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REMARKS ON SEKINE QUANTUM GROUPS

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Abstract. We first describe the Sekine quantum groups $\mathcal{A}_k$ (the finite-dimensional Kac algebra of Kac-Paljutkin type) by generators and relations explicitly, which maybe convenient for further study. Then we classify all irreducible representations of $\mathcal{A}_k$ and describe their representation rings $r(\mathcal{A}_k)$. Finally, we compute the the Frobenius-Perron dimension of the Casimir element and the Casimir number of $r(\mathcal{A}_k)$.

Keywords: Sekine quantum group; representation ring; Casimir number

MSC 2020: 16T05, 16D70, 16G10

1. Introduction

Sekine in [5] introduces a family of finite quantum groups $\mathcal{A}_k$, $k \in \mathbb{N}$, $k \geqslant 2$, referred to as Sekine quantum groups, arising as bicrossed products of classical cyclic groups with the matched pair being $\mathbb{Z}_2$ and $\mathbb{Z}_k \times \mathbb{Z}_k$, see [6] and [8] for more details. Sekine quantum groups form a class of finite quantum groups of Kac-Paljutkin type (see [3]) and they are neither commutative nor cocommutative Hopf algebras for $k > 2$, while $\mathcal{A}_2$ is cocommutative. These quantum groups attract great interest in many fields of mathematics, such as idempotent states, fusion category and so on. In particular, Sekine quantum groups play an important role in the study of idempotent states, since they provide examples that idempotent states on locally compact quantum groups are not necessarily Haar idempotent states. In [2], Franz and Skalski characterise all quantum subgroups of $\mathcal{A}_k$ for a given k and exhibit examples of idempotent states on $\mathcal{A}_k$ for each $k \geqslant 2$

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which are not Haar states on subgroups. In [8], the author computes all the idempotent states on Sekine quantum groups, which completes the work of Franz and Skalski, see [2].

As for the Sekine quantum groups $\mathcal{A}_k$, their algebraic structures are clear, however, the comultiplications are complicated and their applications may result in complex calculation, see [2], [5], and [8]. Can we rewrite the definition of the Sekine quantum groups so that the algebraic and coalgebraic structures are both simple and easy to handle? Inspired by this, in the present short paper we construct a Hopf algebra $\mathcal{H}$ by four generators with a family of relations, which is isomorphic to the Sekine quantum group $\mathcal{A}_k$. Then all the irreducible modules of $\mathcal{H}$ are classified. Furthermore, the representation ring $r(\mathcal{H})$ is established and the Casimir number of $r(\mathcal{H})$ is computed. It is noted that $\mathcal{H}$ has simple defining relations and comultiplications. It may be more convenient to study the Sekine quantum group in the future work.

The note is organized as follows. In Section 1, we first give the definition of the $2k^2$ dimensional Hopf algebra $\mathcal{H}$ by generators and relations, showing that $\mathcal{H}$ is a Hopf algebra. Then we prove that $\mathcal{H}$ is isomorphic to the Sekine quantum group $\mathcal{A}_k$ as a Hopf algebra. In Section 2, we give all the irreducible modules of $\mathcal{H}$ and establish the decomposition formulas of the tensor product of two irreducible $\mathcal{H}$-modules. The representation ring $r(\mathcal{H})$ is also described. Finally, we compute the Frobenius-Perron dimension of the Casimir element and the Casimir number of $r(\mathcal{H})$.

2. Sekine quantum groups

Throughout, we always assume that $\Bbbk$ is an algebraic closed field of characteristic zero. All algebras and modules are over the field $\Bbbk$.

Fix a $k \in \mathbb{Z}$, $k \geqslant 2$, η a primitive kth root of unity, $\mathbb{Z}_k := \{0, 1, \ldots, k-1\}$, $I =: \{1, 2, \ldots, k\}$, and $e_{i,j}$, $i, j \in I$, are the matrix units of $M_k(\Bbbk)$.

Now we introduce an algebra $\mathcal{H}$ by generators and generating relations.

Definition 2.1. Let $\mathcal{H}$ be an associative algebra generated by x, y and σ, τ with the relations

$$x^k = y^k, \quad \sigma^k = \tau^k, \quad x^k + \sigma^k = 1, \quad xy = yx, \quad \sigma\tau = \eta\tau\sigma, \quad ab = ba = 0$$

for $a = x, y$, $b = \sigma, \tau$.

It is easy to see that $\{x^m y^n, \sigma^i \tau^j \colon m, n, i, j \in I\}$ forms a basis of $\mathcal{H}$.

Proposition 2.2. *The algebra $\mathcal{H}$ is a Hopf algebra with the comultiplication Δ, counite ε and antipode S acting on the generators as follows:*

$$\Delta(x) = x \otimes x + \tau \otimes \tau, \quad \Delta(y) = y \otimes y + \sigma \otimes \sigma^{k-1}, \quad \Delta(\sigma) = y \otimes \sigma + \sigma \otimes y^{k-1},$$
$$\Delta(\tau) = \tau \otimes x + x \otimes \tau, \quad \varepsilon(x) = \varepsilon(y) = 1, \quad \varepsilon(\sigma) = \varepsilon(\tau) = 0,$$
$$S(x) = x^{k-1}, \quad S(y) = y^{k-1}, \quad S(\sigma) = \sigma, \quad S(\tau) = \tau^{k-1}.$$

Proof. Firstly, we show that Δ is a homomorphism of the algebra $\mathcal{H}$. Note that $y^k \cdot y^k = y^k(1-\sigma^k) = y^k - y^k\sigma^k = y^k$, we have $(y^{k-1})^k = (y^k)^{k-1} = y^k$, similarly, $(\sigma^{k-1})^k = (\sigma^k)^{k-1} = \sigma^k$. Therefore, we have

$$\begin{aligned}
\Delta(y)^k &= (y \otimes y + \sigma \otimes \sigma^{k-1})^k = y^k \otimes y^k + \sigma^k \otimes (\sigma^{k-1})^k \\
&= y^k \otimes y^k + \sigma^k \otimes \sigma^k = x^k \otimes x^k + \tau^k \otimes \tau^k = \Delta(x)^k, \\
\Delta(\sigma)^k &= (y \otimes \sigma + \sigma \otimes y^{k-1})^k = y^k \otimes \sigma^k + \sigma^k \otimes (y^{k-1})^k \\
&= y^k \otimes \sigma^k + \sigma^k \otimes y^k = x^k \otimes \tau^k + \tau^k \otimes x^k = \Delta(\tau)^k, \\
\Delta(x)\Delta(y) &= xy \otimes xy + \tau\sigma \otimes \tau\sigma^{k-1} = yx \otimes yx + \sigma\tau \otimes \sigma^{k-1}\tau = \Delta(y)\Delta(x), \\
\Delta(\sigma)\Delta(\tau) &= yx \otimes \sigma\tau + \sigma\tau \otimes y^{k-1}x = \eta xy \otimes \tau\sigma + \eta\tau\sigma \otimes xy^{k-1} = \eta\Delta(\tau)\Delta(\sigma), \\
\Delta(x)^k + \Delta(\sigma)^k &= x^k \otimes x^k + \tau^k \otimes \tau^k + y^k \otimes \sigma^k + \sigma^k \otimes (y^{k-1})^k \\
&= x^k \otimes (x^k + \sigma^k) + \tau^k \otimes (\tau^k + (y^{k-1})^k) \\
&= x^k \otimes 1 + \tau^k \otimes 1 = 1 \otimes 1 = \Delta(1).
\end{aligned}$$

It is easy to see that $\Delta(ab) = 0$ for $a = x, y$ and $b = \sigma, \tau$, therefore Δ is a homomorphism of $\mathcal{H}$. The coassociativity of Δ follows from the following identities:

$$\begin{aligned}
(\Delta \otimes \mathrm{id})\Delta(x) &= \Delta(x) \otimes x + \Delta(\tau) \otimes \tau \\
&= x \otimes x \otimes x + \tau \otimes \tau \otimes x + \tau \otimes x \otimes \tau + x \otimes \tau \otimes \tau \\
&= x \otimes \Delta(x) + \tau \otimes \Delta(\tau) = (\mathrm{id} \otimes \Delta)\Delta(x), \\
(\Delta \otimes \mathrm{id})\Delta(y) &= \Delta(y) \otimes y + \Delta(\sigma) \otimes \sigma^{k-1} \\
&= y \otimes y \otimes y + \sigma \otimes \sigma^{k-1} \otimes y + y \otimes \sigma \otimes \sigma^{k-1} + \sigma \otimes y^{k-1} \otimes \sigma^{k-1} \\
&= y \otimes \Delta(y) + \sigma \otimes \Delta(\sigma)^{k-1} = (\mathrm{id} \otimes \Delta)\Delta(y), \\
(\Delta \otimes \mathrm{id})\Delta(\sigma) &= \Delta(y) \otimes \sigma + \Delta(\sigma) \otimes y^{k-1} \\
&= y \otimes y \otimes \sigma + \sigma \otimes \sigma^{k-1} \otimes \sigma + y \otimes \sigma \otimes y^{k-1} + \sigma \otimes y^{k-1} \otimes y^{k-1} \\
&= y \otimes \Delta(\sigma) + \sigma \otimes \Delta(y^{k-1}) = (\mathrm{id} \otimes \Delta)\Delta(\sigma), \\
(\Delta \otimes \mathrm{id})\Delta(\tau) &= \Delta(\tau) \otimes x + \Delta(x) \otimes \tau \\
&= \tau \otimes x \otimes x + x \otimes \tau \otimes x + x \otimes x \otimes \tau + \tau \otimes \tau \otimes \tau \\
&= \tau \otimes \Delta(x) + x \otimes \Delta(\tau) = (\mathrm{id} \otimes \Delta)\Delta(\tau).
\end{aligned}$$

Secondly, we check that S is an anti-homomorphism of the algebra $\mathcal{H}$.

$$\begin{aligned}
S(y)^k &= (y^{k-1})^k = y^k = x^k = (x^{k-1})^k = S(x)^k, \\
S(\tau)^k &= (\tau^{k-1})^k = \tau^k = \sigma^k = S(\sigma)^k, \\
S(x)^k + S(\tau)^k &= (x^{k-1})^k + (\tau^{k-1})^k = x^k + \tau^k = x^k + \sigma^k = 1 = S(1), \\
S(y)S(x) &= y^{k-1}x^{k-1} = x^{k-1}y^{k-1} = S(x)S(y), \\
S(\tau)S(\sigma) &= \tau^{k-1}\sigma = \eta\sigma\tau^{k-1} = \eta S(\sigma)S(\tau), \\
S(a)S(b) &= 0 = S(b)S(a) \quad \text{for } a = x, y,\ b = \sigma, \tau.
\end{aligned}$$

At last, we show that $S * \mathrm{id} = \mathrm{id} * S = u\varepsilon$. As for the generators, we have

$$\begin{aligned}
S * \mathrm{id}(x) &= S(x)x + S(\tau)\tau = x^k + \tau^k = xS(x) + \tau S(\tau) = \mathrm{id} * S(x) = 1 = \varepsilon(x)1, \\
S * \mathrm{id}(y) &= S(y)y + S(\sigma)\sigma^{k-1} = y^k + \sigma^k = yS(y) + \sigma S(\sigma)^{k-1} = \mathrm{id} * S(y) \\
&= 1 = \varepsilon(y)1, \\
S * \mathrm{id}(\sigma) &= S(y)\sigma + S(\sigma)y^{k-1} = 0 = \mathrm{id} * S(\sigma) = \varepsilon(\sigma)1, \\
S * \mathrm{id}(\tau) &= S(\tau)x + S(x)\tau = 0 = \mathrm{id} * S(\tau) = \varepsilon(\tau)1.
\end{aligned}$$

Therefore, $\mathcal{H}$ is a Hopf algebra. $\square$

In fact, the Hopf algebra $\mathcal{H}$ is isomorphic to the Sekine quantum group $\mathcal{A}_k$. To see the fact, we first recall the definition of $\mathcal{A}_k$.

Definition 2.3 ([2]). Let

$$\mathcal{A}_k = \bigoplus_{i,j\in\mathbb{Z}_k} \Bbbk d_{i,j} \oplus M_k(\Bbbk)$$

as an algebra. The comultiplication, counit, and antipode of $\mathcal{A}_k$ are defined by the following formulas:

$$\begin{aligned}
\Delta(d_{i,j}) &= \sum_{m,n\in\mathbb{Z}_k} d_{m,n} \otimes d_{i-m,j-n} + \frac{1}{k}\sum_{m,n\in\mathbb{Z}_k} \eta^{i(m-n)} e_{m,n} \otimes e_{m+j,n+j}, \quad i,j \in \mathbb{Z}_k; \\
\Delta(e_{i,j}) &= \sum_{m,n\in\mathbb{Z}_k} \eta^{m(i-j)} d_{-m,-n} \otimes e_{i-n,j-n} \\
&\quad + \sum_{m,n\in\mathbb{Z}_k} \eta^{m(j-i)} e_{i-n,j-n} \otimes d_{m,n}, \quad i,j \in I; \\
\varepsilon(d_{i,j}) &= \begin{cases} 1, & i = j = 0, \\ 0, & \text{others}, \end{cases} \quad i,j \in \mathbb{Z}_k; \quad \varepsilon(e_{i,j}) = 0,\ i,j \in I; \\
S(d_{i,j}) &= d_{-i,-j}, \quad i,j \in \mathbb{Z}_k, \quad S(e_{i,j}) = e_{j,i}, \quad i,j \in I.
\end{aligned}$$

Then $\mathcal{A}_k$ is a finite dimensional Hopf algebra, called the Sekine quantum group.

Theorem 2.4. *The Sekine quantum group $\mathcal{A}_k$ is isomorphic to the Hopf algebra $\mathcal{H}$.*

P r o o f. Let Φ be the linear map from $\mathcal{H}$ to $\mathcal{A}_k$, where the images of the generators under Φ are

$$\Phi(x)=\sum_{i,j=0}^{k-1}\eta^{-i}d_{i,j},\quad \Phi(y)=\sum_{i,j=0}^{k-1}\eta^{-j}d_{i,j},$$

$$\Phi(\sigma)=\sum_{s=1}^{k}\eta^{s-1}e_{s,s},\quad \Phi(\tau)=\sum_{s=1}^{k}e_{s+1,s}.$$

Firstly, we show that Φ is an algebra homomorphism. Note that

$$\begin{aligned}
(\Phi(x))^k &= \Big(\sum_{i,j=0}^{k-1}\eta^{-i}d_{i,j}\Big)^k=\sum_{i,j=0}^{k-1}d_{i,j}=\Big(\sum_{i,j=0}^{k-1}\eta^{-j}d_{i,j}\Big)^k=(\Phi(y))^k,\\
(\Phi(\sigma))^k &= \Big(\sum_{s=1}^{k}\eta^{s-1}e_{s,s}\Big)^k=\sum_{s=1}^{k}e_{s,s}=\Big(\sum_{s=1}^{k}e_{s+1,s}\Big)^k=(\Phi(\tau))^k,\\
(\Phi(x))^k+(\Phi(\sigma))^k &= \sum_{i,j=0}^{k-1}d_{i,j}+\sum_{s=1}^{k}e_{s,s}=1,\\
\Phi(x)\Phi(y) &= \Big(\sum_{i,j=0}^{k-1}\eta^{-i}d_{i,j}\Big)\Big(\sum_{i,j=0}^{k-1}\eta^{-j}d_{i,j}\Big)=\Phi(y)\Phi(x)\\
\Phi(\sigma)\Phi(\tau) &= \Big(\sum_{s=1}^{k}\eta^{s-1}e_{s,s}\Big)\Big(\sum_{s=1}^{k}e_{s+1,s}\Big)=\sum_{s=1}^{k}\eta^{s}e_{s+1,s},\\
\Phi(\tau)\Phi(\sigma) &= \Big(\sum_{s=1}^{k}e_{s+1,s}\Big)\Big(\sum_{s=1}^{k}\eta^{s-1}e_{s,s}\Big)=\sum_{s=1}^{k}\eta^{s-1}e_{s+1,s},\\
\Phi(a)\Phi(b) &= 0=\Phi(b)\Phi(a)\quad \text{for } a=x,y,\ b=\sigma,\tau.
\end{aligned}$$

It follows that Φ preserves all the relations of Definition 2.1 and therefore Φ is an algebra homomorphism.

Secondly, we prove that Φ is bijective. Note that $\Phi\big(x^my^n\big)=\sum\limits_{i,j=0}^{k-1}\eta^{-(im+jn)}d_{i,j}$ and $\Phi\big(\sigma^i\tau^j\big)=\sum\limits_{s=1}^{k}\eta^{is}e_{s+1,s+1-j}$, therefore if we let

$$\sum_{m,n=1}^{k}a_{m,n}\sum_{i,j=0}^{k-1}\eta^{-(im+jn)}d_{i,j}=0\quad\text{and}\quad\sum_{i,j=1}^{k}b_{ij}\sum_{s=1}^{k}\eta^{is}e_{s+1,s+1-j}=0,$$

it suffices to prove that $a_{m,n} = 0$ for any $m, n = 1, \ldots, k$ and $b_{ij} = 0$ for any $i, j = 1, 2, \ldots, k$. Since $d_{i,j}$ are linearly independent for different i and j, we have

$$\sum_{m,n=1}^{k} a_{m,n} \eta^{-(im+jn)} = 0$$

for any $i, j \in \mathbb{Z}_k$. Let $x_m = \sum_{n=1}^{k} a_{m,n} \eta^{-jn}$, then the above equation can be rewritten as

$$\sum_{m=1}^{k} \eta^{-im} x_m = 0.$$

This is a system of homogeneous linear equations, whose coefficient determinant is a Vandermonde determinant which is not zero, it follows that $x_m = 0$. The same reason forces all the $a_{m,n} = 0$. Similarly, we can prove that all the $b_{ij} = 0$.

Thirdly, we show that Φ is also a coalgebra homomorphism. It is just to prove that for any generators $a = x, y, \sigma, \tau$, we have $\Delta\Phi(a) = (\Phi \otimes \Phi)\Delta(a)$ and $\varepsilon \circ \Phi(a) = \varepsilon(a)$.

For x, we have

$$\begin{aligned}
\Delta\Phi(x) &= \Delta\Bigg(\sum_{i,j=0}^{k-1} \eta^{-i} d_{i,j}\Bigg) \\
&= \sum_{i,j=0}^{k-1} \Bigg(\sum_{m,n=0}^{k-1} \eta^{-i} d_{m,n} \otimes d_{i-m,j-n} + \frac{1}{k} \sum_{s,t=1}^{k} \eta^{i(s-t-1)} e_{s,t} \otimes e_{s+j,t+j}\Bigg), \\
(\Phi \otimes \Phi)\Delta(x) &= (\Phi \otimes \Phi)(x \otimes x + \tau \otimes \tau) \\
&= \sum_{i,j=0}^{k-1} \eta^{-i} d_{i,j} \otimes \sum_{i,j=0}^{k-1} \eta^{-i} d_{i,j} + \sum_{s=1}^{k} e_{s+1,s} \otimes \sum_{s=1}^{k} e_{s+1,s} \\
&= \sum_{i,j=0}^{k-1} \sum_{i',j'=0}^{k-1} \eta^{-(i+i')} d_{i,j} \otimes d_{i',j'} + \sum_{s,t=1}^{k} e_{s+1,s} \otimes e_{t+1,t}.
\end{aligned}$$

Note that if $s - t \neq 1$, then

$$\frac{1}{k} \sum_{i,j=0}^{k-1} \eta^{i(s-t-1)} e_{s,t} \otimes e_{s+j,t+j} = \frac{1}{k} \sum_{j=0}^{k-1} \Bigg(\sum_{i=0}^{k-1} \eta^{i(s-t-1)}\Bigg) e_{s,t} \otimes e_{s+j,t+j} = 0,$$

so we have $\Delta\Phi(x) = (\Phi \otimes \Phi)\Delta(x)$.

For y, we have

$$
\begin{aligned}
\Delta\Phi(y) &= \Delta\Bigg(\sum_{i,j=0}^{k-1} \eta^{-j} d_{i,j}\Bigg) \\
&= \sum_{i,j=0}^{k-1} \Bigg(\sum_{m,n=0}^{k-1} \eta^{-j} d_{m,n} \otimes d_{i-m,j-n} + \frac{1}{k} \sum_{s,t=1}^{k} \eta^{i(s-t)-j} e_{s,t} \otimes e_{s+j,t+j}\Bigg), \\
(\Phi \otimes \Phi)\Delta(y) &= \sum_{i,j=0}^{k-1} \eta^{-j} d_{i,j} \otimes \sum_{i,j=0}^{k-1} \eta^{-j} d_{i,j} + \sum_{s=1}^{k} \eta^{s-1} e_{s,s} \otimes \sum_{s=1}^{k} \eta^{1-s} e_{s,s} \\
&= \sum_{i,j=0}^{k-1} \sum_{i',j'=0}^{k-1} \eta^{-(j+j')} d_{i,j} \otimes d_{i',j'} + \sum_{s,t=1}^{k} \eta^{s-t} e_{s,s} \otimes e_{t,t}.
\end{aligned}
$$

Note that if $s - t \neq 0$, then

$$
\frac{1}{k} \sum_{i,j=0}^{k-1} \eta^{i(s-t)-j} e_{s,t} \otimes e_{s+j,t+j} = \frac{1}{k} \sum_{j=0}^{k-1} \Bigg(\sum_{i=0}^{k-1} \eta^{i(s-t)}\Bigg) \eta^{-j} e_{s,t} \otimes e_{s+j,t+j} = 0,
$$

so we have $\Delta\Phi(y) = (\Phi \otimes \Phi)\Delta(y)$.

For σ, we have

$$
\begin{aligned}
\Delta\Phi(\sigma) &= \Delta\Bigg(\sum_{s=1}^{k} \eta^{s-1} e_{s,s}\Bigg) \\
&= \sum_{s=1}^{k} \eta^{s-1} \Bigg(\sum_{i,j=0}^{k-1} d_{-i,-j} \otimes e_{s-j,s-j} + \sum_{i,j=0}^{k-1} e_{s-j,s-j} \otimes d_{i,j}\Bigg), \\
(\Phi \otimes \Phi)\Delta(\sigma) &= \sum_{i,j=0}^{k-1} \eta^{-j} d_{i,j} \otimes \sum_{s=1}^{k} \eta^{s-1} e_{s,s} + \sum_{s=1}^{k} \eta^{s-1} e_{s,s} \otimes \sum_{i,j=0}^{k-1} (\eta^{-j})^{k-1} d_{i,j} \\
&= \sum_{i,j=0}^{k-1} \sum_{s=1}^{k} \eta^{s-j-1} d_{i,j} \otimes e_{s,s} + \sum_{s=1}^{k} \sum_{i,j=0}^{k-1} \eta^{s+j-1} e_{s,s} \otimes d_{i,j}.
\end{aligned}
$$

Clearly, $\Delta\Phi(\sigma) = (\Phi \otimes \Phi)\Delta(\sigma)$.

For τ, we have

$$
\begin{aligned}
\Delta\Phi(\tau) &= \Delta\Bigg(\sum_{s=1}^{k} e_{s+1,s}\Bigg) \\
&= \sum_{s=1}^{k} \Bigg(\sum_{i,j=0}^{k-1} \eta^{i} d_{-i,-j} \otimes e_{s+1-j,s-j} + \sum_{i,j=0}^{k-1} \eta^{-i} e_{s+1-j,s-j} \otimes d_{i,j}\Bigg), \\
(\Phi \otimes \Phi)\Delta(\tau) &= (\Phi \otimes \Phi)(\tau \otimes x + x \otimes \tau) \\
&= \sum_{s=1}^{k} e_{s+1,s} \otimes \sum_{i,j=0}^{k-1} \eta^{-i} d_{i,j} + \sum_{i,j=0}^{k-1} \eta^{-i} d_{i,j} \otimes \sum_{s=1}^{k} e_{s+1,s}.
\end{aligned}
$$

Clearly, $\Delta\Phi(\tau) = (\Phi \otimes \Phi)\Delta(\tau)$. Note that

$$\varepsilon \circ \Phi(x) = \varepsilon\left(\sum_{i,j=0}^{k-1} \eta^{-i} d_{i,j}\right) = \sum_{i,j=0}^{k-1} \eta^{-i} \varepsilon(d_{i,j}) = 1 = \varepsilon(x);$$

$$\varepsilon \circ \Phi(y) = \varepsilon\left(\sum_{i,j=0}^{k-1} \eta^{-j} d_{i,j}\right) = \sum_{i,j=0}^{k-1} \eta^{-i} \varepsilon(d_{i,j}) = 1 = \varepsilon(y);$$

$$\varepsilon \circ \Phi(\sigma) = \varepsilon\left(\sum_{s=1}^{k} \eta^{s-1} e_{s,s}\right) = 0 = \varepsilon(\sigma);$$

$$\varepsilon \circ \Phi(\tau) = \varepsilon\left(\sum_{s=1}^{k} e_{s+1,s}\right) = 0 = \varepsilon(\tau).$$

Therefore, Φ is a coalgebra homomorphism.

Finally, we prove that for any generators $a \in \mathcal{H}$, $\Phi(S(a)) = S(\Phi(a))$. Note that

$$\Phi(S(x)) = \Phi(x^{k-1}) = \left(\sum_{i,j=0}^{k-1} \eta^{-i} d_{i,j}\right)^{k-1} = \sum_{i,j=0}^{k-1} \eta^{i} d_{i,j};$$

$$S(\Phi(x)) = S\left(\sum_{i,j=0}^{k-1} \eta^{-i} d_{i,j}\right) = \sum_{i,j=0}^{k-1} \eta^{-i} d_{-i,-j} = \Phi(S(x));$$

$$\Phi(S(y)) = \Phi(y^{k-1}) = \left(\sum_{i,j=0}^{k-1} \eta^{-j} d_{i,j}\right)^{k-1} = \sum_{i,j=0}^{k-1} \eta^{j} d_{i,j};$$

$$S(\Phi(y)) = S\left(\sum_{i,j=0}^{k-1} \eta^{-j} d_{i,j}\right) = \sum_{i,j=0}^{k-1} \eta^{-j} d_{-i,-j} = \Phi(S(y));$$

$$\Phi(S(\sigma)) = \Phi(\sigma) = \sum_{s=1}^{k} \eta^{s-1} e_{s,s};$$

$$S(\Phi(\sigma)) = S\left(\sum_{s=1}^{k} \eta^{s-1} e_{s,s}\right) = \sum_{s=1}^{k} \eta^{s-1} e_{s,s} = \Phi(S(\sigma));$$

$$\Phi(S(\tau)) = \Phi(\tau^{k-1}) = \left(\sum_{s=1}^{k} e_{s+1,s}\right)^{k-1} = \sum_{s=1}^{k} e_{s+1,s+2};$$

$$S(\Phi(\tau)) = S\left(\sum_{s=1}^{k} e_{s+1,s}\right) = \sum_{s=1}^{k} e_{s,s+1} = \Phi(S(\tau)).$$

The proof is completed. $\square$

3. Representations of the Sekine quantum groups

Let S_{ij}, $i,j\in\mathbb{Z}_k$, be a one-dimensional irreducible $\mathcal{H}$-module with the basis v_{ij}, the actions of the generators of $\mathcal{H}$ on S_{ij} being

$$x\cdot v_{ij}=\eta^{-i}v_{ij},\quad y\cdot v_{ij}=\eta^{-j}v_{ij},\quad \sigma\cdot v_{ij}=0=\tau\cdot v_{ij}.$$

Let P be the k-dimensional irreducible $\mathcal{H}$-module with the basis $\{v_l\colon\ l\in\mathbb{Z}_k\}$, the module actions being

$$x\cdot v_l=0=y\cdot v_l,\quad \sigma\cdot v_l=\eta^l v_l,\quad \tau\cdot v_l=v_{l+1}.$$

Lemma 3.1. *The following statements hold.*

(1) $S_{ij}\otimes S_{i'j'}\cong S_{i+i',j+j'}$ *for all* $i,j,i',j'\in\mathbb{Z}_k$,

(2) $S_{ij}\otimes P\cong P\cong P\otimes S_{ij}$ *for all* $i,j\in\mathbb{Z}_k$,

(3) $P\otimes P\cong\bigoplus\limits_{i',j'\in\mathbb{Z}_k}S_{i'j'}$.

Proof. (1) Suppose that v_{ij} and $v_{i'j'}$ are the bases of S_{ij} and $S_{i'j'}$, respectively. Then $v_{ij}\otimes v_{i'j'}$ is the basis of $S_{ij}\otimes S_{i'j'}$ and we have

$$\begin{aligned}
x\cdot(v_{ij}\otimes v_{i'j'})&=x\cdot v_{ij}\otimes x\cdot v_{i'j'}+\tau\cdot v_{ij}\otimes\tau\cdot v_{i'j'}=\eta^{-(i+i')}v_{ij}\otimes v_{i'j'},\\
y\cdot(v_{ij}\otimes v_{i'j'})&=y\cdot v_{ij}\otimes y\cdot v_{i'j'}+\sigma\cdot v_{ij}\otimes\tau^{k-1}\cdot v_{i'j'}=\eta^{-(j+j')}v_{ij}\otimes v_{i'j'},\\
\sigma\cdot(v_{ij}\otimes v_{i'j'})&=y\cdot v_{ij}\otimes\sigma\cdot v_{i'j'}+\sigma\cdot v_{ij}\otimes y^{k-1}\cdot v_{i'j'}=0,\\
\tau\cdot(v_{ij}\otimes v_{i'j'})&=\tau\cdot v_{ij}\otimes x\cdot v_{i'j'}+x\cdot v_{ij}\otimes\tau\cdot v_{i'j'}=0.
\end{aligned}$$

So we get (1).

(2) Take $\{v_{ij}\colon\ i,j\in\mathbb{Z}_k\}$ and $\{v_l\colon\ l\in\mathbb{Z}_k\}$ for the bases of S_{ij} and P, respectively. Then

$$v'_l=\eta^{-i(l+j)}v_{ij}\otimes v_{l+j},\quad l\in\mathbb{Z}_k,$$

are the bases of $S_{ij}\otimes P$ and we have

$$\begin{aligned}
x\cdot v'_l&=\eta^{-i(l+j)}(x\cdot v_{ij}\otimes x\cdot v_{l+j}+\tau\cdot v_{ij}\otimes\tau\cdot v_{l+j})=0,\\
y\cdot v'_l&=\eta^{-i(l+j)}(y\cdot v_{ij}\otimes y\cdot v_{l+j}+\sigma\cdot v_{ij}\otimes\sigma^{k-1}\cdot v_{l+j})=0,\\
\sigma\cdot v'_l&=\eta^{-i(l+j)}(y\cdot v_{ij}\otimes\sigma\cdot v_{l+j}+\sigma\cdot v_{ij}\otimes y^{k-1}\cdot v_{l+j})=\eta^l v'_l,\\
\tau\cdot v'_l&=\eta^{-i(l+j)}(\tau\cdot v_{ij}\otimes x\cdot v_{l+j}+x\cdot v_{ij}\otimes\tau\cdot v_{l+j})=\eta^{-i(l+j+1)}v_{ij}\otimes v_{l+j+1}=v'_{l+1}.
\end{aligned}$$

Similarly, let

$$v''_l=\eta^{-i(l+j)}v_{l-j}\otimes v_{ij},\quad l\in\mathbb{Z}_k,$$

then $\{v''_l\colon\ l\in\mathbb{Z}_k\}$ are the bases of $P\otimes S_{ij}$ and we have $P\cong P\otimes S_{ij}$. Therefore, (2) is proved.

(3) For $i, j \in \mathbb{Z}_k$, let $v_{ij} = \sum\limits_{l=0}^{k-1} \eta^{il} v_l \otimes v_{l+j}$, where $\{v_l\colon\ l \in \mathbb{Z}_k\}$ and $\{v_{l+j}\colon\ l+j \in \mathbb{Z}_k\}$ are the bases of P, respectively. Note that

$$x \cdot v_{ij} = \sum_{l=0}^{k-1} \eta^{il} (x \cdot v_l \otimes x \cdot v_{l+j} + \tau \cdot v_l \otimes \tau \cdot v_{l+j}) = \sum_{l=0}^{k-1} \eta^{il} v_{l+1} \otimes v_{l+j+1} = \eta^{-i} v_{ij},$$

$$y \cdot v_{ij} = \sum_{l=0}^{k-1} \eta^{il} (y \cdot v_l \otimes y \cdot v_{l+j} + \sigma \cdot v_l \otimes \sigma^{k-1} \cdot v_{l+j}) = \sum_{l=0}^{k-1} \eta^{il-j} v_l \otimes v_{l+j} = \eta^{-j} v_{ij},$$

$$\sigma \cdot v_{ij} = \sum_{l=0}^{k-1} \eta^{il} (y \cdot v_l \otimes \sigma \cdot v_{l+j} + \sigma \cdot v_l \otimes y^{k-1} \cdot v_{l+j}) = 0,$$

$$\tau \cdot v_{ij} = \sum_{l=0}^{k-1} \eta^{il} (\tau \cdot v_l \otimes x \cdot v_{l+j} + x \cdot v_l \otimes \tau \cdot v_{l+j}) = 0.$$

Hence, $P \otimes P \cong \bigoplus\limits_{i', j' \in \mathbb{Z}_k} S_{i'j'}$. □

Theorem 3.2. *The representation ring $r(\mathcal{H})$ of $\mathcal{H}$ is generated by X_1, X_2 and Y with the relations*

$$X_1 X_2 = X_2 X_1, \quad X_i^k = 1, \quad X_i Y = Y = Y X_i, \quad i = 1, 2, \tag{3.1}$$

$$Y^2 = \sum_{i,j=1}^{k} X_1^i X_2^j. \tag{3.2}$$

The set $\{Y, X_1^i X_2^j\colon\ i, j = 1, 2, \ldots, k\}$ is a $\mathbb{Z}$ basis of $r(\mathcal{H})$.

Proof. Let R be the ring generated by X_1, X_2 and Y with the relations (3.1) and (3.2). It is easy to see that there is a unique ring epimorphism

$$\Phi\colon\ R \to r(\mathcal{H})$$

defined by $\Phi(X_1) = [S_{01}]$, $\Phi(X_2) = [S_{10}]$, $\Phi(Y) = [P]$. (Here and in the following, $[X]$ represents the isomorphism classes of finite dimensional irreducible $\mathcal{H}$ modules X.) Comparing the ranks of R and $r(\mathcal{H})$, we get that Φ is an isomorphism. □

It is well known that the representation ring $r(\mathcal{H})$ or the representation algebra $r(\mathcal{H}) \otimes_{\mathbb{Z}} \Bbbk$ are a Frobenius algebra with the nonsingular, associative and symmetric bilinear form defined by

$$\beta([X], [Y]) = \dim_{\Bbbk} \operatorname{Hom}_{\mathcal{H}}(X, Y^*)$$

for any irreducible representations X, Y of $\mathcal{H}$, $Y^* = \operatorname{Hom}_{\Bbbk}(Y, \Bbbk)$, and the $\mathcal{H}$-actions on the dual Y^* are

$$a \cdot f(y) = f(S(a) \cdot y)$$

for $a \in \mathcal{H}$, $f \in Y^*$ and $y \in Y$.

In the following, we compute the Casimir number of $r(\mathcal{H})$, which can be used to determine whether or not the representation ring $r(\mathcal{H})$ is Jacobson semisimple, namely, it has the zero Jacobson radical, see [7]. Before this, we first recall the definition of the Casimir element and Casimir number in general. Let R be a free Frobenius $\mathbb{Z}$-algebra of finite rank with a nonsingular associative bilinear form $\beta\colon R\times R\to\mathbb{Z}$. If R has two $\mathbb{Z}$-bases $\{x_i\colon 1\leqslant i\leqslant n\}$ and $\{y_i\colon 1\leqslant i\leqslant n\}$ satisfying $\beta(x_i,y_j)=\delta_{ij}$, where δ_{ij} is the Kronecker symbol, then these two bases are called *dual bases* of R with respect to $\beta(-,-)$. Accordingly, any $a\in R$ can be written as

$$a=\sum_{i=1}^{n}\beta(a,y_i)x_i \quad\text{or}\quad a=\sum_{i=1}^{n}\beta(x_i,a)y_i.$$

The Casimir operator of R (see e.g. [4]) is the map c from R to its centre $\mathbb{Z}(R)$ defined by

$$c(a)=\sum_{i=1}^{n}y_iax_i$$

for $a\in R$. The map c is independent of the choice of dual bases $\{x_i\colon 1\leqslant i\leqslant n\}$ and $\{y_i\colon 1\leqslant i\leqslant n\}$. The element $c(1)=\sum\limits_{i=1}^{n}y_ix_i$ is called the *Casimir element* of R and it depends on $\beta(-,-)$ only up to a central unit. The image $\operatorname{Im}c$ of c is an ideal of $\mathbb{Z}(R)$, called the *Casimir ideal* of R. It does not depend on the choice of the bilinear form. Set

$$\mathcal{I}_R:=\{x\in R\colon\ c(x)=n_x1_R\}.$$

Then $0\in\mathcal{I}_R$. Let $\mathcal{I}_{\mathbb{Z}}=\{n_x\colon\ x\in\mathcal{I}_R\}$, then $\mathcal{I}_{\mathbb{Z}}$ is an ideal of $\mathbb{Z}$. Indeed, if $n_x\in\mathcal{I}_{\mathbb{Z}}$ and $n\in\mathbb{Z}$, then there exists $x\in\mathcal{I}_R$ such that $c(nx)=nc(x)=nn_x1_R$, that is $nn_x\in\mathcal{I}_{\mathbb{Z}}$. Therefore, there exists $m\in\mathbb{Z}$ such that $\mathcal{I}_{\mathbb{Z}}=(m)$. Then $n_R:=m$ does not depend on the choice of the bilinear form on R and it is called the *Casimir number* of R.

Furthermore, for each object X of $r(\mathcal{H})$, one can define its Frobenius-Perron dimension $\operatorname{FP}\dim(X)$ (see [1]), which is the largest positive eigenvalue of the matrix of the left or the right multiplication by X. We have the following theorem.

Theorem 3.3. $\operatorname{FP}\dim(c(1))=2k^2$ *and the Casimir number* $n_{r(\mathcal{H})}=2k^4$.

Proof. By the bilinear form

$$\beta([X],[Y])=\dim_{\Bbbk}\operatorname{Hom}_{\mathcal{H}}(X,Y^*),$$

where X, Y are irreducible representations of $\mathcal{H}$ and $Y^*=\operatorname{Hom}_{\Bbbk}(Y,\Bbbk)$, it is easy to see that $S_{ij}^*=S_{-i,-j}$ and $P^*=P$. Therefore, $r(\mathcal{H})$ has the dual basis $\{P,S_{ij}\colon i,j\in\mathbb{Z}_k\}$ and $\{P,S_{-i,-j}\colon i,j\in\mathbb{Z}_k\}$ with respect to the bilinear form above. Accordingly, the Casimir operator of $r(\mathcal{H})$ is

$$c(a) = PaP + \sum_{i,j\in\mathbb{Z}_k} S_{ij} a S_{-i,-j}$$

for any $a \in r(\mathcal{H})$, where the multiplication is given as in Lemma 3.1. In particular, the Casimir element is

$$c(1) = P^2 + \sum_{i,j\in\mathbb{Z}_k} S_{ij} S_{-i,-j} = \sum_{i,j\in\mathbb{Z}_k} S_{i,j} + k^2 S_{0,0}.$$

It follows that

$$c(S_{s,t}) = k^2 S_{s,t} + \sum_{i,j\in\mathbb{Z}_k} S_{i,j}.$$

Furthermore, $c(\sum_{i,j\in\mathbb{Z}_k} S_{i,j}) = 2k^2\Big(\sum_{i,j\in\mathbb{Z}_k} S_{i,j}\Big)$ and $c(P) = k^2P + k^2P = 2k^2P$. Let $\ell_{c(1)}$ be the left multiplication of $c(1)$. Then the matrix of $\ell_{c(1)}$ under the basis $\{S_{i,j} \colon i,j = 0,1,\ldots,n-1, P\}$ is

$$\begin{pmatrix} k^2+1 & 1 & \ldots & 1 & 0 \\ 1 & k^2+1 & \ldots & 1 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 1 & 1 & \ldots & k^2+1 & 0 \\ 0 & 0 & \ldots & 0 & k^2 \end{pmatrix}.$$

It follows that k^2 and $2k^2$ are the different eigenvalues of $\ell_{c(1)}$, hence

$$\operatorname{FP}\dim(c(1)) = 2k^2.$$

Recall that $c(x) = xc(1)$, where $x = \sum_{i,j\in\mathbb{Z}_k} s_{ij}S_{ij} + nP$ with $s_{ij}, n \in \mathbb{Z}$, so we get

$$c(x) = \sum_{i,j\in\mathbb{Z}_k} \Big(k^2 s_{ij} + \sum_{k,l\in\mathbb{Z}_k} s_{kl}\Big) S_{ij} + 2n\, k^2 P.$$

If $c(x) \in \mathbb{Z}1_{r(\mathcal{H})}$, we have

$$\begin{cases} 2nk^2 = 0, \\ k^2 s_{ij} + \sum_{k,l\in\mathbb{Z}_k} s_{kl} = 0, \quad (i,j) \neq (0,0). \end{cases}$$

It follows that $n = 0$ and $s_{ij} = s_{pq} = \mu \in \mathbb{Z}$ for $(i,j) \neq (p,q)$ in $\mathbb{Z}_k \times \mathbb{Z}_k - (0,0)$. Hence, $s_{00} = (1-2k^2)\mu$. In this case, we have

$$x = -2k^2\mu S_{00} + \mu \sum_{i,j\in\mathbb{Z}_k} S_{ij} \in r(\mathcal{H}) \quad \text{and} \quad c(x) = -2k^4\mu S_{00}.$$

It is noted that S_{00} is the identity $1_{r(\mathcal{H})}$ of $r(\mathcal{H})$, hence the Casimir number $n_{r(\mathcal{H})} = 2k^4$. □

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