{j',k'}*\mathcal{R}^1_N(f)(x_{I'},x_{J'})|^2\chi_{I'}(x')\chi_{J'}(y')\Big)^{1/2}\Big\|_{L^p_\omega(\mathbb{R}^{n_1+n_2})}\\&\quad\lesssim 2^{-N}\Big\|\Big(\sum_{j'',k''\in\mathbb{N}}\sum_{I''\times J''\in\Pi^{n_1}_{j''}\times\Pi^{n_2}_{k''}}|\psi_{j'',k''}*f(x_{I''},y_{J''})|^2\chi_{I''}\chi_{J''}\Big)^{1/2}\Big\|_{L^p_\omega(\mathbb{R}^{n_1+n_2})}.\end{aligned}$$

Hence, we obtain (3.5).

Inequalities (3.5) imply $\|\mathcal{R}_N(f)\|_{H^p_{\text{mix}}(\omega,\mathbb{R}^{n_1}\times\mathbb{R}^{n_2})}\leqslant C2^{-N}\|f\|_{H^p_{\text{mix}}(\omega,\mathbb{R}^{n_1}\times\mathbb{R}^{n_2})}$. Thus, the operator $T_N=I-\mathcal{R}_N$ is invertible in $H^p_{\text{mix}}(\omega,\mathbb{R}^{n_1}\times\mathbb{R}^{n_2})$ by choosing N large enough such that $C2^{-N}<1$. Then $T_N^{-1}=\sum\limits_{n=0}^{\infty}(\mathcal{R}_N)^n$ and

$$\|T_N^{-1}(f)\|_{H^p_{\text{mix}}(\omega,\mathbb{R}^{n_1}\times\mathbb{R}^{n_2})}\approx\|f\|_{H^p_{\text{mix}}(\omega,\mathbb{R}^{n_1}\times\mathbb{R}^{n_2})}.$$

Note that

$$T_N(f)(x_1,x_2)=\sum_{\substack{j\in\mathbb{N}\\k\in\mathbb{Z}}}\sum_{I\times J\in\Pi^{n_1}_{j+N}\times\Pi^{n_2}_{k+N}}|I||J|(\psi_{j,k}*f)(u,v)\psi_{j,k}(x_1-u,x_2-v)$$

in $L^2(\mathbb{R}^{n_1+n_2})$. Hence, for any $\varphi\in\mathcal{S}(\mathbb{R}^{n_1+n_2})$,

$$\varphi*T_N(f)(x_1,x_2)=\sum_{\substack{j\in\mathbb{N}\\k\in\mathbb{Z}}}\sum_{I\times J\in\Pi^{n_1}_{j+N}\times\Pi^{n_2}_{k+N}}|I||J|(\psi_{j,k}*f)(u,v)\varphi*\psi_{j,k}(x_1-u,x_2-v).$$

Then, repeating the proof to obtain (3.3), one has

$$\begin{aligned}&\|T_N(f)\|_{H^p_{mix}(\omega,\mathbb{R}^{n_1}\times\mathbb{R}^{n_2})}\\&\quad=\Big\|\sum_{\substack{j\in\mathbb{N}\\k\in\mathbb{Z}}}\sum_{I\times J\in\Pi^{n_1}_{j+N}\times\Pi^{n_2}_{k+N}}|I||J|(\psi_{j,k}*f)(u,v)\psi_{j,k}(x_1-u,x_2-v)\|_{H^p_{\text{mix}}(\omega,\mathbb{R}^{n_1}\times\mathbb{R}^{n_2})}\\&\quad\leqslant C\Big\|\Big(\sum_{\substack{j\in\mathbb{N}\\k\in\mathbb{Z}}}\sum_{I\times J\in\Pi^{n_1}_{j+N}\times\Pi^{n_2}_{k+N}}|(\psi_{j,k}*f)(u,v)|^2\chi_I\chi_J\Big)^{1/2}\Big\|_{L^p_\omega(\mathbb{R}^{n_1+n_2})},\end{aligned}$$

where (u,v) can be any point in $I\times J$. Hence,

$$\begin{aligned}\|f\|_{H^p_{\text{mix}}(\omega,\mathbb{R}^{n_1}\times\mathbb{R}^{n_2})}&=\|T_N^{-1}\circ T_N(f)\|_{H^p_{\text{mix}}(\omega,\mathbb{R}^{n_1}\times\mathbb{R}^{n_2})}\\&\lesssim\|T_N(f)\|_{H^p_{\text{mix}}(\omega,\mathbb{R}^{n_1}\times\mathbb{R}^{n_2})}\\&\lesssim\Big\|\Big(\sum_{\substack{j\in\mathbb{N}\\k\in\mathbb{Z}}}\sum_{I\times J\in\Pi^{n_1}_{j+N}\times\Pi^{n_2}_{k+N}}|(\psi_{j,k}*f)(u,v)|^2\chi_I\chi_J\Big)^{1/2}\Big\|_{L^p_\omega(\mathbb{R}^{n_1+n_2})},\end{aligned}$$

which implies that

$$\|f\|_{H^p_{\mathrm{mix}}(\omega,\mathbb{R}^{n_1}\times\mathbb{R}^{n_2})} \lesssim \Big\|\Big(\sum_{\substack{j\in\mathbb{N}\\ k\in\mathbb{Z}}}\sum_{I\times J\in\Pi^{n_1}_{j+N}\times\Pi^{n_2}_{k+N}}\inf_{(u,v)\in I\times J}|(\psi_{j,k}*f)(u,v)|^2\chi_I\chi_J\Big)^{1/2}\Big\|_{L^p_\omega(\mathbb{R}^{n_1+n_2})},$$

since (u,v) can be any fixed point in $I\times J\in\Pi^{n_1}_{j+N}\times\Pi^{n_2}_{k+N}$. Repeating the proof to obtain (3.3) again, one has

$$\Big\|\Big(\sum_{\substack{j\in\mathbb{N}\\ k\in\mathbb{Z}}}\sum_{I\times J\in\Pi^{n_1}_{j+N}\times\Pi^{n_2}_{k+N}}\sup_{(u,v)\in I\times J}|(\psi_{j,k}*f)(u,v)|^2\chi_I\chi_J\Big)^{1/2}\Big\|_{L^p_\omega(\mathbb{R}^{n_1+n_2})} \lesssim \|f\|_{H^p_{\mathrm{mix}}(\omega,\mathbb{R}^{n_1}\times\mathbb{R}^{n_2})}.$$

Thus, we obtain that

$$\begin{aligned}\|f\|_{H^p_{\mathrm{mix}}(\omega,\mathbb{R}^{n_1}\times\mathbb{R}^{n_2})} &\approx \Big\|\Big(\sum_{\substack{j\in\mathbb{N}\\ k\in\mathbb{Z}}}\sum_{I\times J\in\Pi^{n_1}_{j+N}\times\Pi^{n_2}_{k+N}}\sup_{(u,v)\in I\times J}|(\psi_{j,k}*f)(u,v)|^2\chi_I\chi_J\Big)^{1/2}\Big\|_{L^p_\omega(\mathbb{R}^{n_1+n_2})}\\ &\approx \Big\|\Big(\sum_{\substack{j\in\mathbb{N}\\ k\in\mathbb{Z}}}\sum_{I\times J\in\Pi^{n_1}_{j+N}\times\Pi^{n_2}_{k+N}}\inf_{(u,v)\in I\times J}|(\psi_{j,k}*f)(u,v)|^2\chi_I\chi_J\Big)^{1/2}\Big\|_{L^p_\omega(\mathbb{R}^{n_1+n_2})}.\end{aligned}$$

Thus, we have completed the proof. □

Proposition 3.3. *Let $1<p<\infty$ and $\omega\in A_p(\mathbb{R}^{n_1}\times\mathbb{R}^{n_2})$. Then $H^p_{\mathrm{mix}}(\omega,\mathbb{R}^{n_1}\times\mathbb{R}^{n_2})=L^p_\omega(\mathbb{R}^{n_1+n_2})$.*

P r o o f. By iteration method for all $1<p<\infty$ we have

$$\|S(f)(x)\|_{L^p_\omega(\mathbb{R}^{n_1+n_2})}\lesssim\|f\|_{L^p_\omega(\mathbb{R}^{n_1+n_2})}.$$

Conversely, let p' denote the conjugate of p. Note that if $\omega\in A_p(\mathbb{R}^{n_1}\times\mathbb{R}^{n_2})$, $\omega^{1-p'}\in A_{p'}(\mathbb{R}^{n_1}\times\mathbb{R}^{n_2})$ (see [16]), hence $\omega^{1-p'}\in A_{p'}(\mathbb{R}^{n_1+n_2})$. By classical result, the space $C_0^\infty(\mathbb{R}^{n_1+n_2})$, composed by all smooth functions with compact support, is dense in $L^r_\mu(\mathbb{R}^{n_1+n_2})$ for all $\mu\in A_\infty(\mathbb{R}^{n_1+n_2})$, see [18]. Hence, $L^2(\mathbb{R}^{n_1+n_2})\cap L^{p'}_{\omega^{1-p'}}(\mathbb{R}^{n_1+n_2})$ is dense in $L^{p'}_{\omega^{1-p'}}(\mathbb{R}^{n_1+n_2})$.

By (1.6), for $g \in L^2(\mathbb{R}^{n_1+n_2}) \cap L^{p'}_{\omega^{1-p'}}(\mathbb{R}^{n_1+n_2})$, one has

$$\begin{aligned}\left|\int f(x)g(-x)\,\mathrm{d}x\right| &\leqslant \sum_{\substack{j\in\mathbb{N}\\k\in\mathbb{Z}}} |\psi_{j,k} * \psi_{j,k} * f * g(0)| \\ &\leqslant \sum_{\substack{j\in\mathbb{N}\\k\in\mathbb{Z}}} \int |\psi_{j,k} * f(y)||\psi_{j,k} * g(-y)|\,\mathrm{d}y \\ &\leqslant \int \Bigg(\sum_{\substack{j\in\mathbb{N}\\k\in\mathbb{Z}}} |\psi_{j,k} * f(y)|^2\Bigg)^{1/2} \omega(y)^{1/p} \\ &\quad \times \Bigg(\sum_{\substack{j\in\mathbb{N}\\k\in\mathbb{Z}}} |\psi_{j,k} * g(y)|^2\Bigg)^{1/2} \omega(y)^{-1/p}\,\mathrm{d}y \\ &\leqslant \|S(f)\|_{L^p_\omega(\mathbb{R}^{n_1+n_2})} \|S(g)\|_{L^{p'}_{\omega^{1-p'}}(\mathbb{R}^{n_1+n_2})} \\ &\lesssim \|S(f)\|_{L^p_\omega(\mathbb{R}^{n_1+n_2})} \|g\|_{L^{p'}_{\omega^{1-p'}}(\mathbb{R}^{n_1+n_2})},\end{aligned}$$

which yields our desired result. □

4. Boundedness of operators

To discuss the boundedness of operators on $H^p_{\mathrm{mix}}(\omega, \mathbb{R}^{n_1} \times \mathbb{R}^{n_2})$, we need a new discrete Calderón-type identity composed by some test functions with compact supports. To do this, given a positive integer M large enough, let $\phi_0^{(1)}, \phi^{(1)} \in \mathcal{S}(\mathbb{R}^{n_1})$ with

$$\operatorname{supp} \phi_0^{(1)} \subseteq \{x \in \mathbb{R}^{n_1} \colon |x| \leqslant 1\}; \quad \int \phi_0^{(1)} = 1, \tag{4.1}$$

$$\operatorname{supp} \phi^{(1)} \subseteq \{x \in \mathbb{R}^{n_1} \colon |x| \leqslant 1\}; \quad \int \phi^{(1)}(x) x^\alpha\,\mathrm{d}x = 0 \quad \forall\, |\alpha| \leqslant M, \tag{4.2}$$

and

$$|\widehat{\phi_0^{(1)}}(\xi)|^2 + \sum_{j=1}^{\infty} |\widehat{\phi^{(1)}}(2^{-j}\xi)|^2 = 1 \quad \forall\, \xi \in \mathbb{R}^{n_1}. \tag{4.3}$$

Let $\phi^{(2)} \in \mathcal{S}(\mathbb{R}^{n_2})$ with

$$\operatorname{supp} \phi^{(2)} \subseteq \{x \in \mathbb{R}^{n_2} \colon |x| \leqslant 1\}; \quad \int \phi^{(2)}(x) x^\alpha\,\mathrm{d}x = 0 \quad \forall\, |\alpha| \leqslant M, \tag{4.4}$$

and

$$\sum_{j\in\mathbb{Z}} |\widehat{\phi^{(2)}}(2^{-j}\xi)|^2 = 1 \quad \forall \xi \in \mathbb{R}^{n_2} \setminus \{0\}. \tag{4.5}$$

By taking the Fourier transform, one has the following continuous Calderón-type identity:

$$f(x) = \sum_{j=0}^{\infty} \sum_{k=-\infty}^{\infty} \phi_{j,k} * \phi_{j,k} * f(x), \tag{4.6}$$

where the series converges in $L^2(\mathbb{R}^{n_1+n_2})$, $\mathcal{S}_0(\mathbb{R}^{n_1+n_2})$ and $\mathcal{S}_0'(\mathbb{R}^{n_1+n_2})$. Different from (1.6), the test functions in (4.6) have compact supports.

Lemma 4.1. *Suppose that* $\phi_0^{(1)}$, $\phi^{(1)}$ *and* $\phi^{(2)}$ *satisfy conditions* (4.1)–(4.5). *Let* $\omega \in A_2(\mathbb{R}^{n_1+n_2})$, *then if* $f \in L^2_\omega(\mathbb{R}^{n_1+n_2})$,

$$f(x) = \sum_{j=0}^{\infty} \sum_{k=-\infty}^{\infty} \phi_{j,k} * \phi_{j,k} * f(x), \tag{4.7}$$

where the series converges in $L^2_\omega(\mathbb{R}^{n_1+n_2})$.

Proof. Since the series $\sum\limits_{j=0}^{\infty} \sum\limits_{k=-\infty}^{\infty} \phi_{j,k} * \phi_{j,k} * f(x)$ converges to f in $\mathcal{S}_0(\mathbb{R}^{n_1+n_2})$ and $L^2_\omega(\mathbb{R}^{n_1+n_2})$ norm is dominated by a certain seminorm of $\mathcal{S}(\mathbb{R}^{n_1+n_2})$, (4.7) holds in $L^2_\omega(\mathbb{R}^{n_1+n_2})$ if $f \in \mathcal{S}_0(\mathbb{R}^{n_1+n_2})$.

For general $f \in L^2_\omega(\mathbb{R}^{n_1+n_2})$, given for all $\varepsilon > 0$, by Proposition 3.1, there exists $g \in \mathcal{S}_0(\mathbb{R}^{n_1+n_2})$ and $h \in L^2_\omega(\mathbb{R}^{n_1+n_2})$ with $\|h\|_{L^2_\omega(\mathbb{R}^{n_1+n_2})} < \varepsilon$ such that $f = g + h$. Moreover, for any positive integer N,

$$\begin{aligned}\Big\| f - \sum_{j=0}^{N} \sum_{k=-N}^{N} \phi_{j,k} * \phi_{j,k} * f \Big\|_{L^2_\omega(\mathbb{R}^{n_1+n_2})} &\leqslant \Big\| g - \sum_{j=0}^{N} \sum_{k=-N}^{N} \phi_{j,k} * \phi_{j,k} * g \Big\|_{L^2_\omega(\mathbb{R}^{n_1+n_2})} \\ &\quad + \|h\|_{L^2_\omega(\mathbb{R}^{n_1+n_2})} \\ &\quad + \Big\| \sum_{j=0}^{N} \sum_{k=-N}^{N} \phi_{j,k} * \phi_{j,k} * h \Big\|_{L^2_\omega(\mathbb{R}^{n_1+n_2})}.\end{aligned}$$

Since $\omega \in A_2(\mathbb{R}^{n_1+n_2})$, one has $\omega^{-1} \in A_2(\mathbb{R}^{n_1+n_2})$. Hence, by a duality argument, one has

$$
\begin{aligned}
&\Big\|\sum_{j=0}^{N}\sum_{k=-N}^{N}\phi_{j,k}*\phi_{j,k}*h\Big\|_{L^2_\omega(\mathbb{R}^{n_1+n_2})}\\
&= \sup_{\|u\|_{L^2_{\omega^{-1}}(\mathbb{R}^{n_1+n_2})}\leqslant 1}\Big|\Big\langle \sum_{j=0}^{N}\sum_{k=-N}^{N}\phi_{j,k}*\phi_{j,k}*h, u\Big\rangle\Big|\\
&= \sup_{\|u\|_{L^2_{\omega^{-1}}(\mathbb{R}^{n_1+n_2})}\leqslant 1}\Big|\sum_{j=0}^{N}\sum_{k=-N}^{N}\phi_{j,k}*\phi_{j,k}*h*\tilde{u}(0)\Big|\\
&= \sup_{\|u\|_{L^2_{\omega^{-1}}(\mathbb{R}^{n_1+n_2})}\leqslant 1}\int\sum_{j=0}^{N}\sum_{k=-N}^{N}\phi_{j,k}*h(x)\omega(x)^{1/2}\phi_{j,k}*\tilde{u}(-x)\omega(x)^{-1/2}\,\mathrm{d}x\\
&\leqslant \sup_{\|u\|_{L^2_{\omega^{-1}}(\mathbb{R}^{n_1+n_2})}\leqslant 1}\int\Big(\sum_{j=0}^{N}\sum_{k=-N}^{N}|\phi_{j,k}*h(x)|^2\omega(x)\Big)^{1/2}\\
&\quad\times\Big(\sum_{j=0}^{N}\sum_{k=-N}^{N}|\phi_{j,k}*u(x)|^2\omega(x)^{-1}\Big)^{1/2}\,\mathrm{d}x\\
&\leqslant \|h\|_{L^2_\omega(\mathbb{R}^{n_1+n_2})},
\end{aligned}
$$

where $\tilde{u}(x) = u(-x)$. Since ε is arbitrary, we complete this proof. □

Theorem 4.1. *For $0 < p \leqslant 1$, let $\phi_0^{(1)}$, $\phi^{(1)}$ and $\phi^{(2)}$ satisfy conditions (4.1)–(4.5). Suppose that $\omega \in A_2(\mathbb{R}^{n_1}\times\mathbb{R}^{n_2})$. Then for any $f \in L^2_\omega(\mathbb{R}^{n_1+n_2})\cap H^p_{\mathrm{mix}}(\omega,\mathbb{R}^{n_1}\times\mathbb{R}^{n_2})$,*

$$
\begin{aligned}
(4.8)\qquad &\|f\|_{H^p_{\mathrm{mix}}(\omega,\mathbb{R}^{n_1}\times\mathbb{R}^{n_2})}\\
&\approx \Big\|\Big(\sum_{\substack{j\in\mathbb{N}\\k\in\mathbb{Z}}}|\phi_{j,k}*f(x)|^2\Big)^{1/2}\Big\|_{L^p_\omega(\mathbb{R}^{n_1+n_2})}\\
&\approx \Big\|\Big(\sum_{\substack{j\in\mathbb{N}\\k\in\mathbb{Z}}}\sum_{I\times J\in\Pi^{n_1}_{j+N}\times\Pi^{n_2}_{k+N}}|(\phi_{j,k}*f)(x_I,x_J)|^2\chi_I\chi_J\Big)^{1/2}\Big\|_{L^p_\omega(\mathbb{R}^{n_1+n_2})}.
\end{aligned}
$$

Moreover, there exists a positive integer N such that

$$
(4.9)\qquad f(x) = \sum_{\substack{j\in\mathbb{N}\\k\in\mathbb{Z}}}\sum_{I\times J\in\Pi^{n_1}_{j+N}\times\Pi^{n_2}_{k+N}}|I||J|(\phi_{j,k}*h)(x_I,x_J)\phi_{j,k}(x_1-x_I,x_2-x_J),
$$

where the series converges in $L^2_\omega(\mathbb{R}^{n_1+n_2})$ *and* $H^p_{\text{mix}}(\omega, \mathbb{R}^{n_1} \times \mathbb{R}^{n_2})$, *and* $h \in L^2_\omega(\mathbb{R}^{n_1+n_2}) \cap H^p_{\text{mix}}(\omega, \mathbb{R}^{n_1} \times \mathbb{R}^{n_2})$ *satisfies*

$$\|h\|_{L^2_\omega(\mathbb{R}^{n_1+n_2})} \approx \|f\|_{L^2_\omega(\mathbb{R}^{n_1+n_2})} \quad \textit{and} \quad \|h\|_{H^p_{\text{mix}}(\omega,\mathbb{R}^{n_1}\times\mathbb{R}^{n_2})} \approx \|f\|_{H^p_{\text{mix}}(\omega,\mathbb{R}^{n_1}\times\mathbb{R}^{n_2})}.$$

P r o o f. Let $f \in L^2_\omega(\mathbb{R}^{n_1+n_2}) \cap H^p_{\text{mix}}(\omega, \mathbb{R}^{n_1} \times \mathbb{R}^{n_2})$, then by Lemma 4.1, $f(x) = \sum\limits_{j=0}^{\infty} \sum\limits_{k=-\infty}^{\infty} \phi_{j,k} * \phi_{j,k} * f(x)$ in $L^2_\omega(\mathbb{R}^{n_1+n_2})$. Then for any large positive integer N, similar as in the proof of Proposition 3.2, one has

$$f(x_1, x_2) = T_N(f)(x) + \mathcal{R}_N(f)(x)$$

and

$$\|T_N^{-1}(f)\|_{L^2_\omega(\mathbb{R}^{n_1+n_2})} \approx C2^{-N}\|f\|_{L^2_\omega(\mathbb{R}^{n_1+n_2})}$$

and

$$\|T_N^{-1}(f)\|_{H^p_{\text{mix}}(\omega,\mathbb{R}^{n_1}\times\mathbb{R}^{n_2})} \approx C2^{-N}\|f\|_{H^p_{\text{mix}}(\omega,\mathbb{R}^{n_1}\times\mathbb{R}^{n_2})}.$$

Set $h(x) = T_N^{-1}(f)(x)$, then

$$\begin{aligned} f(x) &= T_N(T_N^{-1}f)(x) \\ &= \sum_{\substack{j\in\mathbb{N} \\ k\in\mathbb{Z}}} \sum_{I\times J\in\Pi^{n_1}_{j+N}\times\Pi^{n_2}_{k+N}} |I||J|(\phi_{j,k} * h)(x_I, x_J)\phi_{j,k}(x_1 - x_I, x_2 - x_J). \end{aligned}$$

It can be seen that this series converges in $H^p_{\text{mix}}(\omega, \mathbb{R}^{n_1} \times \mathbb{R}^{n_2})$ and $L^2_\omega(\mathbb{R}^{n_1+n_2})$.

Thus, we complete the proof. □

P r o o f of Theorem 1.2. Let $f \in L^2_\omega(\mathbb{R}^{n_1+n_2}) \cap H^p_{\text{mix}}(\omega, \mathbb{R}^{n_1} \times \mathbb{R}^{n_2})$. Then by Theorem 4.1,

$$f(x_1, x_2) = \sum_{\substack{j\in\mathbb{N} \\ k\in\mathbb{Z}}} \sum_{I\times J\in\Pi^{n_1}_{j+N}\times\Pi^{n_2}_{k+N}} |I||J|(\varphi_{j,k} * h)(x_I, x_J)\varphi_{j,k}(x_1 - x_I, x_2 - x_J)$$

in $L^2_\omega(\mathbb{R}^{n_1+n_2})$, where $\varphi_{j,k}$ satisfy the same conditions of Theorem 4.1. By the L^2_ω boundedness of T,

$$\begin{aligned} &T(f)(x_1, x_2) \\ &\quad= \sum_{\substack{j\in\mathbb{N} \\ k\in\mathbb{Z}}} \sum_{I\times J\in\Pi^{n_1}_{j+N}\times\Pi^{n_2}_{k+N}} |I||J|(\varphi_{j,k} * h)(x_I, x_J)T(\varphi_{j,k}(\cdot - x_I, \cdot - x_J))(x_1, x_2). \end{aligned}$$

Then by (4.8),

$$\begin{aligned}
(4.10)\qquad & \|T(f)\|_{H^p_{\mathrm{mix}}(\omega,\mathbb{R}^{n_1}\times\mathbb{R}^{n_2})} \\
&\approx C\bigg\|\bigg(\sum_{\substack{j'\in\mathbb{N}\\ k'\in\mathbb{Z}}} |\phi_{j',k'} * (Tf)(x_1,x_2)|^2\bigg)^{1/2}\bigg\|_{L^p_\omega(\mathbb{R}^{n_1+n_2})} \\
&= C\bigg\|\bigg(\sum_{\substack{j'\in\mathbb{N}\\ k'\in\mathbb{Z}}}\bigg|\sum_{\substack{j\in\mathbb{N}\\ k\in\mathbb{Z}}}\sum_{I\times J\in\Pi^{n_1}_{j+N}\times\Pi^{n_2}_{k+N}} |I||J|(\varphi_{j,k}*h)(x_I,x_J) \\
&\qquad \times \phi_{j',k'} * T(\varphi_{j,k}(\cdot - x_I,\cdot - x_J))(x_1,x_2)\bigg|^2\bigg)^{1/2}\bigg\|_{L^p_\omega(\mathbb{R}^{n_1+n_2})},
\end{aligned}$$

where $\phi_{j',k'}$, $\varphi_{j,k}$ satisfy the conditions in Theorem 4.1.

To discuss (4.10), we recall an almost orthogonality estimate from [6].

Lemma 4.2. *Suppose that $\phi_{j,k}$, $\varphi_{j,k}$ satisfy the condition of Theorem* 4.1 *and $j, j'\in\mathbb{N}$, $k, k'\in\mathbb{Z}$. Then for any $0<\varepsilon'<\min\{\delta,\varepsilon\}$ there exists a constant $C>0$ such that*

$$\begin{aligned}
&|\phi_{j',k'} * T(\varphi_{j,k}(\cdot - u_1,\cdot - u_2)(x_1,x_2)| \\
&\quad \leqslant C2^{-|j'-j|\varepsilon'}2^{-|k'-k|\varepsilon'}\frac{2^{-(j\wedge j')\varepsilon}}{(2^{-(j\wedge j')}+|x_1-u_1|)^{n_1+\varepsilon}}\frac{2^{-(k\wedge k')\varepsilon}}{(2^{-(k\wedge k')}+|x_2-u_2|)^{n_2+\varepsilon}}.
\end{aligned}$$

Therefore, by Lemma 2.5 again,

$$\begin{aligned}
&|\phi_{j',k'} * (Tf)(x_1,x_2)| \\
&\quad\lesssim \sum_{\substack{j\in\mathbb{N}\\ k\in\mathbb{Z}}}\sum_{I\times J\in\Pi^{n_1}_{j+N}\times\Pi^{n_2}_{k+N}} |I||J||(\varphi_{j,k}*h)(x_I,x_J)| \\
&\qquad \times 2^{-|j'-j|\varepsilon'}2^{-|k'-k|\varepsilon'}\frac{2^{-(j\wedge j')\varepsilon}}{(2^{-(j\wedge j')}+|x_1-x_I|)^{n_1+\varepsilon}}\frac{2^{-(k\wedge k')\varepsilon}}{(2^{-(k\wedge k')}+|x_2-x_J|)^{n_2+\varepsilon}} \\
&\quad\lesssim \sum_{\substack{j\in\mathbb{N}\\ k\in\mathbb{Z}}} 2^{-|j'-j|\varepsilon'}2^{-|k'-k|\varepsilon'} \\
&\qquad \times C_1\bigg\{\mathcal{M}_s\bigg(\sum_{R\in\Pi^{n_1}_{j+N}\times\Pi^{n_2}_{k+N}} |\varphi_{j,k}*h(x_I,x_J)|^2\chi_I\chi_J\bigg)^{r/2}(u,v)\bigg\}^{1/r}.
\end{aligned}$$

Then, repeating the proof to obtain (3.3), one has

$$\|T(f)\|_{H^p_{\mathrm{mix}}(\omega,\mathbb{R}^{n_1}\times\mathbb{R}^{n_2})} \lesssim \|h\|_{H^p_{\mathrm{mix}}(\omega,\mathbb{R}^{n_1}\times\mathbb{R}^{n_2})} \approx \|f\|_{H^p_{\mathrm{mix}}(\omega,\mathbb{R}^{n_1}\times\mathbb{R}^{n_2})}$$

by Theorem 4.1. Thus, we complete the proof of Theorem 1.2. □

It is well known that the atomic decomposition is the main tool to study the $H^p - L^p$ boundedness of singular operators, see [2], [10], [11], [14], [16], [21], [23], [24], [25]. Different from this, in this paper, we use the approach developed in [22] to prove Theorem 1.3. For this, we prove the following result in mixed multi-parameter setting.

Theorem 4.2. *Let $0 < p \leqslant 1$ and $\omega \in A_2(\mathbb{R}^{n_1} \times \mathbb{R}^{n_2})$. If $f \in L^2_\omega(\mathbb{R}^{n_1+n_2}) \cap H^p_{\rm mix}(\omega, \mathbb{R}^{n_1} \times \mathbb{R}^{n_2})$, then*

$$\|f\|_{L^p_\omega(\mathbb{R}^{n_1+n_2})} \leqslant C\|f\|_{H^p_{\rm mix}(\omega,\mathbb{R}^{n_1}\times\mathbb{R}^{n_2})},$$

where C is a constant independent of f.

Proof. For $f \in L^2_\omega(\mathbb{R}^{n_1+n_2}) \cap H^p_{\rm mix}(\omega, \mathbb{R}^{n_1} \times \mathbb{R}^{n_2})$, let

$$\widetilde{S}(f)(x) = \biggl(\sum_{\substack{j\in\mathbb{N}\\ k\in\mathbb{Z}}} \sum_{I\times J\in\Pi^{n_1}_{j+N}\times\Pi^{n_2}_{k+N}} |(\phi_{j,k} * h)(x_I, x_J)|^2 \chi_I\chi_J\biggr)^{1/2},$$

where $\phi_{j,k}$ and h satisfy the conditions of Theorem 4.1.

For any $i \in \mathbb{Z}$, set

$$\Omega_i = \{x \in \mathbb{R}^{n_1+n_2}\colon\ \widetilde{S}(f)(x) > 2^i\}, \quad \widetilde{\Omega}_i = \Bigl\{x \in \mathbb{R}^{n_1+n_2}\colon\ M^\omega_s(\chi_{\Omega_i})(x) > \frac{1}{10^{n_1+n_2}}\Bigr\}$$

and

$$\mathcal{B}_i = \Bigl\{R\colon\ R \in \Pi,\ \omega(R \cap \Omega_i) > \frac{1}{2}\omega(R),\ \omega(R \cap \Omega_{i+1}) \leqslant \frac{1}{2}\omega(R)\Bigr\}.$$

Here $M^\omega_s(g)(x) = \sup\limits_R(\omega(R))^{-1}\int_R |g(x)|\omega(x)\,\mathrm{d}x$, where the supremum is taken over all cuboids R containing x in $\mathbb{R}^{n_1}\times\mathbb{R}^{n_2}$. It is easy to see that $\bigcup\limits_{R\in\mathcal{B}_i} R \subseteq \widetilde{\Omega}_i$. Moreover, $\omega(\widetilde{\Omega}_i) \leqslant C\omega(\Omega_i)$ by the boundedness of M^ω_s, see [9].

Hence, by (4.9),

$$\begin{aligned} f(x) &= \sum_{\substack{j\in\mathbb{N}\\ k\in\mathbb{Z}}} \sum_{I\times J\in\Pi^{n_1}_{j+N}\times\Pi^{n_2}_{k+N}} |I||J|(\phi_{j,k} * h)(x_I, x_J)\phi_{j,k}(x_1 - x_I, x_2 - x_J) \\ &= \sum_{i=-\infty}^{\infty} \sum_{R\in\mathcal{B}_i} |R|(\phi_R * h)(x_I, x_J)\phi_R(x_1 - x_I, x_2 - x_J), \end{aligned}$$

where the series converges in the L^2_ω norm.

Since $\omega(\widetilde{\Omega}_i) \leqslant C\omega(\Omega_i)$, by Hölder's inequality, one has

$$\begin{aligned} (4.11) \quad & \Big\| \sum_{R\in\mathcal{B}_i} |R|\phi_R(\cdot - x_I, \cdot - x_J)(\phi_R * h)(x_I, x_J) \|^p_{L^p_\omega(\mathbb{R}^{n_1+n_2})} \\ & \leqslant \omega(\Omega_i)^{1-p/2} \Big\| \sum_{R\in\mathcal{B}_i} |R|\phi_R(\cdot - x_I, \cdot - x_J)(\phi_{j,k} * h)(x_I, x_J) \Big\|^p_{L^2_\omega(\mathbb{R}^{n_1+n_2})}. \end{aligned}$$

Using duality argument,

$$\begin{aligned} & \Big\| \sum_{R\in\mathcal{B}_i} |R|\phi_R(\cdot - x_I, \cdot - x_J)(\phi_R * h)(x_I, x_J) \Big\|_{L^2_\omega(\mathbb{R}^{n_1+n_2})} \\ & = \sup_{\|g\|_{L^2_{\omega^{-1}}(\mathbb{R}^{n_1+n_2})}\leqslant 1} \Big| \int \Big(\sum_{R\in\mathcal{B}_i} |R|[(\phi_R * h)(x_I, x_J)]\phi_R(x_1 - x_I, x_2 - x_J) \Big) g(x)\,\mathrm{d}x \Big| \\ & = \sup_{\|g\|_{L^2_{\omega^{-1}}(\mathbb{R}^{n_1+n_2})}\leqslant 1} \Big| \sum_{R\in\mathcal{B}_i} |R|[(\phi_R * h)(x_I, x_J)]\widetilde{\phi}_R * g(x_I, x_J) \Big| \\ & = \sup_{\|g\|_{L^2_{\omega^{-1}}(\mathbb{R}^{n_1+n_2})}\leqslant 1} \Big| \int \sum_{R\in\mathcal{B}_i} [(\phi_R * h)(x_I, x_J)]\widetilde{\phi}_R * g(x_I, x_J)\chi_R(x)\,\mathrm{d}x \Big| \\ & \leqslant \sup_{\|g\|_{L^2_{\omega^{-1}}(\mathbb{R}^{n_1+n_2})}\leqslant 1} \Big\| \Big(\sum_{R\in\mathcal{B}_i} |(\phi_R * h)(x_I, x_J)|^2 \chi_R(x) \Big)^{1/2} \|_{L^2_\omega(\mathbb{R}^{n_1+n_2})} \|g\Big\|_{L^2_{\omega^{-1}}(\mathbb{R}^{n_1+n_2})}. \end{aligned}$$

Hence,

$$\begin{aligned} (4.12) \quad & \Big\| \sum_{R\in\mathcal{B}_i} |R|\phi_R(\cdot - x_I, \cdot - x_J)(\phi_R * h)(x_I, x_J) \Big\|_{L^2_\omega(\mathbb{R}^{n_1+n_2})} \\ & \lesssim \Big\| \Big(\sum_{R\in\mathcal{B}_i} |(\phi_R * h)(x_I, x_J)|^2 \chi_R(x) \Big)^{1/2} \Big\|_{L^2_\omega(\mathbb{R}^{n_1+n_2})} \\ & = \Big(\sum_{R\in\mathcal{B}_i} \omega(R)|(\phi_R * h)(x_I, x_J)|^2 \Big)^{1/2}. \end{aligned}$$

Furthermore, by the definition of B_i, one has $\omega(R \cap \widetilde{\Omega}_i \setminus \Omega_{i+1}) > \frac{1}{2}\omega(R)$. Hence,

$$\begin{aligned} (4.13) \quad & \sum_{R\in\mathcal{B}_i} \omega(R)|(\phi_R * h)(x_I, x_J)|^2 \\ & \lesssim \sum_{R\in\mathcal{B}_i} \omega(R \cap \widetilde{\Omega}_i \setminus \Omega_{i+1})|(\phi_R * h)(x_I, x_J)|^2 \\ & = \int_{\widetilde{\Omega}_i \setminus \Omega_{i+1}} \sum_{R\in\mathcal{B}_i} |(\phi_R * h)(x_I, x_J)|^2 \chi_R(x)\omega(x)\,\mathrm{d}x \\ & \leqslant 2^{2(i+1)}\omega(\widetilde{\Omega}_i) \lesssim 2^{2i}\omega(\Omega_i). \end{aligned}$$

Therefore, by (4.11), (4.12) and (4.13), one has

$$\begin{aligned}
\|f\|^p_{L^p_\omega(\mathbb{R}^{n_1+n_2})} &\leqslant \sum_i \Big\| \sum_{R\in\mathcal{B}_i} |R|\phi_R(\cdot - x_I, \cdot - x_J)(\phi_R * h)(x_I, x_J)\Big\|^p_{L^p_\omega(\mathbb{R}^{n_1+n_2})} \\
&\leqslant \sum_i \omega(\Omega_i)^{1-p/2} \Big\| \sum_{R\in\mathcal{B}_i} |R|\phi_R(\cdot - x_I, \cdot - x_J)(\phi_{j,k} * h)(x_I, x_J)\Big\|^p_{L^2_\omega(\mathbb{R}^{n_1+n_2})} \\
&\lesssim \sum_i \omega(\Omega_i)^{1-p/2} 2^{ip} \omega(\Omega_i)^{p/2} \lesssim \|\widetilde{S}(f)\|^p_{L^p_\omega(\mathbb{R}^{n_1+n_2})},
\end{aligned}$$

which yields that

$$\|f\|^p_{L^p_\omega(\mathbb{R}^{n_1+n_2})} \lesssim \|h\|^p_{H^p_{\rm mix}(\omega,\mathbb{R}^{n_1}\times\mathbb{R}^{n_2})} \approx \|f\|^p_{H^p_{\rm mix}(\omega,\mathbb{R}^{n_1}\times\mathbb{R}^{n_2})}$$

by Theorem 4.1.

Thus, we complete the proof. □

It is easy to see that Theorem 1.3 is a corollary of Theorems 1.2 and 4.2.

References

[1] *R. R. Coifman*: A real variable characterization of H^p. Stud. Math. *51* (1974), 269–274. zbl MR doi

[2] *R. R. Coifman, G. Weiss*: Extensions of Hardy spaces and their use in analysis. Bull. Am. Math. Soc. *83* (1977), 569–645. zbl MR doi

[3] *D. Cruz-Uribe, J. M. Martell, C. Pérez*: Sharp weighted estimates for classical operators. Adv. Math. *229* (2012), 408–441. zbl MR doi

[4] *Y. Ding, Y. Han, G. Lu, X. Wu*: Boundedness of singular integrals on multiparameter weighted Hardy spaces $H^p_w(\mathbb{R}^n \times \mathbb{R}^m)$. Potential Anal. *37* (2012), 31–56. zbl MR doi

[5] *W. Ding, G. Lu, Y. Zhu*: Multi-parameter local Hardy spaces. Nonlinear Anal., Theory Methods Appl., Ser. A *184* (2019), 352–380. zbl MR doi

[6] *W. Ding, M. Qin, Y. Zhu*: The boundedness on mixed Hardy spaces. J. Funct. Spaces *2020* (2020), Article ID 5341674, 12 pages. zbl MR doi

[7] *W. Ding, Y. Zhu*: Mixed Hardy spaces and their applications. Acta Math. Sci., Ser. B, Engl. Ed. *40* (2020), 945–969. MR doi

[8] *C. L. Fefferman, E. M. Stein*: H^p spaces of several variables. Acta Math. *129* (1972), 137–193. zbl MR doi

[9] *R. Fefferman*: Strong differentiation with respect to measures. Am. J. Math. *103* (1981), 33–40. zbl MR doi

[10] *R. Fefferman*: The atomic decomposition of H^1 in product spaces. Adv. Math. *55* (1985), 90–100. zbl MR doi

[11] *R. Fefferman*: Calderón-Zygmund theory for product domains: H^p spaces. Proc. Natl. Acad. Sci. USA *83* (1986), 840–843. zbl MR doi

[12] *R. Fefferman*: Harmonic analysis on product spaces. Ann. Math. (2) *126* (1987), 109–130. zbl MR doi

[13] *R. Fefferman, E. M. Stein*: Singular integrals on product spaces. Adv. Math. *45* (1982), 117–143. zbl MR doi

[14] *G. B. Folland, E. M. Stein*: Hardy Spaces on Homogeneous Groups. Mathematical Notes 28. Princeton University Press, Princeton, 1982. zbl MR doi
[15] *J. Garcia-Cuerva*: Weighted H^p spaces. Diss. Math. *162* (1979), 63 pages. zbl MR
[16] *J. Garcia-Cuerva, J. L. Rubio de Francia*: Weighted Norm Inequalities and Related Topics. North-Holland Mathematics Studies 116. North-Holland, Amsterdam, 1985. zbl MR doi
[17] *D. Goldberg*: A local version of real Hardy spaces. Duke Math. J. *46* (1979), 27–42. zbl MR doi
[18] *L. Grafakos*: Classical and Modern Fourier Analysis. Pearson/Prentice Hall, Upper Saddle River, 2004. zbl MR
[19] *R. F. Gundy, E. M. Stein*: H^p theory for the poly-disk. Proc. Natl. Acad. Sci. USA *76* (1979), 1026–1029. zbl MR doi
[20] *R. F. Gundy, R. L. Wheeden*: Weighted integral inequalities for the nontangential maximal function, Lusin area integral, and Walsh-Paley series. Stud. Math. *49* (1974), 107–124. zbl MR doi
[21] *Y. Han, M.-Y. Lee, C.-C. Lin, Y.-C. Lin*: Calderón-Zygmund operators on product Hardy spaces. J. Funct. Anal. *258* (2010), 2834–2861. zbl MR doi
[22] *Y. Han, C. Lin, G. Lu, Z. Ruan, E. T. Sawyer*: Hardy spaces associated with different homogeneities and boundedness of composition operators. Rev. Mat. Iberoam. *29* (2013), 1127–1157. zbl MR doi
[23] *Y. Han, G. Lu, Z. Ruan*: Boundedness criterion of Journé's class of singular integrals on multiparameter Hardy spaces. J. Funct. Anal. *264* (2013), 1238–1268. zbl MR doi
[24] *Y. Han, G. Lu, Z. Ruan*: Boundedness of singular integrals in Journé's class on weighted multiparameter Hardy spaces. J. Geom. Anal. *24* (2014), 2186–2228. zbl MR doi
[25] *Y. Han, G. Lu, K. Zhao*: Discrete Calderón's identity, atomic decomposition and boundedness criterion of operators on multiparameter Hardy spaces. J. Geom. Anal. *20* (2010), 670–689. zbl MR doi
[26] *G. H. Hardy*: The mean value of the modulus of an analytic function. Proc. Lond. Math. Soc. (2) *14* (1915), 269–277. zbl doi
[27] *J.-L. Journé*: Calderón-Zygmund operators on product spaces. Rev. Mat. Iberoam. *1* (1985), 55–91. zbl MR doi
[28] *J.-L. Journé*: Two problems of Calderón-Zygmund theory on product-spaces. Ann. Inst. Fourier *38* (1988), 111–132. zbl MR doi
[29] *R. H. Latter*: A characterization of $H^p(\mathbb{R}^n)$ in terms of atoms. Stud. Math. *62* (1978), 93–101. zbl MR doi
[30] *G. Lu, Y. Zhu*: Singular integrals and weighted Triebel-Lizorkin and Besov spaces of arbitrary number of parameters. Acta Math. Sin., Engl. Ser. *29* (2013), 39–52. zbl MR doi
[31] *Z. Ruan*: Weighted Hardy spaces in three-parameter case. J. Math. Anal. Appl. *367* (2010), 625–639. zbl MR doi
[32] *E. M. Stein, G. Weiss*: On the theory of harmonic functions of several variables. I. The theory of H^p-spaces. Acta Math. *103* (1960), 25–62. zbl MR doi
[33] *D. Yang, Y. Liang, L. D. Ky*: Real-Variable Theory of Musielak-Orlicz Hardy Spaces. Lecture Notes in Mathematics 2182. Springer, Cham, 2017. zbl MR doi
[34] *D. Yang, S. Yang*: Local Hardy spaces of Musielak-Orlicz type and their applications. Sci. China, Math. *55* (2012), 1677–1720. zbl MR doi

Authors' addresses: Wei Ding (corresponding author), Yun Xu, School of Sciences, Nantong University, Nantong 226007, P. R. China, e-mail: `dingwei@ntu.edu.cn`, `1134721412@qq.com`; Yueping Zhu, Department of Mathematics, Nantong Normal College, Nantong 226010, P. R. China, e-mail: `zhuyueping@ntu.edu.cn`.