Nesrin Güler; Melek Eriş Büyükkaya

Some remarks on comparison of predictors in seemingly unrelated linear mixed models

Applications of Mathematics, Vol. 67 (2022), No. 4, 525–542

Persistent URL: http://dml.cz/dmlcz/150441

SOME REMARKS ON COMPARISON OF PREDICTORS IN SEEMINGLY UNRELATED LINEAR MIXED MODELS

Nesrİn Güler, Sakarya, Melek Erİş Büyükkaya, Trabzon

Received December 21, 2020. Published online October 18, 2021.

Abstract. In this paper, we consider a comparison problem of predictors in the context of linear mixed models. In particular, we assume a set of m different seemingly unrelated linear mixed models (SULMMs) allowing correlations among random vectors across the models. Our aim is to establish a variety of equalities and inequalities for comparing covariance matrices of the best linear unbiased predictors (BLUPs) of joint unknown vectors under SULMMs and their combined model. We use the matrix rank and inertia method for establishing equalities and inequalities. We also give an extensive approach for seemingly unrelated regression models (SURMs) by applying the results obtained for SULMMs to SURMs.

Keywords: BLUP; covariance matrix; inertia; OLSP; rank; SULMM; SURM

MSC 2020: 62J05, 62H12, 15A03

1. Introduction

Linear regression models are one of the most commonly used well-known statistical methods from both theoretical and practical points of view. In linear regression models, regression coefficients are considered fixed, while linear mixed models extend the linear regression models by allowing for the addition of random effects. In statistical analysis, data may be collected from the same individuals over time, or data in some studies may be collected from clusters in related statistical units. Linear mixed models used for both fixed and random effects in the same analysis are useful tools for researchers for modeling such problems encountered and analyzing the data when the correlations exist among the observations. Statistical inference concerning linear mixed models is an important part of data analysis and there have been various books on such models and related topics in the field of statistics and other disciplines; see e.g. [5], [6], [16], [34] among others.

DOI: 10.21136/AM.2021.0366-20

In some statistical problems, we may encounter several number of models, each with different dependent variables. Within the framework of linear mixed models, we can consider the following set of m different linear mixed models, formulated by

$$\mathcal{M}_i\colon\ \mathbf{y}_i = \mathbf{X}_i\boldsymbol{\alpha}_i + \mathbf{Z}_i\boldsymbol{\gamma}_i + \boldsymbol{\varepsilon}_i, \tag{1.1}$$

where $\mathbf{y}_i \in \mathbb{R}^{n_i\times 1}$ is a vector of observable response variables, $\mathbf{X}_i \in \mathbb{R}^{n_i\times k_i}$ and $\mathbf{Z}_i \in \mathbb{R}^{n_i\times p_i}$ are known matrices of arbitrary rank, $\boldsymbol{\alpha}_i \in \mathbb{R}^{k_i\times 1}$ is a vector of fixed but unknown parameters, $\boldsymbol{\gamma}_i \in \mathbb{R}^{p_i\times 1}$ is a vector of unobservable random effects and $\boldsymbol{\varepsilon}_i \in \mathbb{R}^{n_i\times 1}$ is an unobservable vector of random errors, $i = 1,\ldots,m$. Instead of considering the models $\mathcal{M}_i$ in (1.1) individually, a common approach to a set of different models is to consider them jointly since combining information on different models may lead to gain efficiency in prediction or estimation. We can combine these models in the form

$$\begin{bmatrix}\mathbf{y}_1\\ \mathbf{y}_2\\ \vdots\\ \mathbf{y}_m\end{bmatrix} = \begin{bmatrix}\mathbf{X}_1 & \mathbf{0} & \ldots & \mathbf{0}\\ \mathbf{0} & \mathbf{X}_2 & \ldots & \mathbf{0}\\ \vdots & \vdots & \ddots & \vdots\\ \mathbf{0} & \mathbf{0} & \ldots & \mathbf{X}_m\end{bmatrix}\begin{bmatrix}\boldsymbol{\alpha}_1\\ \boldsymbol{\alpha}_2\\ \vdots\\ \boldsymbol{\alpha}_m\end{bmatrix} + \begin{bmatrix}\mathbf{Z}_1 & \mathbf{0} & \ldots & \mathbf{0}\\ \mathbf{0} & \mathbf{Z}_2 & \ldots & \mathbf{0}\\ \vdots & \vdots & \ddots & \vdots\\ \mathbf{0} & \mathbf{0} & \ldots & \mathbf{Z}_m\end{bmatrix}\begin{bmatrix}\boldsymbol{\gamma}_1\\ \boldsymbol{\gamma}_2\\ \vdots\\ \boldsymbol{\gamma}_m\end{bmatrix} + \begin{bmatrix}\boldsymbol{\varepsilon}_1\\ \boldsymbol{\varepsilon}_2\\ \vdots\\ \boldsymbol{\varepsilon}_m\end{bmatrix},$$

or briefly, it can be rewritten in the following compact form:

$$\mathcal{M}\colon\ \mathbf{y} = \mathbf{X}\boldsymbol{\alpha} + \mathbf{Z}\boldsymbol{\gamma} + \boldsymbol{\varepsilon}, \tag{1.2}$$

where $\mathbf{y} \in \mathbb{R}^{n\times 1}$, $\mathbf{X} \in \mathbb{R}^{n\times k}$, $\mathbf{Z} \in \mathbb{R}^{n\times p}$, $\boldsymbol{\alpha} \in \mathbb{R}^{k\times 1}$, $\boldsymbol{\gamma} \in \mathbb{R}^{p\times 1}$, and $\boldsymbol{\varepsilon} \in \mathbb{R}^{n\times 1}$ with $n = n_1 + \ldots + n_m$, $p = p_1 + \ldots + p_m$, $k = k_1 + \ldots + k_m$. We note that the models $\mathcal{M}_i$ are transformed models of $\mathcal{M}$. They are obtained from pre-multiplying the model $\mathcal{M}$ by transformation matrices $\mathbf{T}_i = [\mathbf{0},\ldots,\mathbf{I}_{n_i},\ldots,\mathbf{0}]$, respectively, $i = 1,\ldots,m$.

The models $\mathcal{M}_i$ are said to be seemingly unrelated linear mixed models (SULMMs) since all the matrices and unknown vectors in these models are different. Although the models $\mathcal{M}_i$ seem unrelated due to not having any common unknown vectors, they may have individual relations to each other under certain assumptions such as having correlation among the random vectors between the models. According to this case, the following general assumptions on expectations and dispersion matrices of random vectors are considered for the models $\mathcal{M}_i$ and $\mathcal{M}$,

$$\mathrm{E}\begin{bmatrix}\boldsymbol{\gamma}_i\\ \boldsymbol{\varepsilon}_i\end{bmatrix} = \mathbf{0}, \quad \mathrm{E}\begin{bmatrix}\boldsymbol{\gamma}\\ \boldsymbol{\varepsilon}\end{bmatrix} = \mathbf{0}, \tag{1.3}$$

$$\text{(1.4)} \qquad \mathrm{D}\begin{bmatrix}\gamma_i\\ \varepsilon_i\end{bmatrix} = \mathrm{cov}\left\{\begin{bmatrix}\gamma_i\\ \varepsilon_i\end{bmatrix}, \begin{bmatrix}\gamma_i\\ \varepsilon_i\end{bmatrix}\right\} = \begin{bmatrix}\mathbf{V}_{i,i} & \mathbf{V}_{i,m+i}\\ \mathbf{V}_{m+i,i} & \mathbf{V}_{m+i,m+i}\end{bmatrix},$$

$$\text{(1.5)} \quad \mathrm{D}\begin{bmatrix}\gamma_1\\ \vdots\\ \gamma_m\\ \varepsilon_1\\ \vdots\\ \varepsilon_m\end{bmatrix} = \begin{bmatrix}\mathbf{V}_{1,1} & \cdots & \mathbf{V}_{1,m} & \mathbf{V}_{1,m+1} & \cdots & \mathbf{V}_{1,2m}\\ \vdots & \ddots & \vdots & \vdots & \ddots & \vdots\\ \mathbf{V}_{m,1} & \cdots & \mathbf{V}_{m,m} & \mathbf{V}_{m,m+1} & \cdots & \mathbf{V}_{m,2m}\\ \mathbf{V}_{m+1,1} & \cdots & \mathbf{V}_{m+1,m} & \mathbf{V}_{m+1,m+1} & \cdots & \mathbf{V}_{m+1,2m}\\ \vdots & \ddots & \vdots & \vdots & \ddots & \vdots\\ \mathbf{V}_{2m,1} & \cdots & \mathbf{V}_{2m,m} & \mathbf{V}_{2m,m+1} & \cdots & \mathbf{V}_{2m,2m}\end{bmatrix},$$

where $\mathrm{D}\begin{bmatrix}\gamma_i\\ \varepsilon_i\end{bmatrix} \in \mathbb{R}^{(n_i+p_i)\times(n_i+p_i)}$ and $\mathrm{D}\begin{bmatrix}\gamma\\ \varepsilon\end{bmatrix} \in \mathbb{R}^{(n+p)\times(n+p)}$ are positive semi-definite matrices of arbitrary ranks, $i = 1, \dots, m$. Further, submatrices $\mathbf{V}_{k,j}$ in (1.5) are nonzero for $k \neq j$, $k, j = 1, \dots, 2m$. Let $\mathbf{V}_i = \mathrm{D}\begin{bmatrix}\gamma_i\\ \varepsilon_i\end{bmatrix}$, $\mathbf{V} = \mathrm{D}\begin{bmatrix}\gamma\\ \varepsilon\end{bmatrix}$, $\mathbf{B} = [\,\mathbf{Z}, \mathbf{I}_n\,]$ and $\mathbf{B}_i = [\,\mathbf{Z}_i, \mathbf{I}_{n_i}\,]$ for brevity. According to assumptions in (1.3)–(1.5),

$$\text{(1.6)} \qquad \mathrm{E}(\mathbf{y}) = \mathbf{X}\boldsymbol{\alpha}, \quad \mathrm{E}(\mathbf{y}_i) = \mathbf{X}_i\boldsymbol{\alpha}_i, \quad \mathrm{D}(\mathbf{y}) = \mathbf{B}\mathbf{V}\mathbf{B}', \quad \mathrm{D}(\mathbf{y}_i) = \mathbf{B}_i\mathbf{V}_i\mathbf{B}_i'$$

are obtained, $i = 1, \dots, m$.

In linear regression analysis, establishing relations between two or more different linear models is one of the classical research problems. Consideration of $\mathcal{M}_i$ and their combining model $\mathcal{M}$ together is meaningful for obtaining the results separately or simultaneously for making estimation or prediction on joint unknown vectors $\boldsymbol{\alpha}_i$, γ_i, and $\boldsymbol{\varepsilon}_i$. Thus, we construct the following vector that consists joint unknown vectors in the considered models:

$$\text{(1.7)} \qquad \boldsymbol{\varphi}_i = \widehat{\mathbf{K}}_i\boldsymbol{\alpha} + \widehat{\mathbf{G}}_i\gamma + \widehat{\mathbf{H}}_i\boldsymbol{\varepsilon} \quad \text{or equivalently,} \quad \boldsymbol{\varphi}_i = \mathbf{K}_i\boldsymbol{\alpha}_i + \mathbf{G}_i\gamma_i + \mathbf{H}_i\boldsymbol{\varepsilon}_i$$

for given matrices $\widehat{\mathbf{K}}_i = [\mathbf{0}, \dots, \mathbf{K}_i, \dots, \mathbf{0}] \in \mathbb{R}^{s\times k}$, $\widehat{\mathbf{G}}_i = [\mathbf{0}, \dots, \mathbf{G}_i, \dots, \mathbf{0}] \in \mathbb{R}^{s\times p}$, and $\widehat{\mathbf{H}}_i = [\mathbf{0}, \dots, \mathbf{H}_i, \dots, \mathbf{0}] \in \mathbb{R}^{s\times n}$ with $\mathbf{K}_i \in \mathbb{R}^{s\times k_i}$, $\mathbf{G}_i \in \mathbb{R}^{s\times p_i}$, and $\mathbf{H}_i \in \mathbb{R}^{s\times n_i}$, $i = 1, \dots, m$. Under the assumptions in (1.3)–(1.5), we obtain

$$\text{(1.8)} \qquad \mathrm{E}(\boldsymbol{\varphi}_i) = \widehat{\mathbf{K}}_i\boldsymbol{\alpha} = \mathbf{K}_i\boldsymbol{\alpha}_i, \quad \mathrm{D}(\boldsymbol{\varphi}_i) = \widehat{\mathbf{J}}_i\mathbf{V}\widehat{\mathbf{J}}_i' = \mathbf{J}_i\mathbf{V}_i\mathbf{J}_i',$$

$$\text{(1.9)} \qquad \mathrm{cov}(\boldsymbol{\varphi}_i, \mathbf{y}) = \widehat{\mathbf{J}}_i\mathbf{V}\mathbf{B}', \quad \mathrm{cov}(\boldsymbol{\varphi}_i, \mathbf{y}_i) = \mathbf{J}_i\mathbf{V}_i\mathbf{B}_i',$$

where $\mathbf{J}_i = [\,\mathbf{G}_i, \mathbf{H}_i\,]$ and $\widehat{\mathbf{J}}_i = [\,\widehat{\mathbf{G}}_i, \widehat{\mathbf{H}}_i\,]$, $i = 1, \dots, m$. Detail studies on algebraic and statistical properties on estimation/prediction of separately or jointly considered fixed effects, random effects, and error terms in linear mixed models can be found in, e.g., [2], [13], [14], [18], [19], [22], [35] among others.

Comparison of predictors of unknown vectors in different models is one of the main problems encountered in the theory of regression analysis. Predictors of unknown vectors under different models have different algebraic expressions, properties and performances. However, observable random vectors may preserve enough information for predicting and/or estimating unknown vectors in connected models. Thus, it is natural to consider possible connections and comparisons between inference results derived from different models since there may be connections between inference results obtained from all these models. Best linear unbiased predictors (BLUPs) of unknown vectors play a significant role in statistical inference from linear regression models and their covariance matrices are mostly used as a criterion in comparisons for optimality with other predictors. Matrix theory is an essential key for establishing equalities and inequalities while working on characterization and comparison of algebraic or statistical properties of predictors/estimators. Especially, the rank and inertia formulas of matrices have been used for simplifying various complicated matrix expressions including Moore-Penrose generalized inverses of matrices. We may mention the studies [10], [11], [12], [28], [29], [30], [32] on comparison of covariance matrices of predictors/estimators by using matrix inertia and rank method. For more details on inertias and ranks of symmetric matrices and relations between inertias, ranks and Löwner partial ordering of symmetric matrices, we may refer to [20], [25], [26], and [31].

In this study, we consider prediction problems under SULMMs and we give several results on properties of BLUPs of unknown vectors. Our main purpose is to give a new insight into the comparison of BLUPs under SULMMs and their combined model. We establish a variety of equalities and inequalities for comparison of covariance matrices of BLUPs in SULMMs. For doing this, we use an approach consisting of formulas of inertias and ranks of block matrices for simplifying heavy matrix operations including Moore-Penrose generalized inverses of matrices. Further, we consider seemingly unrelated regression models (SURMs) which are special versions of SULMMs. As an application, the results obtained for SULMMs are also presented for SURMs.

Let $\mathbb{R}^{m\times n}$ stand for the set of all $m\times n$ real matrices. $\mathbf{A}'$, $\boldsymbol{r}(\mathbf{A})$, $\mathscr{C}(\mathbf{A})$ and $\mathbf{A}^{+}$ denote the transpose, the rank, the column space and the Moore-Penrose generalized inverse of $\mathbf{A}\in\mathbb{R}^{m\times n}$, respectively. $\mathbf{I}_m$ denotes the identity matrix of order m. Symbol $\mathbf{E}_{\mathbf{A}}=\mathbf{A}^{\perp}=\mathbf{I}_m-\mathbf{A}\mathbf{A}^{+}$ is used for the orthogonal projector. The positive inertia and the negative inertia of $\mathbf{A}$ are denoted by $\boldsymbol{i}_{+}(\mathbf{A})$ and $\boldsymbol{i}_{-}(\mathbf{A})$, respectively, which are defined, respectively, as the numbers of the positive and negative eigenvalues of symmetric matrix $\mathbf{A}$ counted with multiplicities, and also for brief, $\boldsymbol{i}_{\pm}(\mathbf{A})$ is used for both numbers. For the symmetric matrices $\mathbf{A}_1$ and $\mathbf{A}_2$ of the same size, the inequalities $\mathbf{A}_1 - \mathbf{A}_2 \prec \mathbf{0}$ ($\mathbf{A}_1 \prec \mathbf{A}_2$), $\mathbf{A}_1 - \mathbf{A}_2 \preccurlyeq \mathbf{0}$ ($\mathbf{A}_1 \preccurlyeq \mathbf{A}_2$), $\mathbf{A}_1 - \mathbf{A}_2 \succ \mathbf{0}$ ($\mathbf{A}_1 \succ \mathbf{A}_2$) and $\mathbf{A}_1 - \mathbf{A}_2 \succcurlyeq \mathbf{0}$ ($\mathbf{A}_1 \succcurlyeq \mathbf{A}_2$) mean that the difference $\mathbf{A}_1 - \mathbf{A}_2$ is negative definite, negative semi-definite, positive definite and positive semi-definite matrix in the Löwner partial ordering, respectively.

2. Preliminary results

In this section, we first give well-known results on BLUPs of all unknown vectors under the models, and then we give some inertia and rank formulas of block matrices. In what follows, we assume that the model $\mathcal{M}$ is consistent, i.e. $\mathbf{y} \in \mathscr{C}[\mathbf{X}, \mathbf{BVB}']$ holds with probability 1, see, e.g. [21]. The consistency assumption of $\mathcal{M}_i$ is provided under the assumption of consistency of $\mathcal{M}$.

The predictability condition of $\boldsymbol{\varphi}_i$ under $\mathcal{M}$ is expressed that there exists a linear statistic $\mathbf{L}_i\mathbf{y}$, $\mathbf{L}_i \in \mathbb{R}^{s\times n}$, such that $\mathrm{E}(\mathbf{L}_i\mathbf{y} - \boldsymbol{\varphi}_i) = \mathbf{0}$, or equivalently, $\mathscr{C}(\widehat{\mathbf{K}}_i') \subseteq \mathscr{C}(\mathbf{X}')$ holds; see [28]. This requirement also corresponds to the estimability condition of vector $\widehat{\mathbf{K}}_i\boldsymbol{\alpha}$ under $\mathcal{M}$; see e.g. [1]. The predictability condition of $\boldsymbol{\varphi}_i$ under $\mathcal{M}_i$ is $\mathscr{C}(\mathbf{K}_i') \subseteq \mathscr{C}(\mathbf{X}_i')$. It is obvious that $\boldsymbol{\varphi}_i$ is predictable under $\mathcal{M}$ if it is predictable under $\mathcal{M}_i$. Let $\boldsymbol{\varphi}_i$ be predictable under $\mathcal{M}$. If there exists $\mathbf{L}_i$ such that

$$\mathrm{D}(\mathbf{L}_i\mathbf{y} - \boldsymbol{\varphi}_i) = \min \text{ subject to } \mathrm{E}(\mathbf{L}_i\mathbf{y} - \boldsymbol{\varphi}_i) = \mathbf{0}, \quad i = 1, \ldots, m, \tag{2.1}$$

holds in the Löwner partial ordering, the linear statistic $\mathbf{L}_i\mathbf{y}$ is defined to be the BLUP of $\boldsymbol{\varphi}_i$, denoted by $\mathbf{L}_i\mathbf{y} = \mathrm{BLUP}_{\mathcal{M}}(\boldsymbol{\varphi}_i) = \mathrm{BLUP}_{\mathcal{M}}(\widehat{\mathbf{K}}_i\boldsymbol{\alpha} + \widehat{\mathbf{G}}_i\boldsymbol{\gamma} + \widehat{\mathbf{H}}_i\boldsymbol{\varepsilon})$; see [8]. If $\widehat{\mathbf{G}}_i = \mathbf{0}$ and $\widehat{\mathbf{H}}_i = \mathbf{0}$, BLUP of $\boldsymbol{\varphi}_i$ reduces the best linear unbiased estimator (BLUE) of $\widehat{\mathbf{K}}_i\boldsymbol{\alpha}$, denoted by $\mathrm{BLUE}_{\mathcal{M}}(\widehat{\mathbf{K}}_i\boldsymbol{\alpha})$, under $\mathcal{M}$.

The following two lemmas are derived from [20], Proposition 10.6 and [27], Theorem 1.

Lemma 2.1. *Let $\mathcal{M}$ be as given in* (1.2) *and let $\boldsymbol{\varphi}_i$ in* (1.7) *be predictable under $\mathcal{M}$, $i = 1, \ldots, m$. In this case,*

$$\mathrm{BLUP}_{\mathcal{M}}(\boldsymbol{\varphi}_i) = \mathbf{L}_i\mathbf{y} \Leftrightarrow \mathbf{L}_i[\mathbf{X}, \mathbf{BVB}'\mathbf{X}^{\perp}] = [\widehat{\mathbf{K}}_i, \widehat{\mathbf{J}}_i\mathbf{VB}'\mathbf{X}^{\perp}]. \tag{2.2}$$

Then

$$\mathrm{BLUP}_{\mathcal{M}}(\boldsymbol{\varphi}_i) = \mathbf{L}_i\mathbf{y} = ([\widehat{\mathbf{K}}_i, \widehat{\mathbf{J}}_i\mathbf{VB}'\mathbf{X}^{\perp}]\mathbf{W}^{+} + \mathbf{U}_i\mathbf{W}^{\perp})\mathbf{y}, \tag{2.3}$$

where $\mathbf{W} = [\mathbf{X}, \mathbf{BVB}'\mathbf{X}^{\perp}]$ and $\mathbf{U}_i \in \mathbb{R}^{s\times n}$ is arbitrary. Furthermore,

(a) $\boldsymbol{r}[\mathbf{X}, \mathbf{BVB'X^{\perp}}] = \boldsymbol{r}[\mathbf{X}, \mathbf{BVB'}]$, $\mathscr{C}[\mathbf{X}, \mathbf{BVB'X^{\perp}}] = \mathscr{C}[\mathbf{X}, \mathbf{BVB'}]$, *and* $\mathscr{C}(\mathbf{X}) \cap \mathscr{C}(\mathbf{BVB'X^{\perp}}) = \{\mathbf{0}\}$;

(b) $\mathbf{L}_i$ *is unique* $\Leftrightarrow \boldsymbol{r}[\mathbf{X}, \mathbf{BVB'}] = n$ *and* $\mathbf{L}_i\mathbf{y}$ *is unique* $\Leftrightarrow \mathcal{M}$ *is consistent*;

(c) $\mathrm{BLUP}_{\mathcal{M}}(\boldsymbol{\varphi}_i)$ *satisfies*

$$\mathrm{D}[\mathrm{BLUP}_{\mathcal{M}}(\boldsymbol{\varphi}_i)] = [\,\widehat{\mathbf{K}}_i, \widehat{\mathbf{J}}_i\mathbf{VB'X^{\perp}}\,]\,\mathbf{W}^{+}\mathbf{BVB'}([\,\widehat{\mathbf{K}}_i, \widehat{\mathbf{J}}_i\mathbf{VB'X^{\perp}}\,]\,\mathbf{W}^{+})', \tag{2.4}$$

$$\mathrm{cov}\{\mathrm{BLUP}_{\mathcal{M}}(\boldsymbol{\varphi}_i), \boldsymbol{\varphi}_i\} = [\,\widehat{\mathbf{K}}_i, \widehat{\mathbf{J}}_i\mathbf{VB'X^{\perp}}\,]\,\mathbf{W}^{+}\mathbf{BV}\widehat{\mathbf{J}}_i', \tag{2.5}$$

$$\begin{aligned}\mathrm{D}[\boldsymbol{\varphi}_i - \mathrm{BLUP}_{\mathcal{M}}(\boldsymbol{\varphi}_i)] &= ([\,\widehat{\mathbf{K}}_i, \widehat{\mathbf{J}}_i\mathbf{VB'X^{\perp}}\,]\,\mathbf{W}^{+}\mathbf{B} - \widehat{\mathbf{J}}_i)\\ &\quad\times \mathbf{V}([\,\widehat{\mathbf{K}}_i, \widehat{\mathbf{J}}_i\mathbf{VB'X^{\perp}}\,]\,\mathbf{W}^{+}\mathbf{B} - \widehat{\mathbf{J}}_i)'.\end{aligned} \tag{2.6}$$

Lemma 2.2. *Let* $\mathcal{M}_i$ *be as given in* (1.1) *and let* $\boldsymbol{\varphi}_i$ *in* (1.7) *be predictable under* $\mathcal{M}_i$, $i = 1, \ldots, m$. *In this case,*

$$\mathrm{BLUP}_{\mathcal{M}_i}(\boldsymbol{\varphi}_i) = \mathbf{L}_i\mathbf{y}_i \Leftrightarrow \mathbf{L}_i\,[\,\mathbf{X}_i, \mathbf{B}_i\mathbf{V}_i\mathbf{B}_i'\mathbf{X}_i^{\perp}\,] = [\,\mathbf{K}_i, \mathbf{J}_i\mathbf{V}_i\mathbf{B}_i'\mathbf{X}_i^{\perp}\,]\,. \tag{2.7}$$

Then

$$\mathrm{BLUP}_{\mathcal{M}_i}(\boldsymbol{\varphi}_i) = \mathbf{L}_i\mathbf{y}_i = ([\,\mathbf{K}_i, \mathbf{J}_i\mathbf{V}_i\mathbf{B}_i'\mathbf{X}_i^{\perp}\,]\,\mathbf{W}_i^{+} + \mathbf{U}_i\mathbf{W}_i^{\perp})\mathbf{y}_i, \tag{2.8}$$

where $\mathbf{W}_i = [\,\mathbf{X}_i, \mathbf{B}_i\mathbf{V}_i\mathbf{B}_i'\mathbf{X}_i^{\perp}\,]$ *and* $\mathbf{U}_i \in \mathbb{R}^{s\times n_i}$ *is arbitrary. Furthermore,*

(a) $\boldsymbol{r}\,[\,\mathbf{X}_i, \mathbf{B}_i\mathbf{V}_i\mathbf{B}_i'\mathbf{X}_i^{\perp}\,] = \boldsymbol{r}\,[\,\mathbf{X}_i, \mathbf{B}_i\mathbf{V}_i\mathbf{B}_i'\,]$, $\mathscr{C}\,[\,\mathbf{X}_i, \mathbf{B}_i\mathbf{V}_i\mathbf{B}_i'\mathbf{X}_i^{\perp}\,] = \mathscr{C}\,[\,\mathbf{X}_i, \mathbf{B}_i\mathbf{V}_i\mathbf{B}_i'\,]$, *and* $\mathscr{C}(\mathbf{X}_i) \cap \mathscr{C}(\mathbf{B}_i\mathbf{V}_i\mathbf{B}_i'\mathbf{X}_i^{\perp}) = \{\mathbf{0}\}$;

(b) $\mathbf{L}_i$ *is unique* $\Leftrightarrow \boldsymbol{r}\,[\,\mathbf{X}_i, \mathbf{B}_i\mathbf{V}_i\mathbf{B}_i'\,] = n_i$ *and* $\mathbf{L}_i\mathbf{y}_i$ *is unique* $\Leftrightarrow \mathcal{M}_i$ *is consistent*;

(c) $\mathrm{BLUP}_{\mathcal{M}_i}(\boldsymbol{\varphi}_i)$ *satisfies*

$$\begin{aligned}\mathrm{D}[\mathrm{BLUP}_{\mathcal{M}_i}(\boldsymbol{\varphi}_i)] &= [\,\mathbf{K}_i, \mathbf{J}_i\mathbf{V}_i\mathbf{B}_i'\mathbf{X}_i^{\perp}\,]\,\mathbf{W}_i^{+}\mathbf{B}_i\mathbf{V}_i\mathbf{B}_i'\\ &\quad\times([\,\mathbf{K}_i, \mathbf{J}_i\mathbf{V}_i\mathbf{B}_i'\mathbf{X}_i^{\perp}\,]\,\mathbf{W}_i^{+})',\end{aligned} \tag{2.9}$$

$$\mathrm{cov}\{\mathrm{BLUP}_{\mathcal{M}_i}(\boldsymbol{\varphi}_i), \boldsymbol{\varphi}_i\} = [\,\mathbf{K}_i, \mathbf{J}_i\mathbf{V}_i\mathbf{B}_i'\mathbf{X}_i^{\perp}\,]\,\mathbf{W}_i^{+}\mathbf{B}_i\mathbf{V}_i\mathbf{J}_i', \tag{2.10}$$

$$\begin{aligned}\mathrm{D}[\boldsymbol{\varphi}_i - \mathrm{BLUP}_{\mathcal{M}_i}(\boldsymbol{\varphi}_i)] &= ([\,\mathbf{K}_i, \mathbf{J}_i\mathbf{V}_i\mathbf{B}_i'\mathbf{X}_i^{\perp}\,]\,\mathbf{W}_i^{+}\mathbf{B}_i - \mathbf{J}_i)\\ &\quad\times \mathbf{V}_i([\,\mathbf{K}_i, \mathbf{J}_i\mathbf{V}_i\mathbf{B}_i'\mathbf{X}_i^{\perp}\,]\,\mathbf{W}_i^{+}\mathbf{B}_i - \mathbf{J}_i)'.\end{aligned} \tag{2.11}$$

We collect some inertia and rank formulas of block matrices in the following three lemmas; see [25].

Lemma 2.3. *Let* $\mathbf{A}_1$, $\mathbf{A}_2 \in \mathbb{R}^{m\times n}$ *or let* $\mathbf{A}_1 = \mathbf{A}_1'$, $\mathbf{A}_2 = \mathbf{A}_2' \in \mathbb{R}^{m\times m}$. *Then*

(a) $\mathbf{A}_1 = \mathbf{A}_2 \Leftrightarrow \boldsymbol{r}(\mathbf{A}_1 - \mathbf{A}_2) = 0$;

(b) $\mathbf{A}_1 \succ \mathbf{A}_2 \Leftrightarrow \boldsymbol{i}_{+}(\mathbf{A}_1 - \mathbf{A}_2) = m$ *and* $\mathbf{A}_1 \prec \mathbf{A}_2 \Leftrightarrow \boldsymbol{i}_{-}(\mathbf{A}_1 - \mathbf{A}_2) = m$;

(c) $\mathbf{A}_1 \succcurlyeq \mathbf{A}_2 \Leftrightarrow \boldsymbol{i}_{-}(\mathbf{A}_1 - \mathbf{A}_2) = 0$ *and* $\mathbf{A}_1 \preccurlyeq \mathbf{A}_2 \Leftrightarrow \boldsymbol{i}_{+}(\mathbf{A}_1 - \mathbf{A}_2) = 0$.

Lemma 2.4. *Let* $\mathbf{A}_1 = \mathbf{A}_1' \in \mathbb{R}^{m\times m}$, $\mathbf{A}_2 = \mathbf{A}_2' \in \mathbb{R}^{n\times n}$, $\mathbf{Q} \in \mathbb{R}^{m\times n}$ *and* $k \in \mathbb{R}$. *Then*

$$\boldsymbol{r}(\mathbf{A}_1) = \boldsymbol{i}_{+}(\mathbf{A}_1) + \boldsymbol{i}_{-}(\mathbf{A}_1), \tag{2.12}$$

$$\boldsymbol{i}_{\pm}(k\mathbf{A}_1) = \begin{cases} \boldsymbol{i}_{\pm}(\mathbf{A}_1) & \text{if } k > 0 \\ \boldsymbol{i}_{\mp}(\mathbf{A}_1) & \text{if } k < 0 \end{cases}, \tag{2.13}$$

$$\boldsymbol{i}_{\pm}\begin{bmatrix} \mathbf{A}_1 & \mathbf{Q} \\ \mathbf{Q}' & \mathbf{A}_2 \end{bmatrix} = \boldsymbol{i}_{\pm}\begin{bmatrix} \mathbf{A}_1 & -\mathbf{Q} \\ -\mathbf{Q}' & \mathbf{A}_2 \end{bmatrix} = \boldsymbol{i}_{\mp}\begin{bmatrix} -\mathbf{A}_1 & \mathbf{Q} \\ \mathbf{Q}' & -\mathbf{A}_2 \end{bmatrix}, \tag{2.14}$$

$$\boldsymbol{i}_{\pm}\begin{bmatrix} \mathbf{A}_1 & \mathbf{0} \\ \mathbf{0} & \mathbf{A}_2 \end{bmatrix} = \boldsymbol{i}_{\pm}(\mathbf{A}_1) + \boldsymbol{i}_{\pm}(\mathbf{A}_2), \quad \boldsymbol{i}_{+}\begin{bmatrix} \mathbf{0} & \mathbf{Q} \\ \mathbf{Q}' & \mathbf{0} \end{bmatrix} = \boldsymbol{i}_{-}\begin{bmatrix} \mathbf{0} & \mathbf{Q} \\ \mathbf{Q}' & \mathbf{0} \end{bmatrix} = \boldsymbol{r}(\mathbf{Q}). \tag{2.15}$$

Lemma 2.5. *Let* $\mathbf{A}_1 = \mathbf{A}_1' \in \mathbb{R}^{m\times m}$, $\mathbf{B} = \mathbf{B}' \in \mathbb{R}^{n\times n}$ *and* $\mathbf{A}_2 \in \mathbb{R}^{m\times n}$. *Then*

$$\boldsymbol{i}_{\pm}\begin{bmatrix} \mathbf{A}_1 & \mathbf{A}_2 \\ \mathbf{A}_2' & \mathbf{0} \end{bmatrix} = \boldsymbol{r}(\mathbf{A}_2) + \boldsymbol{i}_{\pm}(\mathbf{E}_{\mathbf{A}_2}\mathbf{A}_1\mathbf{E}_{\mathbf{A}_2}). \tag{2.16}$$

In particular,

$$\boldsymbol{i}_{+}\begin{bmatrix} \mathbf{A}_1\mathbf{A}_1' & \mathbf{A}_2 \\ \mathbf{A}_2' & \mathbf{0} \end{bmatrix} = \boldsymbol{r}\,[\,\mathbf{A}_1, \mathbf{A}_2\,], \quad \boldsymbol{i}_{-}\begin{bmatrix} \mathbf{A}_1\mathbf{A}_1' & \mathbf{A}_2 \\ \mathbf{A}_2' & \mathbf{0} \end{bmatrix} = \boldsymbol{r}(\mathbf{A}_2), \tag{2.17}$$

$$\boldsymbol{i}_{\pm}\begin{bmatrix} \mathbf{A}_1 & \mathbf{A}_2 \\ \mathbf{A}_2' & \mathbf{B} \end{bmatrix} = \boldsymbol{i}_{\pm}(\mathbf{A}_1) + \boldsymbol{i}_{\pm}(\mathbf{B} - \mathbf{A}_2'\mathbf{A}_1^{+}\mathbf{A}_2) \quad \text{if } \mathscr{C}(\mathbf{A}_2) \subseteq \mathscr{C}(\mathbf{A}_1). \tag{2.18}$$

3. Comparisons of BLUPs in SULMMs

In this section, some results on the comparison of covariance matrices of predictors under SULMMs are derived and related conclusions are established for special cases by using block matrices' rank and inertia formulas.

Theorem 3.1. *Let* $\mathcal{M}_i$ *and* $\mathcal{M}$ *be as given in* (1.1) *and* (1.2), *respectively, and assume that* $\boldsymbol{\varphi}_i$ *is predictable under* $\mathcal{M}_i$ *(also predictable under* $\mathcal{M}$*),* $i = 1, \ldots, m$. *Let* $\mathrm{BLUP}_{\mathcal{M}}(\boldsymbol{\varphi}_i)$ *and* $\mathrm{BLUP}_{\mathcal{M}_i}(\boldsymbol{\varphi}_i)$ *be as given in* (2.3) *and* (2.8), *respectively. Denote*

$$\mathbf{N} = \begin{bmatrix} \mathbf{BVB}' & \mathbf{0} & \mathbf{BV}\widehat{\mathbf{J}}_i' & \mathbf{X} & \mathbf{0} \\ \mathbf{0} & -\mathbf{B}_i\mathbf{V}_i\mathbf{B}_i' & \mathbf{B}_i\mathbf{V}_i\mathbf{J}_i' & \mathbf{0} & \mathbf{X}_i \\ \widehat{\mathbf{J}}_i\mathbf{VB}' & \mathbf{J}_i\mathbf{V}_i\mathbf{B}_i' & \mathbf{0} & \widehat{\mathbf{K}}_i & -\mathbf{K}_i \\ \mathbf{X}' & \mathbf{0} & \widehat{\mathbf{K}}_i' & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{X}_i' & -\mathbf{K}_i' & \mathbf{0} & \mathbf{0} \end{bmatrix}. \tag{3.1}$$

Then

$$\text{(3.2)} \qquad \boldsymbol{i}_{+}(\mathrm{D}[\boldsymbol{\varphi}_i - \mathrm{BLUP}_{\mathcal{M}}(\boldsymbol{\varphi}_i)] - \mathrm{D}[\boldsymbol{\varphi}_i - \mathrm{BLUP}_{\mathcal{M}_i}(\boldsymbol{\varphi}_i)]) = \boldsymbol{i}_{+}(\mathbf{N}) - \boldsymbol{r}\,[\,\mathbf{X}, \mathbf{BVB}'\,] - \boldsymbol{r}(\mathbf{X}_i),$$

$$\text{(3.3)} \qquad \boldsymbol{i}_{-}(\mathrm{D}[\boldsymbol{\varphi}_i - \mathrm{BLUP}_{\mathcal{M}}(\boldsymbol{\varphi}_i)] - \mathrm{D}[\boldsymbol{\varphi}_i - \mathrm{BLUP}_{\mathcal{M}_i}(\boldsymbol{\varphi}_i)]) = \boldsymbol{i}_{-}(\mathbf{N}) - \boldsymbol{r}\,[\,\mathbf{X}_i, \mathbf{B}_i\mathbf{V}_i\mathbf{B}_i'\,] - \boldsymbol{r}(\mathbf{X}),$$

$$\text{(3.4)} \qquad \boldsymbol{r}(\mathrm{D}[\boldsymbol{\varphi}_i - \mathrm{BLUP}_{\mathcal{M}}(\boldsymbol{\varphi}_i)] - \mathrm{D}[\boldsymbol{\varphi}_i - \mathrm{BLUP}_{\mathcal{M}_i}(\boldsymbol{\varphi}_i)]) = \boldsymbol{r}(\mathbf{N}) - \boldsymbol{r}\,[\,\mathbf{X}, \mathbf{BVB}'\,] - \boldsymbol{r}(\mathbf{X}_i) - \boldsymbol{r}\,[\,\mathbf{X}_i, \mathbf{B}_i\mathbf{V}_i\mathbf{B}_i'\,] - \boldsymbol{r}(\mathbf{X}).$$

In consequence, the following results hold.

(a) $\mathrm{D}[\boldsymbol{\varphi}_i - \mathrm{BLUP}_{\mathcal{M}_i}(\boldsymbol{\varphi}_i)] \succ \mathrm{D}[\boldsymbol{\varphi}_i - \mathrm{BLUP}_{\mathcal{M}}(\boldsymbol{\varphi}_i)] \Leftrightarrow \boldsymbol{i}_{-}(\mathbf{N}) = \boldsymbol{r}\,[\,\mathbf{X}_i, \mathbf{B}_i\mathbf{V}_i\mathbf{B}_i'\,] + \boldsymbol{r}(\mathbf{X}) + s;$

(b) $\mathrm{D}[\boldsymbol{\varphi}_i - \mathrm{BLUP}_{\mathcal{M}_i}(\boldsymbol{\varphi}_i)] \prec \mathrm{D}[\boldsymbol{\varphi}_i - \mathrm{BLUP}_{\mathcal{M}}(\boldsymbol{\varphi}_i)] \Leftrightarrow \boldsymbol{i}_{+}(\mathbf{N}) = \boldsymbol{r}\,[\,\mathbf{X}, \mathbf{BVB}'\,] + \boldsymbol{r}(\mathbf{X}_i) + s;$

(c) $\mathrm{D}[\boldsymbol{\varphi}_i - \mathrm{BLUP}_{\mathcal{M}_i}(\boldsymbol{\varphi}_i)] \succcurlyeq \mathrm{D}[\boldsymbol{\varphi}_i - \mathrm{BLUP}_{\mathcal{M}}(\boldsymbol{\varphi}_i)] \Leftrightarrow \boldsymbol{i}_{+}(\mathbf{N}) = \boldsymbol{r}\,[\,\mathbf{X}, \mathbf{BVB}'\,] + \boldsymbol{r}(\mathbf{X}_i);$

(d) $\mathrm{D}[\boldsymbol{\varphi}_i - \mathrm{BLUP}_{\mathcal{M}_i}(\boldsymbol{\varphi}_i)] \preccurlyeq \mathrm{D}[\boldsymbol{\varphi}_i - \mathrm{BLUP}_{\mathcal{M}}(\boldsymbol{\varphi}_i)] \Leftrightarrow \boldsymbol{i}_{-}(\mathbf{N}) = \boldsymbol{r}\,[\,\mathbf{X}_i, \mathbf{B}_i\mathbf{V}_i\mathbf{B}_i'\,] + \boldsymbol{r}(\mathbf{X});$

(e) $\mathrm{D}[\boldsymbol{\varphi}_i - \mathrm{BLUP}_{\mathcal{M}_i}(\boldsymbol{\varphi}_i)] = \mathrm{D}[\boldsymbol{\varphi}_i - \mathrm{BLUP}_{\mathcal{M}}(\boldsymbol{\varphi}_i)] \Leftrightarrow \boldsymbol{r}(\mathbf{N}) = \boldsymbol{r}\,[\,\mathbf{X}, \mathbf{BVB}'\,] + \boldsymbol{r}(\mathbf{X}_i) + \boldsymbol{r}\,[\,\mathbf{X}_i, \mathbf{B}_i\mathbf{V}_i\mathbf{B}_i'\,] + \boldsymbol{r}(\mathbf{X}).$

Proof. By using the relation (2.11) and applying (2.18) to the difference between $\mathrm{D}[\boldsymbol{\varphi}_i - \mathrm{BLUP}_{\mathcal{M}}(\boldsymbol{\varphi}_i)]$ and $\mathrm{D}[\boldsymbol{\varphi}_i - \mathrm{BLUP}_{\mathcal{M}_i}(\boldsymbol{\varphi}_i)]$,

(3.5)

$$\begin{aligned}
&\boldsymbol{i}_{\pm}(\mathrm{D}[\boldsymbol{\varphi}_i - \mathrm{BLUP}_{\mathcal{M}}(\boldsymbol{\varphi}_i)] - \mathrm{D}[\boldsymbol{\varphi}_i - \mathrm{BLUP}_{\mathcal{M}_i}(\boldsymbol{\varphi}_i)]) \\
&\quad = \boldsymbol{i}_{\pm}(\mathrm{D}[\boldsymbol{\varphi}_i - \mathrm{BLUP}_{\mathcal{M}}(\boldsymbol{\varphi}_i)] \\
&\qquad - ([\,\mathbf{K}_i, \mathbf{J}_i\mathbf{V}_i\mathbf{B}_i'\mathbf{X}_i^{\perp}\,]\,\mathbf{W}_i^{+}\mathbf{B}_i - \mathbf{J}_i)\mathbf{V}_i\mathbf{V}_i^{+}\mathbf{V}_i([\,\mathbf{K}_i, \mathbf{J}_i\mathbf{V}_i\mathbf{B}_i'\mathbf{X}_i^{\perp}\,]\,\mathbf{W}_i^{+}\mathbf{B}_i - \mathbf{J}_i)') \\
&\quad = \boldsymbol{i}_{\pm}\begin{bmatrix} \mathbf{V}_i & \mathbf{V}_i([\,\mathbf{K}_i, \mathbf{J}_i\mathbf{V}_i\mathbf{B}_i'\mathbf{X}_i^{\perp}\,]\,\mathbf{W}_i^{+}\mathbf{B}_i - \mathbf{J}_i)' \\ ([\,\mathbf{K}_i, \mathbf{J}_i\mathbf{V}_i\mathbf{B}_i'\mathbf{X}_i^{\perp}\,]\,\mathbf{W}_i^{+}\mathbf{B}_i - \mathbf{J}_i)\mathbf{V}_i & \mathrm{D}[\boldsymbol{\varphi}_i - \mathrm{BLUP}_{\mathcal{M}}(\boldsymbol{\varphi}_i)] \end{bmatrix} \\
&\qquad - \boldsymbol{i}_{\pm}(\mathbf{V}_i) \\
&\quad = \boldsymbol{i}_{\pm}\left(\begin{bmatrix} \mathbf{V}_i & -\mathbf{V}_i\mathbf{J}_i' \\ -\mathbf{J}_i\mathbf{V}_i & \mathrm{D}[\boldsymbol{\varphi}_i - \mathrm{BLUP}_{\mathcal{M}}(\boldsymbol{\varphi}_i)] \end{bmatrix} + \begin{bmatrix} \mathbf{V}_i\mathbf{B}_i' & \mathbf{0} \\ \mathbf{0} & [\,\mathbf{K}_i, \mathbf{J}_i\mathbf{V}_i\mathbf{B}_i'\mathbf{X}_i^{\perp}\,] \end{bmatrix}\right. \\
&\qquad \left.\times \begin{bmatrix} \mathbf{0} & \mathbf{W}_i \\ \mathbf{W}_i' & \mathbf{0} \end{bmatrix}^{+} \begin{bmatrix} \mathbf{B}_i\mathbf{V}_i & \mathbf{0} \\ \mathbf{0} & [\,\mathbf{K}_i, \mathbf{J}_i\mathbf{V}_i\mathbf{B}_i'\mathbf{X}_i^{\perp}\,]' \end{bmatrix}\right) - \boldsymbol{i}_{\pm}(\mathbf{V}_i)
\end{aligned}$$

is obtained. We can reapply (2.18) to (3.5), since

$$\mathscr{C}(\mathbf{B}_i\mathbf{V}_i) = \mathscr{C}(\mathbf{B}_i\mathbf{V}_i\mathbf{B}_i') \subseteq \mathscr{C}(\mathbf{W}_i) \quad \text{and} \quad \mathscr{C}([\,\mathbf{K}_i, \mathbf{J}_i\mathbf{V}_i\mathbf{B}_i'\mathbf{X}_i^{\perp}\,]') \subseteq \mathscr{C}(\mathbf{W}_i'),$$

where $\mathbf{W}_i = [\,\mathbf{X}_i, \mathbf{B}_i\mathbf{V}_i\mathbf{B}_i'\mathbf{X}_i^{\perp}\,]$. Then (3.5) is equivalently written as follows:

(3.6)

$$
\begin{aligned}
&\boldsymbol{i}_{\pm}\begin{bmatrix} \mathbf{0} & -\mathbf{X}_i & -\mathbf{B}_i\mathbf{V}_i\mathbf{B}_i'\mathbf{X}_i^{\perp} & \mathbf{B}_i\mathbf{V}_i & \mathbf{0} \\ -\mathbf{X}_i' & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{K}_i' \\ -\mathbf{X}_i^{\perp}\mathbf{B}_i\mathbf{V}_i\mathbf{B}_i' & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{X}_i^{\perp}\mathbf{B}_i\mathbf{V}_i\mathbf{J}_i' \\ \mathbf{V}_i\mathbf{B}_i' & \mathbf{0} & \mathbf{0} & \mathbf{V}_i & -\mathbf{V}_i\mathbf{J}_i' \\ \mathbf{0} & \mathbf{K}_i & \mathbf{J}_i\mathbf{V}_i\mathbf{B}_i'\mathbf{X}_i^{\perp} & -\mathbf{J}_i\mathbf{V}_i & \mathrm{D}[\boldsymbol{\varphi}_i - \mathrm{BLUP}_{\mathcal{M}}(\boldsymbol{\varphi}_i)] \end{bmatrix} \\
&\quad - \boldsymbol{r}\,[\,\mathbf{X}_i, \mathbf{B}_i\mathbf{V}_i\mathbf{B}_i'\mathbf{X}_i^{\perp}\,] - \boldsymbol{i}_{\pm}(\mathbf{V}_i) \\
&= \boldsymbol{i}_{\pm}\begin{bmatrix} -\mathbf{B}_i\mathbf{V}_i\mathbf{B}_i' & -\mathbf{X}_i & -\mathbf{B}_i\mathbf{V}_i\mathbf{B}_i'\mathbf{X}_i^{\perp} & \mathbf{B}_i\mathbf{V}_i\mathbf{J}_i' \\ -\mathbf{X}_i' & \mathbf{0} & \mathbf{0} & \mathbf{K}_i' \\ -\mathbf{X}_i^{\perp}\mathbf{B}_i\mathbf{V}_i\mathbf{B}_i' & \mathbf{0} & \mathbf{0} & \mathbf{X}_i^{\perp}\mathbf{B}_i\mathbf{V}_i\mathbf{J}_i' \\ \mathbf{J}_i\mathbf{V}_i\mathbf{B}_i' & \mathbf{K}_i & \mathbf{J}_i\mathbf{V}_i\mathbf{B}_i'\mathbf{X}_i^{\perp} & \mathrm{D}[\boldsymbol{\varphi}_i - \mathrm{BLUP}_{\mathcal{M}}(\boldsymbol{\varphi}_i)] - \mathbf{J}_i\mathbf{V}_i\mathbf{J}_i' \end{bmatrix} \\
&\quad - \boldsymbol{r}\,[\,\mathbf{X}_i, \mathbf{B}_i\mathbf{V}_i\mathbf{B}_i'\mathbf{X}_i^{\perp}\,] \\
&= \boldsymbol{i}_{\pm}\begin{bmatrix} -\mathbf{B}_i\mathbf{V}_i\mathbf{B}_i' & -\mathbf{X}_i & \mathbf{B}_i\mathbf{V}_i\mathbf{J}_i' \\ -\mathbf{X}_i' & \mathbf{0} & \mathbf{K}_i' \\ \mathbf{J}_i\mathbf{V}_i\mathbf{B}_i' & \mathbf{K}_i & \mathrm{D}[\boldsymbol{\varphi}_i - \mathrm{BLUP}_{\mathcal{M}}(\boldsymbol{\varphi}_i)] - \mathbf{J}_i\mathbf{V}_i\mathbf{J}_i' \end{bmatrix} \\
&\quad - \boldsymbol{r}\,[\,\mathbf{X}_i, \mathbf{B}_i\mathbf{V}_i\mathbf{B}_i'\,] + \boldsymbol{i}_{\pm}(\mathbf{X}_i^{\perp}\mathbf{B}_i\mathbf{V}_i\mathbf{B}_i'\mathbf{X}_i^{\perp}) \\
&= \boldsymbol{i}_{\mp}\begin{bmatrix} \mathbf{B}_i\mathbf{V}_i\mathbf{B}_i' & \mathbf{B}_i\mathbf{V}_i\mathbf{J}_i' & \mathbf{X}_i \\ \mathbf{J}_i\mathbf{V}_i\mathbf{B}_i' & \mathbf{J}_i\mathbf{V}_i\mathbf{J}_i' - \mathrm{D}[\boldsymbol{\varphi}_i - \mathrm{BLUP}_{\mathcal{M}}(\boldsymbol{\varphi}_i)] & \mathbf{K}_i \\ \mathbf{X}_i' & \mathbf{K}_i' & \mathbf{0} \end{bmatrix} \\
&\quad - \boldsymbol{r}\,[\,\mathbf{X}_i, \mathbf{B}_i\mathbf{V}_i\mathbf{B}_i'\,] + \boldsymbol{i}_{\pm}(\mathbf{X}_i^{\perp}\mathbf{B}_i\mathbf{V}_i\mathbf{B}_i'\mathbf{X}_i^{\perp}) \\
&= \boldsymbol{i}_{\mp}\left(\begin{bmatrix} \mathbf{B}_i\mathbf{V}_i\mathbf{B}_i' & \mathbf{B}_i\mathbf{V}_i\mathbf{J}_i' & \mathbf{X}_i \\ \mathbf{J}_i\mathbf{V}_i\mathbf{B}_i' & \mathbf{J}_i\mathbf{V}_i\mathbf{J}_i' & \mathbf{K}_i \\ \mathbf{X}_i' & \mathbf{K}_i' & \mathbf{0} \end{bmatrix} - \begin{bmatrix} \mathbf{0} \\ \mathbf{I}_s \\ \mathbf{0} \end{bmatrix} \mathrm{D}[\boldsymbol{\varphi}_i - \mathrm{BLUP}_{\mathcal{M}}(\boldsymbol{\varphi}_i)]\,[\,\mathbf{0} \quad \mathbf{I}_s \quad \mathbf{0}\,]\right) \\
&\quad - \boldsymbol{r}\,[\,\mathbf{X}_i, \mathbf{B}_i\mathbf{V}_i\mathbf{B}_i'\,] + \boldsymbol{i}_{\pm}(\mathbf{X}_i^{\perp}\mathbf{B}_i\mathbf{V}_i\mathbf{B}_i'\mathbf{X}_i^{\perp}).
\end{aligned}
$$

We can apply (2.18) to (3.6) after setting $\mathrm{D}[\boldsymbol{\varphi}_i - \mathrm{BLUP}_{\mathcal{M}}(\boldsymbol{\varphi}_i)]$ in (2.6). Then in a similar way to obtaining (3.5), (3.6) is equivalently written as

$$
\begin{aligned}
(3.7)\qquad &\boldsymbol{i}_{\mp}\left(\begin{bmatrix} \mathbf{V} & \mathbf{0} & -\mathbf{V}\widehat{\mathbf{J}}_i' & \mathbf{0} \\ \mathbf{0} & \mathbf{B}_i\mathbf{V}_i\mathbf{B}_i' & \mathbf{B}_i\mathbf{V}_i\mathbf{J}_i' & \mathbf{X}_i \\ -\widehat{\mathbf{J}}_i\mathbf{V} & \mathbf{J}_i\mathbf{V}_i\mathbf{B}_i' & \mathbf{J}_i\mathbf{V}_i\mathbf{J}_i' & \mathbf{K}_i \\ \mathbf{0} & \mathbf{X}_i' & \mathbf{K}_i' & \mathbf{0} \end{bmatrix} + \begin{bmatrix} \mathbf{V}\mathbf{B}' & \mathbf{0} \\ \mathbf{0} & \mathbf{0} \\ \mathbf{0} & [\,\widehat{\mathbf{K}}_i, \widehat{\mathbf{J}}_i\mathbf{V}\mathbf{B}'\mathbf{X}^{\perp}\,] \\ \mathbf{0} & \mathbf{0} \end{bmatrix}\right. \\
&\qquad\left. \times \begin{bmatrix} \mathbf{0} & \mathbf{W} \\ \mathbf{W}' & \mathbf{0} \end{bmatrix}^{+} \begin{bmatrix} \mathbf{B}\mathbf{V} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & [\,\widehat{\mathbf{K}}_i, \widehat{\mathbf{J}}_i\mathbf{V}\mathbf{B}'\mathbf{X}^{\perp}\,]' & \mathbf{0} \end{bmatrix}\right) \\
&\qquad - \boldsymbol{r}\,[\,\mathbf{X}_i, \mathbf{B}_i\mathbf{V}_i\mathbf{B}_i'\,] + \boldsymbol{i}_{\pm}(\mathbf{X}_i^{\perp}\mathbf{B}_i\mathbf{V}_i\mathbf{B}_i'\mathbf{X}_i^{\perp}) - \boldsymbol{i}_{\mp}(\mathbf{V}).
\end{aligned}
$$

We can apply (2.18) to (3.7), since

$$\mathscr{C}(\mathbf{BV}) = \mathscr{C}(\mathbf{BVB}') \subseteq \mathscr{C}(\mathbf{W}) \quad \text{and} \quad \mathscr{C}([\,\widehat{\mathbf{K}}_i, \widehat{\mathbf{J}}_i\mathbf{VB}'\mathbf{X}^{\perp}\,]') \subseteq \mathscr{C}(\mathbf{W}'),$$

where $\mathbf{W} = [\,\mathbf{X}, \mathbf{BVB}'\mathbf{X}^{\perp}\,]$. From Lemma 2.4 and 2.5 and some congruence operations, (3.7) is equivalently written as

(3.8)

$$\begin{aligned}
&\boldsymbol{i}_{\mp}\begin{bmatrix} \mathbf{0} & -\mathbf{X} & -\mathbf{BVB}'\mathbf{X}^{\perp} & \mathbf{BV} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ -\mathbf{X}' & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \widehat{\mathbf{K}}_i' & \mathbf{0} \\ -\mathbf{X}^{\perp}\mathbf{BVB}' & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{X}^{\perp}\mathbf{BV}\widehat{\mathbf{J}}_i' & \mathbf{0} \\ \mathbf{VB}' & \mathbf{0} & \mathbf{0} & \mathbf{V} & \mathbf{0} & -\mathbf{V}\widehat{\mathbf{J}}_i' & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{B}_i\mathbf{V}_i\mathbf{B}_i' & \mathbf{B}_i\mathbf{V}_i\mathbf{J}_i' & \mathbf{X}_i \\ \mathbf{0} & \widehat{\mathbf{K}}_i & \widehat{\mathbf{J}}_i\mathbf{VB}'\mathbf{X}^{\perp} & -\widehat{\mathbf{J}}_i\mathbf{V} & \mathbf{J}_i\mathbf{V}_i\mathbf{B}_i' & \mathbf{J}_i\mathbf{V}_i\mathbf{J}_i' & \mathbf{K}_i \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{X}_i' & \mathbf{K}_i' & \mathbf{0} \end{bmatrix} \\
&\quad - \boldsymbol{r}\,[\,\mathbf{X}_i, \mathbf{B}_i\mathbf{V}_i\mathbf{B}_i'\,] + \boldsymbol{i}_{\pm}(\mathbf{X}_i^{\perp}\mathbf{B}_i\mathbf{V}_i\mathbf{B}_i'\mathbf{X}_i^{\perp}) - \boldsymbol{i}_{\mp}(\mathbf{V}) - \boldsymbol{r}\,[\,\mathbf{X}, \mathbf{BVB}'\mathbf{X}^{\perp}\,] \\
&= \boldsymbol{i}_{\mp}\begin{bmatrix} -\mathbf{BVB}' & -\mathbf{X} & -\mathbf{BVB}'\mathbf{X}^{\perp} & \mathbf{0} & \mathbf{BV}\widehat{\mathbf{J}}_i' & \mathbf{0} \\ -\mathbf{X}' & \mathbf{0} & \mathbf{0} & \mathbf{0} & \widehat{\mathbf{K}}_i' & \mathbf{0} \\ -\mathbf{X}^{\perp}\mathbf{BVB}' & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{X}^{\perp}\mathbf{BV}\widehat{\mathbf{J}}_i' & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{B}_i\mathbf{V}_i\mathbf{B}_i' & \mathbf{B}_i\mathbf{V}_i\mathbf{J}_i' & \mathbf{X}_i \\ \widehat{\mathbf{J}}_i\mathbf{VB}' & \widehat{\mathbf{K}}_i & \widehat{\mathbf{J}}_i\mathbf{VB}'\mathbf{X}^{\perp} & \mathbf{J}_i\mathbf{V}_i\mathbf{B}_i' & \mathbf{J}_i\mathbf{V}_i\mathbf{J}_i' - \widehat{\mathbf{J}}_i\mathbf{V}\widehat{\mathbf{J}}_i' & \mathbf{K}_i \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{X}_i' & \mathbf{K}_i' & \mathbf{0} \end{bmatrix} \\
&\quad - \boldsymbol{r}\,[\,\mathbf{X}_i, \mathbf{B}_i\mathbf{V}_i\mathbf{B}_i'\,] + \boldsymbol{i}_{\pm}(\mathbf{X}_i^{\perp}\mathbf{B}_i\mathbf{V}_i\mathbf{B}_i'\mathbf{X}_i^{\perp}) - \boldsymbol{r}\,[\,\mathbf{X}, \mathbf{BVB}'\,] \\
&= \boldsymbol{i}_{\mp}\begin{bmatrix} -\mathbf{BVB}' & -\mathbf{X} & \mathbf{0} & \mathbf{BV}\widehat{\mathbf{J}}_i' & \mathbf{0} \\ -\mathbf{X}' & \mathbf{0} & \mathbf{0} & \widehat{\mathbf{K}}_i' & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{B}_i\mathbf{V}_i\mathbf{B}_i' & \mathbf{B}_i\mathbf{V}_i\mathbf{J}_i' & \mathbf{X}_i \\ \widehat{\mathbf{J}}_i\mathbf{VB}' & \widehat{\mathbf{K}}_i & \mathbf{J}_i\mathbf{V}_i\mathbf{B}_i' & \mathbf{0} & \mathbf{K}_i \\ \mathbf{0} & \mathbf{0} & \mathbf{X}_i' & \mathbf{K}_i' & \mathbf{0} \end{bmatrix} \\
&\quad - \boldsymbol{r}\,[\,\mathbf{X}_i, \mathbf{B}_i\mathbf{V}_i\mathbf{B}_i'\,] + \boldsymbol{i}_{\pm}(\mathbf{X}_i^{\perp}\mathbf{B}_i\mathbf{V}_i\mathbf{B}_i'\mathbf{X}_i^{\perp}) - \boldsymbol{r}\,[\,\mathbf{X}, \mathbf{BVB}'\,] + \boldsymbol{i}_{\mp}(\mathbf{X}^{\perp}\mathbf{BVB}'\mathbf{X}^{\perp}) \\
&= \boldsymbol{i}_{\pm}\begin{bmatrix} \mathbf{BVB}' & \mathbf{0} & \mathbf{BV}\widehat{\mathbf{J}}_i' & \mathbf{X} & \mathbf{0} \\ \mathbf{0} & -\mathbf{B}_i\mathbf{V}_i\mathbf{B}_i' & \mathbf{B}_i\mathbf{V}_i\mathbf{J}_i' & \mathbf{0} & \mathbf{X}_i \\ \widehat{\mathbf{J}}_i\mathbf{VB}' & \mathbf{J}_i\mathbf{V}_i\mathbf{B}_i' & \mathbf{0} & \widehat{\mathbf{K}}_i & -\mathbf{K}_i \\ \mathbf{X}' & \mathbf{0} & \widehat{\mathbf{K}}_i' & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{X}_i' & -\mathbf{K}_i' & \mathbf{0} & \mathbf{0} \end{bmatrix} \\
&\quad - \boldsymbol{r}\,[\,\mathbf{X}_i, \mathbf{B}_i\mathbf{V}_i\mathbf{B}_i'\,] + \boldsymbol{i}_{\pm}(\mathbf{X}_i^{\perp}\mathbf{B}_i\mathbf{V}_i\mathbf{B}_i'\mathbf{X}_i^{\perp}) - \boldsymbol{r}\,[\,\mathbf{X}, \mathbf{BVB}'\,] + \boldsymbol{i}_{\mp}(\mathbf{X}^{\perp}\mathbf{BVB}'\mathbf{X}^{\perp}).
\end{aligned}$$

In consequence, by using (2.16) and (2.17), we obtain (3.2) and (3.3). According to (2.12), adding the equalities in (3.2) and (3.3) yields (3.4). Applying Lemma 2.3 to (3.2)–(3.4) yields (a)–(e). □

The conclusions obtained in Theorem 3.1 can be considered for certain specific forms of $\boldsymbol{\varphi}_i$ as presented in the following corollaries.

Corollary 3.1. *Let* $\mathbf{K}_i\boldsymbol{\alpha}_i$ *be estimable under* $\mathcal{M}_i$ *(also estimable under* $\mathcal{M}$*),* $i = 1, \ldots, m$. *Denote*

$$\mathbf{N} = \begin{bmatrix} \mathbf{BVB}' & \mathbf{0} & \mathbf{0} & \mathbf{X} & \mathbf{0} \\ \mathbf{0} & -\mathbf{B}_i\mathbf{V}_i\mathbf{B}_i' & \mathbf{0} & \mathbf{0} & \mathbf{X}_i \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \widehat{\mathbf{K}}_i & -\mathbf{K}_i \\ \mathbf{X}' & \mathbf{0} & \widehat{\mathbf{K}}_i' & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{X}_i' & -\mathbf{K}_i' & \mathbf{0} & \mathbf{0} \end{bmatrix}. \tag{3.9}$$

Then

$$\begin{aligned} &\boldsymbol{i}_{+}(\mathrm{D}[\mathrm{BLUE}_{\mathcal{M}}(\mathbf{K}_i\boldsymbol{\alpha}_i)] - \mathrm{D}[\mathrm{BLUE}_{\mathcal{M}_i}(\mathbf{K}_i\boldsymbol{\alpha}_i)]) \\ &\qquad = \boldsymbol{i}_{+}(\mathbf{N}) - \boldsymbol{r}\,[\,\mathbf{X}, \mathbf{BVB}'\,] - \boldsymbol{r}(\mathbf{X}_i), \end{aligned} \tag{3.10}$$

$$\begin{aligned} &\boldsymbol{i}_{-}(\mathrm{D}[\mathrm{BLUE}_{\mathcal{M}}(\mathbf{K}_i\boldsymbol{\alpha}_i)] - \mathrm{D}[\mathrm{BLUE}_{\mathcal{M}_i}(\mathbf{K}_i\boldsymbol{\alpha}_i)]) \\ &\qquad = \boldsymbol{i}_{-}(\mathbf{N}) - \boldsymbol{r}\,[\,\mathbf{X}_i, \mathbf{B}_i\mathbf{V}_i\mathbf{B}_i'\,] - \boldsymbol{r}(\mathbf{X}), \end{aligned} \tag{3.11}$$

$$\begin{aligned} &\boldsymbol{r}(\mathrm{D}[\mathrm{BLUE}_{\mathcal{M}}(\mathbf{K}_i\boldsymbol{\alpha}_i)] - \mathrm{D}[\mathrm{BLUE}_{\mathcal{M}_i}(\mathbf{K}_i\boldsymbol{\alpha}_i)]) \\ &\qquad = \boldsymbol{r}(\mathbf{N}) - \boldsymbol{r}(\mathbf{X}) - \boldsymbol{r}(\mathbf{X}_i) - \boldsymbol{r}\,[\,\mathbf{X}, \mathbf{BVB}'\,] - \boldsymbol{r}\,[\,\mathbf{X}_i, \mathbf{B}_i\mathbf{V}_i\mathbf{B}_i'\,]\,. \end{aligned} \tag{3.12}$$

Consequently, the following results hold:

(a) $$\begin{aligned} &\mathrm{D}[\mathrm{BLUE}_{\mathcal{M}_i}(\mathbf{K}_i\boldsymbol{\alpha}_i)] \succ \mathrm{D}[\mathrm{BLUE}_{\mathcal{M}}(\mathbf{K}_i\boldsymbol{\alpha}_i)] \\ &\quad \Leftrightarrow \boldsymbol{i}_{-}(\mathbf{N}) = \boldsymbol{r}\,[\,\mathbf{X}_i, \mathbf{B}_i\mathbf{V}_i\mathbf{B}_i'\,] + \boldsymbol{r}(\mathbf{X}) + s; \end{aligned}$$

(b) $$\begin{aligned} &\mathrm{D}[\mathrm{BLUE}_{\mathcal{M}_i}(\mathbf{K}_i\boldsymbol{\alpha}_i)] \prec \mathrm{D}[\mathrm{BLUE}_{\mathcal{M}}(\mathbf{K}_i\boldsymbol{\alpha}_i)] \\ &\quad \Leftrightarrow \boldsymbol{i}_{+}(\mathbf{N}) = \boldsymbol{r}\,[\,\mathbf{X}, \mathbf{BVB}'\,] + \boldsymbol{r}(\mathbf{X}_i) + s; \end{aligned}$$

(c) $$\begin{aligned} &\mathrm{D}[\mathrm{BLUE}_{\mathcal{M}_i}(\mathbf{K}_i\boldsymbol{\alpha}_i)] \succcurlyeq \mathrm{D}[\mathrm{BLUE}_{\mathcal{M}}(\mathbf{K}_i\boldsymbol{\alpha}_i)] \\ &\quad \Leftrightarrow \boldsymbol{i}_{+}(\mathbf{N}) = \boldsymbol{r}\,[\,\mathbf{X}, \mathbf{BVB}'\,] + \boldsymbol{r}(\mathbf{X}_i); \end{aligned}$$

(d) $$\begin{aligned} &\mathrm{D}[\mathrm{BLUE}_{\mathcal{M}_i}(\mathbf{K}_i\boldsymbol{\alpha}_i)] \preccurlyeq \mathrm{D}[\mathrm{BLUE}_{\mathcal{M}}(\mathbf{K}_i\boldsymbol{\alpha}_i)] \\ &\quad \Leftrightarrow \boldsymbol{i}_{-}(\mathbf{N}) = \boldsymbol{r}\,[\,\mathbf{X}_i, \mathbf{B}_i\mathbf{V}_i\mathbf{B}_i'\,] + \boldsymbol{r}(\mathbf{X}); \end{aligned}$$

(e) $$\begin{aligned} &\mathrm{D}[\mathrm{BLUE}_{\mathcal{M}_i}(\mathbf{K}_i\boldsymbol{\alpha}_i)] = \mathrm{D}[\mathrm{BLUE}_{\mathcal{M}}(\mathbf{K}_i\boldsymbol{\alpha}_i)] \\ &\quad \Leftrightarrow \boldsymbol{r}(\mathbf{N}) = \boldsymbol{r}\,[\,\mathbf{X}, \mathbf{BVB}'\,] + \boldsymbol{r}(\mathbf{X}_i) + \boldsymbol{r}\,[\,\mathbf{X}_i, \mathbf{B}_i\mathbf{V}_i\mathbf{B}_i'\,] + \boldsymbol{r}(\mathbf{X}). \end{aligned}$$

Corollary 3.2. $\mathbf{X}_i\boldsymbol{\alpha}_i$ *is always estimable under* $\mathcal{M}_i$ *(and thereby under* $\mathcal{M}$*). Let* $\widehat{\mathbf{X}}_i = [\,\mathbf{0}, \ldots, \mathbf{X}_i, \ldots, \mathbf{0}\,]$ *and denote*

$$\mathbf{N} = \begin{bmatrix} \mathbf{BVB}' & \mathbf{0} & \mathbf{X} \\ \mathbf{0} & -\mathbf{B}_i\mathbf{V}_i\mathbf{B}_i' & \widehat{\mathbf{X}}_i \\ \mathbf{X}' & \widehat{\mathbf{X}}_i' & \mathbf{0} \end{bmatrix}. \tag{3.13}$$

Then

$$(3.14)\qquad \boldsymbol{i}_{+}(\mathrm{D}[\mathrm{BLUE}_{\mathcal{M}}(\mathbf{X}_i\boldsymbol{\alpha}_i)] - \mathrm{D}[\mathrm{BLUE}_{\mathcal{M}_i}(\mathbf{X}_i\boldsymbol{\alpha}_i)]) = \boldsymbol{i}_{+}(\mathbf{N}) - \boldsymbol{r}\,[\,\mathbf{X}, \mathbf{BVB}'\,],$$

$$(3.15)\qquad \boldsymbol{i}_{-}(\mathrm{D}[\mathrm{BLUE}_{\mathcal{M}}(\mathbf{X}_i\boldsymbol{\alpha}_i)] - \mathrm{D}[\mathrm{BLUE}_{\mathcal{M}_i}(\mathbf{X}_i\boldsymbol{\alpha}_i)])$$
$$= \boldsymbol{i}_{-}(\mathbf{N}) - \boldsymbol{r}\,[\,\mathbf{X}_i, \mathbf{B}_i\mathbf{V}_i\mathbf{B}_i'\,] + \boldsymbol{r}(\mathbf{X}_i) - \boldsymbol{r}(\mathbf{X}),$$

$$(3.16)\qquad \boldsymbol{r}(\mathrm{D}[\mathrm{BLUE}_{\mathcal{M}}(\mathbf{X}_i\boldsymbol{\alpha}_i)] - \mathrm{D}[\mathrm{BLUE}_{\mathcal{M}_i}(\mathbf{X}_i\boldsymbol{\alpha}_i)])$$
$$= \boldsymbol{r}(\mathbf{N}) - \boldsymbol{r}\,[\,\mathbf{X}_i, \mathbf{B}_i\mathbf{V}_i\mathbf{B}_i'\,] - \boldsymbol{r}\,[\,\mathbf{X}, \mathbf{BVB}'\,] + \boldsymbol{r}(\mathbf{X}_i) - \boldsymbol{r}(\mathbf{X}).$$

Consequently,

(a) $\mathrm{D}[\mathrm{BLUE}_{\mathcal{M}_i}(\mathbf{X}_i\boldsymbol{\alpha}_i)] \succ \mathrm{D}[\mathrm{BLUE}_{\mathcal{M}}(\mathbf{X}_i\boldsymbol{\alpha}_i)]$
$\Leftrightarrow \boldsymbol{i}_{-}(\mathbf{N}) = \boldsymbol{r}\,[\,\mathbf{X}_i, \mathbf{B}_i\mathbf{V}_i\mathbf{B}_i'\,] - \boldsymbol{r}(\mathbf{X}_i) + \boldsymbol{r}(\mathbf{X}) + s;$

(b) $\mathrm{D}[\mathrm{BLUE}_{\mathcal{M}_i}(\mathbf{X}_i\boldsymbol{\alpha}_i)] \prec \mathrm{D}[\mathrm{BLUE}_{\mathcal{M}}(\mathbf{X}_i\boldsymbol{\alpha}_i)] \Leftrightarrow \boldsymbol{i}_{+}(\mathbf{N}) = \boldsymbol{r}\,[\,\mathbf{X}, \mathbf{BVB}'\,] + s;$

(c) $\mathrm{D}[\mathrm{BLUE}_{\mathcal{M}_i}(\mathbf{X}_i\boldsymbol{\alpha}_i)] \succcurlyeq \mathrm{D}[\mathrm{BLUE}_{\mathcal{M}}(\mathbf{X}_i\boldsymbol{\alpha}_i)] \Leftrightarrow \boldsymbol{i}_{+}(\mathbf{N}) = \boldsymbol{r}\,[\,\mathbf{X}, \mathbf{BVB}'\,];$

(d) $\mathrm{D}[\mathrm{BLUE}_{\mathcal{M}_i}(\mathbf{X}_i\boldsymbol{\alpha}_i)] \preccurlyeq \mathrm{D}[\mathrm{BLUE}_{\mathcal{M}}(\mathbf{X}_i\boldsymbol{\alpha}_i)]$
$\Leftrightarrow \boldsymbol{i}_{-}(\mathbf{N}) = \boldsymbol{r}\,[\,\mathbf{X}_i, \mathbf{B}_i\mathbf{V}_i\mathbf{B}_i'\,] - \boldsymbol{r}(\mathbf{X}_i) + \boldsymbol{r}(\mathbf{X});$

(e) $\mathrm{D}[\mathrm{BLUE}_{\mathcal{M}_i}(\mathbf{X}_i\boldsymbol{\alpha}_i)] = \mathrm{D}[\mathrm{BLUE}_{\mathcal{M}}(\mathbf{X}_i\boldsymbol{\alpha}_i)]$
$\Leftrightarrow \boldsymbol{r}(\mathbf{N}) = \boldsymbol{r}\,[\,\mathbf{X}, \mathbf{BVB}'\,] - \boldsymbol{r}(\mathbf{X}_i) + \boldsymbol{r}\,[\,\mathbf{X}_i, \mathbf{B}_i\mathbf{V}_i\mathbf{B}_i'\,] + \boldsymbol{r}(\mathbf{X}).$

4. Comparisons of BLUPs in SURMs

SULMMs are an extension of SURMs to include random effects. In this section, as an application, some of the results obtained for SULMMs are also presented for SURMs. We consider a set of m different SURMs and their combined model which correspond to $\mathcal{M}_i$ and $\mathcal{M}$, respectively. We derived necessary and sufficient conditions on equalities and inequalities for comparing covariance matrices of predictors of joint unknown vectors under SURMs and their combined model by using Theorem 3.1.

SURMs can have correlated error terms among each other although they seem unrelated if the same data or an amount of the same independent variables are used for the models. These models were originally proposed by [36]. Such models have found many applications in different fields of sciences and they have gained considerable interest in recent years. As previous and recent works on SURMs, we may refer to [3], [4], [7], [9], [15], [17], [23], [24], [37] among others and for seemingly unrelated linear random effects models, see [33].

In accordance with the models $\mathcal{M}_i$ and $\mathcal{M}$, we consider a set of m different SURMs and their combined model as

$$\mathcal{S}_i\colon\ \mathbf{y}_i = \mathbf{X}_i\boldsymbol{\beta}_i + \boldsymbol{\varepsilon}_i \text{ with } \mathrm{E}(\boldsymbol{\varepsilon}_i) = \mathbf{0} \text{ and } \mathrm{cov}(\boldsymbol{\varepsilon}_i, \boldsymbol{\varepsilon}_j) = \sigma_{i,j}\mathbf{I}_n, \tag{4.1}$$

$$\mathcal{S}\colon\ \mathbf{y} = \mathbf{X}\boldsymbol{\beta} + \boldsymbol{\varepsilon} \text{ with } \mathrm{E}(\boldsymbol{\varepsilon}) = \mathbf{0} \text{ and } \mathrm{D}(\boldsymbol{\varepsilon}) = \boldsymbol{\Sigma} \otimes \mathbf{I}_n, \tag{4.2}$$

respectively, where $\mathbf{y}_i \in \mathbb{R}^{n\times 1}$ is a vector of observable response variables, $\mathbf{X}_i \in \mathbb{R}^{n\times k_i}$ is a known matrix of arbitrary rank, $\boldsymbol{\beta}_i \in \mathbb{R}^{k_i\times 1}$ is a vector of fixed but unknown parameters, $\boldsymbol{\varepsilon}_i \in \mathbb{R}^{n\times 1}$ is an error vector, $\mathbf{y} \in \mathbb{R}^{nm\times 1}$, $\mathbf{X} \in \mathbb{R}^{nm\times k}$, $\boldsymbol{\beta} \in \mathbb{R}^{k\times 1}$, $\boldsymbol{\varepsilon} \in \mathbb{R}^{nm\times 1}$, $\boldsymbol{\Sigma} = (\sigma_{i,j}) \in \mathbb{R}^{m\times m}$ is a positive semi-definite matrix of arbitrary rank, $i, j = 1, \ldots, m$, $k_1 + \ldots + k_m = k$. To establish the results, we consider the vector

$$\boldsymbol{\varphi}_i = \widehat{\mathbf{K}}_i\boldsymbol{\beta} + \widehat{\mathbf{H}}_i\boldsymbol{\varepsilon} \quad \text{or equivalently} \quad \boldsymbol{\varphi}_i = \mathbf{K}_i\boldsymbol{\beta}_i + \mathbf{H}_i\boldsymbol{\varepsilon}_i \tag{4.3}$$

for given matrices $\mathbf{K}_i \in \mathbb{R}^{s\times k_i}$ and $\mathbf{H}_i \in \mathbb{R}^{s\times n}$ with $\widehat{\mathbf{K}}_i = [\,\mathbf{0}, \ldots, \mathbf{K}_i, \ldots, \mathbf{0}\,]$ and $\widehat{\mathbf{H}}_i = [\,\mathbf{0}, \ldots, \mathbf{H}_i, \ldots, \mathbf{0}\,]$, $i = 1, \ldots, m$. In this case,

$$\mathrm{E}(\boldsymbol{\varphi}_i) = \widehat{\mathbf{K}}_i\boldsymbol{\beta} = \mathbf{K}_i\boldsymbol{\beta}_i, \quad \mathrm{D}(\boldsymbol{\varphi}_i) = \sigma_{i,i}\mathbf{H}_i\mathbf{H}_i' = \widehat{\mathbf{H}}_i(\boldsymbol{\Sigma} \otimes \mathbf{I}_n)\widehat{\mathbf{H}}_i', \tag{4.4}$$

$$\mathrm{cov}(\boldsymbol{\varphi}_i, \mathbf{y}) = \widehat{\mathbf{H}}_i(\boldsymbol{\Sigma} \otimes \mathbf{I}_n), \quad \mathrm{cov}(\boldsymbol{\varphi}_i, \mathbf{y}_i) = \sigma_{i,i}\mathbf{H}_i = \widehat{\mathbf{H}}_i(\boldsymbol{\Sigma} \otimes \mathbf{I}_n)\mathbf{T}_i', \tag{4.5}$$

where $\mathbf{T}_i = [\mathbf{0}, \ldots, \mathbf{I}_n, \ldots, \mathbf{0}]$, $i = 1, \ldots, m$.

According to Lemma 2.1, we obtain the following results for models $\mathcal{S}$. Assume that the vector $\boldsymbol{\varphi}_i$ in (4.3) is predictable under $\mathcal{S}$. In this case,

$$\mathbf{L}_i\mathbf{y} = \mathrm{BLUP}_{\mathcal{S}}(\boldsymbol{\varphi}_i) \Leftrightarrow \mathbf{L}_i\,[\,\mathbf{X}, (\boldsymbol{\Sigma} \otimes \mathbf{I}_n)\mathbf{X}^{\perp}\,] = [\,\widehat{\mathbf{K}}_i, \widehat{\mathbf{H}}_i(\boldsymbol{\Sigma} \otimes \mathbf{I}_n)\mathbf{X}^{\perp}\,]. \tag{4.6}$$

Then

$$\mathrm{BLUP}_{\mathcal{S}}(\boldsymbol{\varphi}_i) = \mathbf{L}_i\mathbf{y} = ([\,\widehat{\mathbf{K}}_i, \widehat{\mathbf{H}}_i(\boldsymbol{\Sigma} \otimes \mathbf{I}_n)\mathbf{X}^{\perp}\,]\,\mathbf{W}^{+} + \mathbf{U}_i\mathbf{W}^{\perp})\mathbf{y}, \tag{4.7}$$

$$\begin{aligned}\mathrm{D}[\boldsymbol{\varphi}_i - \mathrm{BLUP}_{\mathcal{S}}(\boldsymbol{\varphi}_i)] = {} & ([\,\widehat{\mathbf{K}}_i, \widehat{\mathbf{H}}_i(\boldsymbol{\Sigma} \otimes \mathbf{I}_n)\mathbf{X}^{\perp}\,]\,\mathbf{W}^{+} - \widehat{\mathbf{H}}_i) \\ & \times (\boldsymbol{\Sigma} \otimes \mathbf{I}_n)([\,\widehat{\mathbf{K}}_i, \widehat{\mathbf{H}}_i(\boldsymbol{\Sigma} \otimes \mathbf{I}_n)\mathbf{X}^{\perp}\,]\,\mathbf{W}^{+} - \widehat{\mathbf{H}}_i)',\end{aligned} \tag{4.8}$$

where $\mathbf{U}_i \in \mathbb{R}^{s\times nm}$ is arbitrary and $\mathbf{W} = [\,\mathbf{X}, (\boldsymbol{\Sigma} \otimes \mathbf{I}_n)\mathbf{X}^{\perp}\,]$.

Assume that the vector $\boldsymbol{\varphi}_i$ in (4.3) is predictable under $\mathcal{S}_i$. From Lemma 2.2, BLUP of $\boldsymbol{\varphi}_i$ under $\mathcal{S}_i$ is written as

$$\mathrm{BLUP}_{\mathcal{S}_i}(\boldsymbol{\varphi}_i) = \mathbf{L}_i\mathbf{y}_i = ([\,\mathbf{K}_i, \sigma_{i,i}\mathbf{H}_i\mathbf{X}_i^{\perp}\,]\,\mathbf{W}_i^{+} + \mathbf{U}_i\mathbf{W}_i^{\perp})\mathbf{y}_i = (\mathbf{K}_i\mathbf{X}_i^{+} + \mathbf{H}_i\mathbf{X}_i^{\perp})\mathbf{y}_i, \tag{4.9}$$

where $\mathbf{U}_i \in \mathbb{R}^{s\times n}$ and $\mathbf{W}_i = [\,\mathbf{X}_i, \sigma_{i,i}\mathbf{X}_i^{\perp}\,]$, $i = 1, \ldots, m$. The last expression in (4.9) is the ordinary least-squares predictor (OLSP) of $\boldsymbol{\varphi}_i$ under $\mathcal{S}_i$, see Definition 1.3

in [28]. We note that the BLUP of $\boldsymbol{\varphi}_i$ and the OLSP of $\boldsymbol{\varphi}_i$ under $\mathcal{S}_i$ coincide since $\mathrm{D}(\mathbf{y}_i) = \sigma_{i,i}\mathbf{I}_n$. Further, we can write

(4.10)
$$\mathrm{D}[\boldsymbol{\varphi}_i - \mathrm{BLUP}_{\mathcal{S}_i}(\boldsymbol{\varphi}_i)] = \sigma_{i,i}([\,\mathbf{K}_i, \sigma_{i,i}\mathbf{H}_i\mathbf{X}_i^{\perp}\,]\,\mathbf{W}_i^{+} - \mathbf{H}_i)([\,\mathbf{K}_i, \sigma_{i,i}\mathbf{H}_i\mathbf{X}_i^{\perp}\,]\,\mathbf{W}_i^{+} - \mathbf{H}_i)'.$$

Now, we can give the following results for comparison of covariance matrices of the BLUPs and OLSPs of $\boldsymbol{\varphi}_i$ under SURMs and their combined model.

Theorem 4.1. *Let $\mathcal{S}_i$ and $\mathcal{S}$ be as given in (4.1) and (4.2), respectively, and assume that $\boldsymbol{\varphi}_i$ in (4.3) is predictable under $\mathcal{S}_i$ (also predictable under $\mathcal{S}$), $i = 1, \ldots, m$. Let $\mathrm{BLUP}_{\mathcal{S}}(\boldsymbol{\varphi}_i)$ and $\mathrm{OLSP}_{\mathcal{S}_i}(\boldsymbol{\varphi}_i)$ be as given in (4.7) and (4.9), respectively. Denote*

$$\mathbf{N} = \begin{bmatrix} \boldsymbol{\Sigma}\otimes\mathbf{I}_n & \mathbf{X} & \mathbf{0} & \mathbf{0} \\ \mathbf{X}' & \mathbf{0} & \widehat{\mathbf{K}}_i' - \mathbf{X}'\widehat{\mathbf{H}}_i' & \mathbf{0} \\ \mathbf{0} & \widehat{\mathbf{K}}_i - \widehat{\mathbf{H}}_i\mathbf{X} & \mathbf{0} & \mathbf{K}_i - \mathbf{H}_i\mathbf{X}_i \\ \mathbf{0} & \mathbf{0} & \mathbf{K}_i' - \mathbf{X}_i'\mathbf{H}_i' & \sigma_{i,i}^{-1}\mathbf{X}_i'\mathbf{X}_i \end{bmatrix}. \tag{4.11}$$

Then

$$\boldsymbol{i}_{+}(\mathrm{D}[\boldsymbol{\varphi}_i - \mathrm{BLUP}_{\mathcal{S}}(\boldsymbol{\varphi}_i)] - \mathrm{D}[\boldsymbol{\varphi}_i - \mathrm{OLSP}_{\mathcal{S}_i}(\boldsymbol{\varphi}_i)]) = \boldsymbol{i}_{+}(\mathbf{N}) - \boldsymbol{r}\,[\,\mathbf{X}, \boldsymbol{\Sigma}\otimes\mathbf{I}_n\,] - \boldsymbol{r}(\mathbf{X}_i), \tag{4.12}$$

$$\boldsymbol{i}_{-}(\mathrm{D}[\boldsymbol{\varphi}_i - \mathrm{BLUP}_{\mathcal{S}}(\boldsymbol{\varphi}_i)] - \mathrm{D}[\boldsymbol{\varphi}_i - \mathrm{OLSP}_{\mathcal{S}_i}(\boldsymbol{\varphi}_i)]) = \boldsymbol{i}_{-}(\mathbf{N}) - \boldsymbol{r}(\mathbf{X}), \tag{4.13}$$

$$\boldsymbol{r}(\mathrm{D}[\boldsymbol{\varphi}_i - \mathrm{BLUP}_{\mathcal{S}}(\boldsymbol{\varphi}_i)] - \mathrm{D}[\boldsymbol{\varphi}_i - \mathrm{OLSP}_{\mathcal{S}_i}(\boldsymbol{\varphi}_i)]) = \boldsymbol{r}(\mathbf{N}) - \boldsymbol{r}\,[\,\mathbf{X}, \boldsymbol{\Sigma}\otimes\mathbf{I}_n\,] - \boldsymbol{r}(\mathbf{X}_i) - \boldsymbol{r}(\mathbf{X}). \tag{4.14}$$

Consequently, the following results hold:

(a) $\mathrm{D}[\boldsymbol{\varphi}_i - \mathrm{OLSP}_{\mathcal{S}_i}(\boldsymbol{\varphi}_i)] \succ \mathrm{D}[\boldsymbol{\varphi}_i - \mathrm{BLUP}_{\mathcal{S}}(\boldsymbol{\varphi}_i)] \Leftrightarrow \boldsymbol{i}_{-}(\mathbf{N}) = \boldsymbol{r}(\mathbf{X}) + s;$

(b) $\mathrm{D}[\boldsymbol{\varphi}_i - \mathrm{OLSP}_{\mathcal{S}_i}(\boldsymbol{\varphi}_i)] \prec \mathrm{D}[\boldsymbol{\varphi}_i - \mathrm{BLUP}_{\mathcal{S}}(\boldsymbol{\varphi}_i)] \Leftrightarrow \boldsymbol{i}_{+}(\mathbf{N}) = \boldsymbol{r}\,[\,\mathbf{X}, \boldsymbol{\Sigma}\otimes\mathbf{I}_n\,] + \boldsymbol{r}(\mathbf{X}_i) + s;$

(c) $\mathrm{D}[\boldsymbol{\varphi}_i - \mathrm{OLSP}_{\mathcal{S}_i}(\boldsymbol{\varphi}_i)] \succcurlyeq \mathrm{D}[\boldsymbol{\varphi}_i - \mathrm{BLUP}_{\mathcal{S}}(\boldsymbol{\varphi}_i)] \Leftrightarrow \boldsymbol{i}_{+}(\mathbf{N}) = \boldsymbol{r}\,[\,\mathbf{X}, \boldsymbol{\Sigma}\otimes\mathbf{I}_n\,] + \boldsymbol{r}(\mathbf{X}_i);$

(d) $\mathrm{D}[\boldsymbol{\varphi}_i - \mathrm{OLSP}_{\mathcal{S}_i}(\boldsymbol{\varphi}_i)] \preccurlyeq \mathrm{D}[\boldsymbol{\varphi}_i - \mathrm{BLUP}_{\mathcal{S}}(\boldsymbol{\varphi}_i)] \Leftrightarrow \boldsymbol{i}_{-}(\mathbf{N}) = \boldsymbol{r}(\mathbf{X});$

(e) $\mathrm{D}[\boldsymbol{\varphi}_i - \mathrm{OLSP}_{\mathcal{S}_i}(\boldsymbol{\varphi}_i)] = \mathrm{D}[\boldsymbol{\varphi}_i - \mathrm{BLUP}_{\mathcal{S}}(\boldsymbol{\varphi}_i)] \Leftrightarrow \boldsymbol{r}(\mathbf{N}) = \boldsymbol{r}\,[\,\mathbf{X}, \boldsymbol{\Sigma}\otimes\mathbf{I}_n\,] + \boldsymbol{r}(\mathbf{X}_i) + \boldsymbol{r}(\mathbf{X}).$

Proof. According to (3.8) in Theorem 3.1 and by using (4.8) and (4.10), we can write

$$
\begin{aligned}
(4.15)\qquad & \boldsymbol{i}_{\pm}(\mathrm{D}[\boldsymbol{\varphi}_i - \mathrm{BLUP}_{\mathcal{S}}(\boldsymbol{\varphi}_i)] - \mathrm{D}[\boldsymbol{\varphi}_i - \mathrm{OLSP}_{\mathcal{S}_i}(\boldsymbol{\varphi}_i)]) \\
& = \boldsymbol{i}_{\pm}\begin{bmatrix} \boldsymbol{\Sigma}\otimes\mathbf{I}_n & \mathbf{0} & (\boldsymbol{\Sigma}\otimes\mathbf{I}_n)\widehat{\mathbf{H}}_i' & \mathbf{X} & \mathbf{0} \\ \mathbf{0} & -\sigma_{i,i}\mathbf{I}_n & \sigma_{i,i}\mathbf{H}_i' & \mathbf{0} & \mathbf{X}_i \\ \widehat{\mathbf{H}}_i(\boldsymbol{\Sigma}\otimes\mathbf{I}_n) & \sigma_{i,i}\mathbf{H}_i & \mathbf{0} & \widehat{\mathbf{K}}_i & -\mathbf{K}_i \\ \mathbf{X}' & \mathbf{0} & \widehat{\mathbf{K}}_i' & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{X}_i' & -\mathbf{K}_i' & \mathbf{0} & \mathbf{0} \end{bmatrix} \\
& \quad - \boldsymbol{r}\,[\,\mathbf{X}_i, \sigma_{i,i}\mathbf{I}_n\,] + \boldsymbol{i}_{\pm}(\mathbf{X}_i^{\perp}\sigma_{i,i}\mathbf{I}_n\mathbf{X}_i^{\perp}) \\
& \quad - \boldsymbol{r}\,[\,\mathbf{X}, \boldsymbol{\Sigma}\otimes\mathbf{I}_n\,] + \boldsymbol{i}_{\mp}(\mathbf{X}^{\perp}(\boldsymbol{\Sigma}\otimes\mathbf{I}_n)\mathbf{X}^{\perp}).
\end{aligned}
$$

From Lemma 2.4 and 2.5 and some congruence operations, (4.15) is equivalently written as

(4.16)

$$
\begin{aligned}
& \boldsymbol{i}_{\pm}\begin{bmatrix} \boldsymbol{\Sigma}\otimes\mathbf{I}_n & \mathbf{0} & \mathbf{0} & \mathbf{X} & \mathbf{0} \\ \mathbf{0} & -\sigma_{i,i}\mathbf{I}_n & \sigma_{i,i}\mathbf{H}_i' & \mathbf{0} & \mathbf{X}_i \\ \mathbf{0} & \sigma_{i,i}\mathbf{H}_i & -\widehat{\mathbf{H}}_i(\boldsymbol{\Sigma}\otimes\mathbf{I}_n)\widehat{\mathbf{H}}_i' & \widehat{\mathbf{K}}_i - \widehat{\mathbf{H}}_i\mathbf{X} & -\mathbf{K}_i \\ \mathbf{X}' & \mathbf{0} & \widehat{\mathbf{K}}_i' - \mathbf{X}'\widehat{\mathbf{H}}_i' & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{X}_i' & -\mathbf{K}_i' & \mathbf{0} & \mathbf{0} \end{bmatrix} \\
& \quad - n + \boldsymbol{i}_{\pm}\begin{bmatrix} \sigma_{i,i}\mathbf{I}_n & \mathbf{X}_i \\ \mathbf{X}_i' & \mathbf{0} \end{bmatrix} - \boldsymbol{r}(\mathbf{X}_i) - \boldsymbol{r}\,[\,\mathbf{X}, \boldsymbol{\Sigma}\otimes\mathbf{I}_n\,] + \boldsymbol{i}_{\mp}\begin{bmatrix} \boldsymbol{\Sigma}\otimes\mathbf{I}_n & \mathbf{X} \\ \mathbf{X}' & \mathbf{0} \end{bmatrix} - \boldsymbol{r}(\mathbf{X}) \\
& = \boldsymbol{i}_{\pm}\begin{bmatrix} \boldsymbol{\Sigma}\otimes\mathbf{I}_n & \mathbf{0} & \mathbf{0} & \mathbf{X} & \mathbf{0} \\ \mathbf{0} & -\sigma_{i,i}\mathbf{I}_n & \mathbf{0} & \mathbf{0} & \mathbf{X}_i \\ \mathbf{0} & \mathbf{0} & \sigma_{i,i}\mathbf{H}_i\mathbf{H}_i' - \widehat{\mathbf{H}}_i(\boldsymbol{\Sigma}\otimes\mathbf{I}_n)\widehat{\mathbf{H}}_i' & \widehat{\mathbf{K}}_i - \widehat{\mathbf{H}}_i\mathbf{X} & -\mathbf{K}_i + \mathbf{H}_i\mathbf{X}_i \\ \mathbf{X}' & \mathbf{0} & \widehat{\mathbf{K}}_i' - \mathbf{X}'\widehat{\mathbf{H}}_i' & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{X}_i' & -\mathbf{K}_i' + \mathbf{X}_i'\mathbf{H}_i' & \mathbf{0} & \mathbf{0} \end{bmatrix} \\
& \quad - n + \boldsymbol{i}_{\pm}\begin{bmatrix} \sigma_{i,i}\mathbf{I}_n & \mathbf{X}_i \\ \mathbf{X}_i' & \mathbf{0} \end{bmatrix} - \boldsymbol{r}(\mathbf{X}_i) - \boldsymbol{r}\,[\,\mathbf{X}, \boldsymbol{\Sigma}\otimes\mathbf{I}_n\,] + \boldsymbol{i}_{\mp}\begin{bmatrix} \boldsymbol{\Sigma}\otimes\mathbf{I}_n & \mathbf{X} \\ \mathbf{X}' & \mathbf{0} \end{bmatrix} - \boldsymbol{r}(\mathbf{X}) \\
& = \boldsymbol{i}_{\pm}\begin{bmatrix} \boldsymbol{\Sigma}\otimes\mathbf{I}_n & \mathbf{X} & \mathbf{0} & \mathbf{0} \\ \mathbf{X}' & \mathbf{0} & \widehat{\mathbf{K}}_i' - \mathbf{X}'\widehat{\mathbf{H}}_i' & \mathbf{0} \\ \mathbf{0} & \widehat{\mathbf{K}}_i - \widehat{\mathbf{H}}_i\mathbf{X} & \mathbf{0} & \mathbf{K}_i - \mathbf{H}_i\mathbf{X}_i \\ \mathbf{0} & \mathbf{0} & \mathbf{K}_i' - \mathbf{X}_i'\mathbf{H}_i' & \sigma_{i,i}^{-1}\mathbf{X}_i'\mathbf{X}_i \end{bmatrix} + \boldsymbol{i}_{\mp}(\sigma_{i,i}\mathbf{I}_n) - n \\
& \quad + \boldsymbol{i}_{\pm}\begin{bmatrix} \sigma_{i,i}\mathbf{I}_n & \mathbf{X}_i \\ \mathbf{X}_i' & \mathbf{0} \end{bmatrix} - \boldsymbol{r}(\mathbf{X}_i) - \boldsymbol{r}\,[\,\mathbf{X}, \boldsymbol{\Sigma}\otimes\mathbf{I}_n\,] + \boldsymbol{i}_{\mp}\begin{bmatrix} \boldsymbol{\Sigma}\otimes\mathbf{I}_n & \mathbf{X} \\ \mathbf{X}' & \mathbf{0} \end{bmatrix} - \boldsymbol{r}(\mathbf{X}).
\end{aligned}
$$

In consequence, by using (2.16), (2.17) and Lemma 2.3, the required results are obtained. □

5. Concluding remarks

We have presented a comprehensive investigation of the comparison problem of predictors under SULMMs by making use of formulas of inertias and ranks of matrices which are effective algebraic tools in matrix theory. In particular, we have established a variety of equalities and inequalities on the comparison of BLUPs of joint unknown vectors in SULMMs and their combined model under most general assumptions. Although $\mathcal{M}_i$ and their combined model $\mathcal{M}$ can be considered simultaneously or separately for prediction of joint unknown parameters, it is worth considering certain links and comparisons among predictors/estimators under these models since there may be connections between inference results obtained from all these models. We have also applied some of the results to SURMs, which are a special form of SULMMs. The results obtained in this paper are general and we believe that they have provided useful aspects in the theoretical point of view for describing performances of BLUPs of joint unknown vectors under SULMMs and their combined model, and also under SURMs.

Acknowledgments. The authors would like to thank the anonymous referees for their careful reading of the paper and their valuable remarks.

References

[1] *I. S. Alalouf, G. P. H. Styan*: Characterizations of estimability in the general linear model. Ann. Stat. *7* (1979), 194–200. zbl MR doi
[2] *B. Arendacká, S. Puntanen*: Further remarks on the connection between fixed linear model and mixed linear model. Stat. Pap. *56* (2015), 1235–1247. zbl MR doi
[3] *J. K. Baksalary, R. Kala*: On the prediction problem in the seemingly unrelated regression equations model. Math. Operationsforsch. Stat., Ser. Stat. *10* (1979), 203–208. zbl MR doi
[4] *R. Bartels, D. G. Fiebig*: A simple characterization of seemingly unrelated regressions models in which OLS is BLUE. Am. Stat. *45* (1991), 137–140. MR doi
[5] *H. Brown, R. Prescott*: Applied Mixed Models in Medicine. Statistics in Practice. John Wiley & Sons, Hoboken, 2006. zbl doi
[6] *E. Demidenko*: Mixed Models: Theory and Applications. Wiley Series in Probability and Statistics. John Wiley & Sons, New York, 2004. zbl MR doi
[7] *T. D. Dwivedi, V. K. Srivastava*: Optimality of least squares in the seemingly unrelated regression equation model. J. Econom. *7* (1978), 391–395. zbl MR doi
[8] *A. S. Goldberger*: Best linear unbiased prediction in the generalized linear regression model. J. Am. Stat. Assoc. *57* (1962), 369–375. zbl MR doi
[9] *L. Gong*: Establishing equalities of OLSEs and BLUEs under seemingly unrelated regression models. J. Stat. Theory Pract. *13* (2019), Article ID 5, 10 pages. zbl MR doi
[10] *N. Güler*: On relations between BLUPs under two transformed linear random-effects models. To appear in Commun. Stat., Simulation Comput. doi
[11] *N. Güler, M. E. Büyükkaya*: Notes on comparison of covariance matrices of BLUPs under linear random-effects model with its two subsample models. Iran. J. Sci. Technol., Trans. A, Sci. *43* (2019), 2993–3002. MR doi

[12] *N. Güler, M. E. Büyükkaya*: Rank and inertia formulas for covariance matrices of BLUPs in general linear mixed models. Commun. Stat., Theory Methods *50* (2021), 4997–5012. MR doi
[13] *S. J. Haslett, S. Puntanen*: On the equality of the BLUPs under two linear mixed models. Metrika *74* (2011), 381–395. zbl MR doi
[14] *S. J. Haslett, S. Puntanen, B. Arendacká*: The link between the mixed and fixed linear models revisited. Stat. Pap. *56* (2015), 849–861. zbl MR doi
[15] *J. Hou, Y. Zhao*: Some remarks on a pair of seemingly unrelated regression models. Open Math. *17* (2019), 979–989. zbl MR doi
[16] *J. Jiang*: Linear and Generalized Linear Mixed Models and Their Applications. Springer Series in Statistics. Springer, New York, 2007. zbl MR doi
[17] *H. Jiang, J. Qian, Y. Sun*: Best linear unbiased predictors and estimators under a pair of constrained seemingly unrelated regression models. Stat. Probab. Lett. *158* (2020), Article ID 108669, 7 pages. zbl MR doi
[18] *X.-Q. Liu, J.-Y. Rong, X.-Y. Liu*: Best linear unbiased prediction for linear combinations in general mixed linear models. J. Multivariate Anal. *99* (2008), 1503–1517. zbl MR doi
[19] *X. Liu, Q.-W. Wang*: Equality of the BLUPs under the mixed linear model when random components and errors are correlated. J. Multivariate Anal. *116* (2013), 297–309. zbl MR doi
[20] *S. Puntanen, G. P. H. Styan, J. Isotalo*: Matrix Tricks for Linear Statistical Models: Our Personal Top Twenty. Springer, Berlin, 2011. zbl MR doi
[21] *C. R. Rao*: Representations of best linear unbiased estimators in the Gauss-Markoff model with a singular dispersion matrix. J. Multivariate Anal. *3* (1973), 276–292. zbl MR doi
[22] *S. R. Searle*: The matrix handling of BLUE and BLUP in the mixed linear model. Linear Algebra Appl. *264* (1997), 291–311. zbl MR doi
[23] *V. K. Srivastava, D. E. A. Giles*: Seemingly Unrelated Regression Equations Models: Estimation and Inference. Statistics: Textbooks and Monographs 80. Marcel Dekker, New York, 1987. zbl MR doi
[24] *Y. Sun, R. Ke, Y. Tian*: Some overall properties of seemingly unrelated regression models. AStA, Adv. Stat. Anal. *98* (2014), 103–120. zbl MR
[25] *Y. Tian*: Equalities and inequalities for inertias of Hermitian matrices with applications. Linear Algebra Appl. *433* (2010), 263–296. zbl MR doi
[26] *Y. Tian*: Solving optimization problems on ranks and inertias of some constrained nonlinear matrix functions via an algebraic linearization method. Nonlinear Anal., Theory Methods Appl., Ser. A *75* (2012), 717–734. zbl MR doi
[27] *Y. Tian*: A new derivation of BLUPs under random-effects model. Metrika *78* (2015), 905–918. zbl MR doi
[28] *Y. Tian*: Matrix rank and inertia formulas in the analysis of general linear models. Open Math. *15* (2017), 126–150. zbl MR doi
[29] *Y. Tian*: Some equalities and inequalities for covariance matrices of estimators under linear model. Stat. Pap. *58* (2017), 467–484. zbl MR doi
[30] *Y. Tian, W. Guo*: On comparison of dispersion matrices of estimators under a constrained linear model. Stat. Methods Appl. *25* (2016), 623–649. zbl MR doi
[31] *Y. Tian, B. Jiang*: Matrix rank/inertia formulas for least-squares solutions with statistical applications. Spec. Matrices *4* (2016), 130–140. zbl MR doi
[32] *Y. Tian, J. Wang*: Some remarks on fundamental formulas and facts in the statistical analysis of a constrained general linear model. Commun. Stat., Theory Methods *49* (2020), 1201–1216. MR doi
[33] *Y. Tian, P. Xie*: Simultaneous optimal predictions under two seemingly unrelated linear random-effects models. To appear in J. Ind. Manag. Optim. doi
[34] *G. Verbeke, G. Molenberghs*: Linear Mixed Models for Longitudinal Data. Springer Series in Statistics. Springer, New York, 2000. zbl MR doi

[35] *Q.-W. Wang, X. Liu*: The equalities of BLUPs for linear combinations under two general linear mixed models. Commun. Stat., Theory Methods *42* (2013), 3528–3543. zbl MR doi
[36] *A. Zellner*: An efficient method of estimating seemingly unrelated regressions and tests for aggregation bias. J. Am. Stat. Assoc. *57* (1962), 348–368. zbl MR doi
[37] *A. Zellner, D. S. Huang*: Further properties of efficient estimators for seemingly unrelated regression equations. Int. Econ. Rev. *3* (1962), 300–313. zbl doi

Authors' addresses: *Nesrin Güler* (corresponding author), Department of Econometrics, Sakarya University, TR-54187, Sakarya, Turkey, e-mail: `nesring@sakarya.edu.tr`; *Melek Eriş Büyükkaya*, Department of Statistics and Computer Sciences, Karadeniz Technical University, TR-61080 Trabzon, Turkey, e-mail: `melekeris@ktu.edu.tr`.