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EXTENSION OF SEMICLEAN RINGS

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Abstract. This paper aims at the study of the notions of periodic, UU and semiclean properties in various context of commutative rings such as trivial ring extensions, amalgamations and pullbacks. The results obtained provide new original classes of rings subject to various ring theoretic properties.

Keywords: amalgamated algebra; nil-clean ring; periodic ring; pullback; UU ring; semiclean ring

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1. Introduction

Throughout the whole paper, we assume that all rings are commutative with unity. For the convenience of the reader, we fix notation for the rest of the paper. For a ring A, we denote by $U(A)$, $\mathrm{Idem}(A)$, $\mathrm{Nilp}(A)$, $\mathrm{Jac}(A)$ and $\mathrm{Per}(A)$, the multiplicative group of units of A, the set of all idempotent elements of A, the ideal of all nilpotent elements of A, the Jacobson radical of A and the set of all periodic elements of A. An element x of a ring A is *nilpotent* if there exists a positive integer $n \geqslant 1$ such that $x^n = 0$, an element a of a ring A is *periodic* if there exist distinct positive integers $m, n > 0$ such that $a^m = a^n$. Note that every idempotent, potent and nilpotent element is periodic. It is well known that a ring A is periodic if each $x \in A$ satisfies the Chacron criterion (see [7]), that is, there exist a positive integer m and $P(T) \in \mathbb{Z}[T]$ such that $x^m = x^{m+1}P(x)$. It follows that A is periodic if for each $x \in A$, there exist distinct positive integers m, n for which $x^n - x^m \in \mathrm{Nilp}(A)$. Recall that a ring A is UU (i.e., every Unit is Unipotent) if every unit $u \in A$ is unipotent, that is, u can be expressed as $u = 1 + n$ for some nilpotent element n of A. Let A be a ring and M be an A-module. The trivial ring extension of A by M

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(also called the *idealization* of M over A) is the ring $R = A \propto E$ whose underlying group is $A \times M$ with multiplication given by $(a, m)(b, n) = (ab, an + bm)$ for each $a, b \in A$ and $m, n \in M$. Let (A, B) be a pair of rings, J be an ideal of B and $f\colon A \to B$ be a ring homomorphism. In this setting, D'Anna, Finocchiaro, and Fontana introduced in [9], the subring of $A \times B$:

$$A \bowtie^f J := \{(a, f(a) + j)\colon\ a \in A,\ j \in J\},$$

called the *amalgamation* of A with B along J with respect to f. For more details about trivial ring extension and amalgamation, we refer the reader to [5], [14], [16], [17], [18].

In [20], Nicholson introduced the clean ring (that is a ring in which every element is a sum of a unit and an idempotent). It was proved that an element $a \in A$ is clean if and only if $1 - a$ is clean. In [23], Ye introduced the concept of a semiclean ring (that is a ring in which every element is a sum of a unit and a periodic element) as a generalization of the notion of clean ring. In [10], Diesl introduced a new class of rings and called it the *nil-clean ring* (that is a ring in which every element can be expressed as a sum of an idempotent and a nilpotent). Every nil-clean element is clean and every clean element is semiclean. Notice that the nil-clean ring is a periodic ring, furthermore, it was proved in [21], Theorem 3 (1) that a ring A is periodic if and only if each element x of A can be expressed in the form $x = p + n$, where n is a nilpotent element and $p^k = p$ for some positive integer $k \geqslant 2$. Han and Nicholson showed in [11] that the group ring $\mathbb{Z}_{(7)}[C_3]$ is not clean, where $\mathbb{Z}_{(7)} = \{m/n\colon\ m, n \in \mathbb{Z}$ and $\gcd(7, n) = 1\}$ and C_3 is a cyclic group of order 3, while Ye proved in [23], that for every prime number p and for every cyclic group C_3 of order 3, the group ring $\mathbb{Z}_{(p)}[C_3]$ is semiclean, where $\mathbb{Z}_{(p)} = \{m/n\colon\ m, n \in \mathbb{Z}$ and $\gcd(p, n) = 1\}$.

The following diagram of the implications summarizes the relations between the main notions involved in this paper

$$\begin{tikzcd}
 & & \text{+UU} \arrow[dll, Rightarrow] \arrow[dr, equal] & \\
\text{Nil-clean} \arrow[r, Rightarrow] \arrow[dr, Rightarrow] & \text{Periodic} \arrow[r, Rightarrow] & \text{Clean} \arrow[r, Rightarrow] & \text{Semiclean} \\
\text{+Semiclean} \arrow[u, Rightarrow] \arrow[r, equal] & \text{UU} & &
\end{tikzcd}$$

Notice that the above implications are not reversible in general. This paper studies the notions of periodic, UU and semiclean properties in various context of commutative rings such as trivial ring extensions, amalgamations and pullbacks. The obtained results provide new original classes of rings subject to various ring theoretic properties.

2. On periodic rings

Recall that a ring A is periodic provided for every element a of a ring A there exist distinct positive integers $m, n > 0$ such that $a^m = a^n$. It was proved in [21], Theorem 3 (1), that a ring A is periodic if and only if each element x of A can be expressed in the form $x = p + n$, where n is a nilpotent element and $p^k = p$ for some positive integer $k \geqslant 2$. Our first proposition examines the perodic property of a homomorphic image.

Proposition 2.1. *Let R be a ring and I be an ideal of R. If R is a periodic ring, then R/I is a periodic ring. The converse holds if I is a nil ideal of R.*

P r o o f. Assume that a ring R is periodic and let $x \in R$, let $f\colon R \to B$ be a ring homomorphism, where B is a commutative ring. Then, there exist positive integers $m, n > 0$ such that $x^m = x^n$ and so $f(x)^m = f(x^m) = f(x^n) = f(x)^n$, then $f(x)$ is a periodic element in B. Conversely, assume that I is a nil ideal of R. Let $\overline{x} \in R/I$ such that $\overline{x - x^n} \in \mathrm{Nilp}(R/I)$ for some positive integer $n \geqslant 1$ and $\bar{x} \in R/I$. Then $x - x^n + I \in \mathrm{Nilp}(R/I)$. Since I is a nil ideal of R, then it follows that $x - x^n \in \mathrm{Nilp}(R)$. Hence, R is a periodic ring. $\square$

Remark 2.2. It is worthwhile noting that the condition "I is a nil ideal of R" in Proposition 2.1 is essential for the converse. Indeed, let A be any periodic ring. It is well known that $A[X]$ is never periodic. However, $(A[X]/(X)) \simeq A$ and A is periodic.

Recall that a commutative ring R is said to be π-regular if for every element $x \in R$, there exist a positive integer n and $y \in R$ such that $x^n = x^{n+1}y$. It is worthwhile noting that the properties "Krull dimension 0" and "π-regular" are equivalent, see [13], Theorem 3.1, page 10. Also, recall from [4], Corollary 3.4 that a commutative ring R is periodic if and only if R is zero-dimensional and $U(R)$ is torsion. Our next proposition establishes a characterization of the relationship between periodic rings, rings of Krull dimension 0.

Proposition 2.3. *Let R be a ring. Then the following assertions are equivalent:*

(1) *R is a periodic ring,*

(2) *R is a ring of Krull dimension 0 satisfying every unit u of R can be expressed as $u = f + n$ with $f^k = f \in R$ for some positive integer $k \geqslant 2$ and $n \in \mathrm{Nilp}(R)$.*

P r o o f. (1) $\Rightarrow$ (2) Assume that R is a periodic ring. Then it is well known that R is zero-dimensional. Next, let $u \in U(R)$. From [21], Theorem 3 (1) it follows that $u = f + n$ with $f^k = f \in R$ for some positive integer $k \geqslant 2$ and $n \in \mathrm{Nilp}(R)$.

(2) $\Rightarrow$ (1) Assume that R is a ring of Krull dimension 0 satisfying every unit u of R can be expressed as $u = f + n$ with $f^k = f \in R$ for some positive integer $k \geqslant 2$ and $n \in \mathrm{Nilp}(R)$. Then R is π-regular. Let $a \in R$. From [3], Corollary 1, $a = eu + w$ for some $e \in \mathrm{Idem}(R)$, $u \in U(R)$ and $w \in \mathrm{Nilp}(R)$. From the assumption, $u = f + n$ with $f^k = f \in R$ for some positive integer $k \geqslant 2$ and $n \in \mathrm{Nilp}(R)$. Therefore, $f = u - n \in U(R)$ and so $f^{k-1} = 1$. It easily follows that the group $U(R/\mathrm{Nilp}(R))$ is a torsion. By [4], Corollary 3.4, the ring $R/\mathrm{Nilp}(R)$ is periodic, and so is R by Proposition 2.1. $\square$

Now, we introduce the notion of almost Boolean ring.

Definition 2.4. A ring R is said to be almost Boolean if for every $x \in R$ there exists an integer $n \geqslant 2$ such that $x^n = x$.

Notice that from the previous definition, every Boolean ring is almost Boolean. However, the converse is not true in general. For instance, $R := \mathbb{Z}/6\mathbb{Z}$ is almost Boolean (as $x^3 = x$ for all $x \in R$) which is not Boolean. The next proposition establishes a characterization of periodic rings.

Proposition 2.5. *Let R be a ring. Then the following assertions are equivalent:*
(1) *R is periodic,*
(2) *$R/\mathrm{Jac}(R)$ is almost Boolean and $\mathrm{Jac}(R)$ is a nil ideal of R.*

Proof. (1) $\Rightarrow$ (2) Assume that R is periodic. Let $\bar{x} \in R/\mathrm{Jac}(R)$. Then $\bar{x} = x + \mathrm{Jac}(R)$ such that $x \in R$. The fact that R is periodic gives $x - x^n \in \mathrm{Nilp}(R)$ for some positive integer $n \geqslant 2$. Since $\mathrm{Nilp}(R) \subseteq \mathrm{Jac}(R)$, then it follows that $\overline{x - x^n} = \bar{0}$. Therefore, $\bar{x}^n = \bar{x}$ for some positive integer $n \geqslant 2$. Consequently, $R/\mathrm{Jac}(R)$ is almost Boolean. The fact that $\mathrm{Jac}(R)$ is a nil ideal of R follows from [6], (P-1), the last line of the first page, as R is periodic.

(2) $\Rightarrow$ (1) Assume that $R/\mathrm{Jac}(R)$ is almost Boolean and $\mathrm{Jac}(R)$ is a nil ideal of R. We claim that R is periodic. Indeed, let $x \in R$. Then $\bar{x} \in R/\mathrm{Jac}(R)$ which is almost Boolean. So, there exists an integer $n \geqslant 2$ such that $\bar{x}^n = \bar{x}$. Therefore, $x - x^n \in \mathrm{Jac}(R) \subseteq \mathrm{Nilp}(R)$. Hence, R is periodic as desired. $\square$

A ring R is called a *torsion* if it has positive characteristic, that is to say, if $l \neq 0 \in R$ for some positive integer $l > 0$ (not necessarily prime). In particular, in a torsion ring, the set of periodic elements is closed under addition. Also, notice that a periodic ring is a torsion. The next theorem gives a characterization of the periodic property in an amalgamated algebra and is equivalent to [15], Theorem 2.6 (except for the condition (3)).

Theorem 2.6. *Let (A, B) be a pair of rings, J be an ideal of B and $f\colon A \to B$ be a ring homomorphism. Then the following assertions are equivalent:*

(1) *$A \bowtie^f J$ is periodic,*
(2) *A and $f(A) + J$ are periodic,*
(3) *$A \times (f(A) + J)$ is periodic,*
(4) *A is periodic and $J \subseteq \mathrm{Per}(B)$.*

The proof of this theorem requires the following lemma.

Lemma 2.7.
(1) *Any subring of a periodic ring is periodic.*
(2) *The ring $A = A_1 \times A_2$ is periodic if and only if so are A_1 and A_2.*

Proof. (1) Let $A \subseteq B$ be two rings, B be periodic and let $x \in A$. Then x is periodic in B and so x is periodic in A. Hence, A is periodic.

(2) In view of [2], Lemma 2.4 for $N = 2$, we obtain the result. □

Proof of Theorem 2.6. (1) ⇔ (2) Assume that $A \bowtie^f J$ is periodic. From [9], Proposition 5.1, A and $f(A) + J$ are homomorphic images of $A \bowtie^f J$. Then by Proposition 2.1, A and $f(A)+J$ are periodic. Conversely, assume that A and $f(A)+J$ are periodic, then $A \times (f(A) + J)$ is periodic by Lemma 2.7 (2). Since $A \bowtie^f J$ is a subring of $A \times (f(A) + J)$, then $A \bowtie^f J$ is periodic by Lemma 2.7 (1).

(2) ⇔ (3) This follows from Lemma 2.7 (2).

(2) ⇒ (4) Assume that A and $f(A) + J$ are periodic. Since $J \subseteq f(A) + J$, $J \subseteq \mathrm{Per}(B)$.

(4) ⇒ (1) This follows from the proof of (3) ⇒ (1) of [15], Theorem 2.6. □

The next corollary is a consequence of Theorem 2.6 and Proposition 2.5.

Corollary 2.8. *Let (A, B) be a pair of rings, $f\colon A \to B$ be a ring homomorphism and J be an ideal of B such that $J \subseteq \mathrm{Jac}(B)$. Then the following assertions are equivalent:*
(1) *$A \bowtie^f J$ is periodic,*
(2) *$A/\mathrm{Jac}(A)$ is almost Boolean, $\mathrm{Jac}(A)$ is nil ideal.*

Proof. This follows from the statement that $A \bowtie^f J$ is periodic if and only if A is periodic and $J \subseteq \mathrm{Per}(B)$ (see Theorem 2.6), using Propositions 2.1 and 2.5 with the fact that a subideal J of $\mathrm{Jac}(B)$ is nil if and only if $J \subseteq \mathrm{Per}(B)$. □

Corollary 2.8 provides a new original class of periodic rings that are not nil-clean.

Example 2.9. Let $A := \mathbb{Z}_6$ be a ring, E be an A-module and $B := A \propto E$ be the trivial ring extension of A by E, $f\colon A \to B$ be the canonical injection defined by $f(a) = (a, 0)$ and $J = 0 \propto E$ be an ideal of B. Then:
(1) $A \bowtie^f J$ is periodic,
(2) $A \bowtie^f J$ is not nil-clean.

P r o o f. (1) First, observe that $\mathrm{Jac}(A) = \mathbb{Z}_3 \cap \mathbb{Z}_5 = 0$ and so $A/\mathrm{Jac}(A) = \mathbb{Z}_6$ which is almost Boolean, $\mathrm{Jac}(A) = 0 = \mathrm{Nilp}(A)$ (as A is reduced). On the other hand, $J = 0 \propto E \subseteq \mathrm{Jac}(B) = \mathrm{Jac}(A) \propto E$. Hence, by Corollary 2.8, $A \bowtie^f J$ is periodic.

(2) By [12], Theorem 3, $A \bowtie^f J$ is not nil-clean since $(\bar{2},\bar{2}) - (\bar{2},\bar{2})^2 = (\bar{4},\bar{4}) \notin \mathrm{Nilp}(A \bowtie^f J)$, as $\bar{4} \notin \mathrm{Nilp}(A_1)$. Hence, $A \bowtie^f J$ is not nil-clean. $\square$

Let I be a proper ideal of A. The (*amalgamated*) *duplication* of A along I is a special amalgamation given by

$$A \bowtie I := A \bowtie^{\mathrm{id}_A} I = \{(a, a+i)\colon\ a \in A,\, i \in I\}.$$

The next corollary is an immediate consequence of Theorem 2.6 on the transfer of periodic property to duplications and is equivalent to [15], Corollary 2.8.

Corollary 2.10. *Let A be a ring and I be an ideal of A. Then $A \bowtie I$ is periodic if and only if so is A.*

Theorem 2.6 recovers a known result for trivial ring extensions which is given in [15], Theorem 2.1.

Corollary 2.11. *Let A be a ring, E be an A-module and $R := A \propto E$ be the trivial ring extension of A by E. Then R is periodic if and only if so is A.*

P r o o f. Consider the injective ring homomorphism $f\colon A \hookrightarrow B$ defined by $f(a) = (a, 0)$ for every $a \in A$, let $J := 0 \propto E$ be an ideal of B. Clearly, $f^{-1}(J) = 0$. Therefore, by [9], Proposition 5.1 (3), $f(A) + J = A \propto 0 + 0 \propto E = A \propto E = B \simeq A \bowtie^f J$. On the other hand, $J := 0 \propto E \subseteq \mathrm{Nilp}(B)$ and so by application to Theorem 2.6, we have the desired result. $\square$

Now we show how one may use Corollary 2.10 to generate a new class of periodic rings that are not nil-clean.

Example 2.12. Let A be a finite ring (for instance, take $A := \mathbb{Z}_{12}$) and $I := 6\mathbb{Z}_{12}$ be an ideal of A. Then:

(1) $A \bowtie I$ is periodic,

(2) $A \bowtie I$ is not nil-clean.

P r o o f. (1) By Corollary 2.10, $A \bowtie I$ is periodic since A is periodic.

(2) From [12], Theorem 3, $A \bowtie I$ is not nil-clean since $((\bar{2},\bar{2}) - ((\bar{2},\bar{2})^2 = (\overline{10},\overline{10}) \notin \mathrm{Nilp}A \bowtie I)$, as $\overline{10} \notin \mathrm{Nilp}(A)$. Hence, $A \bowtie I$ is not nil-clean. $\square$

3. On UU and semiclean rings

It is worthwhile recalling that every clean ring is semiclean. However, the converse is not true in general. In [23], Theorem 5.1, Ye gave a condition to have this converse. Our first proposition of this section establishes the relationship between the notions of nil-clean, UU and semiclean rings.

Proposition 3.1. *Let A be a nil-clean ring. Then the following assertions hold:*
(1) *A is clean, in particular, A is semiclean,*
(2) *A is periodic.*

Proof. (1) By [10], Proposition 3.4, A is clean. Hence, it follows that A is semiclean.

(2) This follows from [21], Theorem 3 (1). $\square$

The following example shows that clean and periodic rings need not be UU.

Example 3.2. Let $A := \mathbb{Z}/6\mathbb{Z}$ be the ring of integers modulo 6. Then:
(1) A is clean,
(2) A is periodic,
(3) A is not UU.

Proof. As a finite ring, A is clean and periodic. However, $\bar{5} = \bar{1} + \bar{4}$ and $\bar{4}$ is not nilpotent. Hence, A is not UU. $\square$

It is well known that for any ring A, the polynomial ring $A[X]$ is never nil-clean nor periodic. The next example shows that UU rings need not be nil-clean nor periodic in general. Recall that an element $f(X) = a_0 + a_1X + a_2X^2 + \ldots + a_nX^n \in A[X]$ is nilpotent if and only if each a_i is nilpotent for $i = 0, \ldots, n$. Also, recall that $f(X) = a_0 + a_1X + a_2X^2 + \ldots + a_nX^n$ is a unit of $A[X]$ if and only if a_0 is a unit of A and each a_i is a nilpotent element of A for $i = 1, \ldots, n$.

Example 3.3. Let $A := \mathbb{Z}/4\mathbb{Z}$ be the ring of integers modulo 4. Then:
(1) $A[X]$ is UU,
(2) $A[X]$ is not nil-clean,
(3) $A[X]$ is not periodic.

Proof. (1) Since A is UU, then it follows that $A[X]$ is UU.

(2) By [12], Theorem 3 $A[X]$ is not nil-clean since $X - X^2 \notin \mathrm{Nilp}(A[X])$.

(3) $A[X]$ is not periodic since $X \in A[X]$ and $X^m \neq X^n$ for all integers $m, n \geqslant 2$ with $m \neq n$. $\square$

The next theorem gives a new characterization of nil-clean rings.

Theorem 3.4. *Let A be a ring. Then the following conditions are equivalent:*
(1) *A is nil-clean.*
(2) *For each $a \in A$, we have $a = e + u$, where $e \in \mathrm{Idem}(A)$ and $u \in \mathrm{UU}(A)$.*
(3) *A is clean and* UU.
(4) *A is semiclean and* UU.

P r o o f. (1) $\Leftrightarrow$ (2) Assume that A is nil-clean. Let $a \in A$. Using a similar argument as in the proof of [10], Proposition 3.4 it follows that $a = e + 1 + n$ with $e \in \mathrm{Idem}(A)$ and $n \in \mathrm{Nilp}(A)$. Hence, $a = e+u$ with $u = 1+n \in \mathrm{UU}(A)$. Conversely, assume that for each $a \in A$, we have $a = e + u$, where $e \in \mathrm{Idem}(A)$ and $u \in \mathrm{UU}(A)$. Let $a \in A$. Then $a+1 = e+u$ with $u \in \mathrm{UU}(A)$ and $e \in \mathrm{Idem}(A)$. So, $a+1 = e+1+n$ with $n \in \mathrm{Nilp}(A)$. Therefore, $a = e + n$ which is nil-clean. Finally, A is nil-clean, as desired.

(1) $\Rightarrow$ (3) Assume that A is nil-clean. Then by Proposition 3.1, A is semiclean. We claim that A is UU. Indeed, let $u \in U(A)$. Then $u = e + n$, where $e \in \mathrm{Idem}(A)$ and $n \in \mathrm{Nilp}(A)$. So, $u - n = e$ is both a unit and idempotent. Therefore, $u - n = 1$ and so $u = 1 + n$. Hence, A is UU.

(3) $\Rightarrow$ (4) Trivial.

(4) $\Rightarrow$ (1) Assume that A is semiclean and UU. Let $a \in A$. Then $a + 2 = p + u$ for some $u \in U(A)$ and $p \in \mathrm{Per}(A)$. The fact that every periodic element is clean gives $p = v+e$ for some $v \in U(A)$ and $e \in \mathrm{Idem}(A)$. So, $a+2 = u+v+e = 1+n+1+m+e = (2 + n + m) + e$, where $m, n \in \mathrm{Nilp}(A)$. Consequently, $a = (n + m) + e$, which is nil-clean. Hence, A is nil-clean, as desired. □

The next theorem examines the semiclean property in indecomposable rings. Recall that a ring A is indecomposable if $\mathrm{Idem}(A) = \{0, 1\}$.

Theorem 3.5. *Let A be an indecomposable ring. Then every semiclean element of A is either a unit or a sum of a unit and a root of unity.*

P r o o f. Assume that A is an indecomposable ring and let $x \in A$ be a semiclean element. Then there exist $u \in U(A)$ and $p \in \mathrm{Per}(A)$ such that $x = u + p$. From [23] Lemma 5.2, there exists a positive integer $k \geqslant 1$ such that $p^k \in \mathrm{Idem}(A) = \{0, 1\}$ since A is indecomposable. So, $p^k = 0$ or $p^k = 1$. Thus, p is either a nilpotent or a root of unity. Two cases are then possible:

Case 1: p is nilpotent. Then $x = u + p$ is a unit.

Case 2: p is root of unity. Then p is a unit and therefore $x = u + p$ is sum of two units.

Hence, it follows that x is either a unit or a sum of a unit and a root of unity. □

The following corollary is an immediate consequence of Theorem 3.5.

Corollary 3.6. *Let A be an integral domain. Then every semiclean element is either a unit or sum of two units.*

Now, we give a characterization for the amalgamation ring to inherit the UU property.

Theorem 3.7. *Let (A, B) be a pair of rings, J be an ideal of B and $f\colon A \to B$ be a ring homomorphism. Set $J_0 := \{j \in J\colon f(u) + j \in U(B)$ for some $u \in U(A)\}$. Then the following assertions hold:*
(1) *$A \bowtie^f J$ is* UU *if and only if A is* UU *and $J_0 \subseteq \mathrm{Nilp}(B)$.*
(2) *If A and $f(A) + J$ are* UU*, then so is $A \bowtie^f J$.*

Proof. (1) Assume that $A \bowtie^f J$ is UU. We claim that A is UU. Let $u \in U(A)$. Then $(u, f(u)) \in U(A \bowtie^f J)$ which is UU. So, $(u, f(u)) = (1, 1) + (n, f(n) + k) = (1 + n, 1 + f(n) + k)$ with $(n, f(n) + k) \in \mathrm{Nilp}(A \bowtie^f J)$. By [8], Lemma 2.10 $n \in \mathrm{Nilp}(A)$ and $k \in \mathrm{Nilp}(B) \cap J$. It follows that $u = 1 + n$, which is an UU element of A. Hence, A is UU. It remains to show that $J_0 \subseteq \mathrm{Nilp}(B)$. Let $k \in J_0$. Then $(v, f(v) + k) \in U(A \bowtie^f J)$ for some $v \in U(A)$. So, $v = 1 + n$ for some $n \in \mathrm{Nilp}(A)$, and $f(v)+k = 1+f(x)+m$ for some $x \in A$ and $m \in J$ such that $f(x)+m \in \mathrm{Nilp}(B)$. First, we have $f(v) = f(1 + n) = 1 + f(n)$. Then $1 + f(n) + k = 1 + f(x) + m$. Thus, $f(n) + k = f(x) + m$. Hence, $(v, f(v) + k) = (1, 1) + (n, f(x) + m)$ with $(n, f(x) + m) \in \mathrm{Nilp}(A \times B)$. Since $f(n) + k = f(x) + m$, then $(v, f(v) + k) = (1, 1) + (n, f(n) + k)$ and $(n, f(n) + k) \in \mathrm{Nilp}(A \bowtie^f J)$. By [8], Lemma 2.10 it follows that $k \in \mathrm{Nilp}(B)$. Conversely, assume that A is UU and $J_0 \subseteq \mathrm{Nilp}(B)$. Let $(u, f(u) + j) \in U(A \bowtie^f J)$. Then u is a unit of A and $j \in J_0 \subseteq \mathrm{Nilp}(B)$. So, $u = 1 + n$ for some $n \in \mathrm{Nilp}(A)$. Hence, $(u, f(u) + j) = (1, 1) + (n, f(n) + j)$. One can easily check that $(n, f(n)+j) \in \mathrm{Nilp}(A \bowtie^f J)$, making $(u, f(u)+j)$ a unipotent element of $A \bowtie^f J$. Thus, $A \bowtie^f J$ is UU.

(2) Assume that A and $f(A)+J$ are UU. Let $x \in A$, $j \in J$ such that $(x, f(x)+j) \in U(A \bowtie^f J)$. Then $x \in U(A)$ and $f(x) + j \in U(f(A) + J)$. So,

(i) $x = 1 + b$ for some $b \in \mathrm{Nilp}(A)$,

(ii) $f(x) + j = 1 + b'$ for some $b' \in \mathrm{Nilp}(f(A) + J)$.

From (i), it follows that $f(x) = f(1 + b) = 1 + f(b)$. Substituting (i) into (ii) we get $1 + f(b) + j = 1 + b'$. Hence, $f(b) + j = b'$. Consequently, $(x, f(x) + j) = (1 + b, 1 + b') = (1, 1) + (b, b') = (1, 1) + (b, f(b) + j)$ which is a unipotent element of $A \bowtie^f J$. Hence, $A \bowtie^f J$ is UU, as desired. □

The next result is a consequence of Theorem 3.7 for the transfer of UU property to a trivial ring extension.

Corollary 3.8. *Let A be a ring, E be an A-module and $R := A \propto E$ be the trivial ring extension of A by E. Then R is* UU *if and only if so is A.*

Proof. Set $B := A \propto E$, $f\colon A \hookrightarrow B$ is the injective ring homomorphism defined by $f(a) = (a, 0)$ and $J := 0 \propto E$ is an ideal of B. Observe that $J = 0 \propto E \subseteq \mathrm{Nilp}(B)$ and so $J_0 := \{j \in J\colon f(u) + j \in U(B) \text{ for some } u \in U(A)\} \subseteq \mathrm{Nilp}(B)$. Clearly, $f(A) + J = A \propto 0 + 0 \propto E = B = A \propto E \simeq A \bowtie^f J$, as $f^{-1}(J) = 0$, see [9], Proposition 5.1 (3). Hence, by Theorem 3.7 (1), $A \bowtie^f J$ is UU if and only if so is A. Now, the result is straightforward. $\square$

Recall that a ring A is semiclean if every element of A can be written as the sum of a unit and a periodic element. Also, recall from [23], Proposition 2.1 that the semiclean property is stable under homomorphic image. In the next theorem, we investigate about when the amalgamation is semiclean.

Theorem 3.9. *Let (A, B) be a pair of rings, $f\colon A \to B$ be a ring homomorphism and J be an ideal of B. Then:*

(1) *If $A \bowtie^f J$ is semiclean, then so are A and $f(A) + J$.*

(2) *Assume that $f(u) + j \in U(B)$ for all $u \in U(A)$ and $j \in J$. Then $A \bowtie^f J$ is semiclean if and only if so is A.*

(3) *Assume that $f(p) + j \in \mathrm{Per}(B)$ for all $p \in \mathrm{Per}(A)$ and $j \in J$. Then $A \bowtie^f J$ is semiclean if and only if so is A.*

Proof. (1) If $A \bowtie^f J$ is semiclean, then so are A and $f(A) + J$ as homomorphic images of $A \bowtie^f J$, see [9], Proposition 5.1.

(2) If $A \bowtie^f J$ is semiclean then, by assertion (1) above, so is A. Conversely, assume that A is semiclean and let $(a, j) \in A \times J$. Then $a = u + x$ with $u \in U(A)$ and $x \in \mathrm{Per}(A)$. So, $(a, f(a) + j) = (u, f(u) + j) + (x, f(x))$. One can easily check that $(x, f(x)) \in \mathrm{Per}(A \bowtie^f J)$ as $x \in \mathrm{Per}(A)$. From assumption, it follows that $(u, f(u) + j)$ is a unit of $A \bowtie^f J$. Therefore, $(a, f(a) + j)$ is semiclean, as desired.

(3) Assume that A is semiclean and let $(a, j) \in A \times J$. Then $a = u + x$ with $u \in U(A)$ and $x \in \mathrm{Per}(A)$. Observe that $(a, f(a) + j) = (u, f(u)) + (x, f(x) + j)$ with $(u, f(u)) \in U(A \bowtie^f J)$. From the assumption, $f(x) + j \in \mathrm{Per}(B)$. Therefore, by Lemma 2.7, it follows that $(x, f(x) + j)$ is a periodic element of $A \bowtie^f J$. Hence, $(a, f(a) + j)$ is semiclean, making $A \bowtie^f J$, a semiclean ring, as desired. $\square$

The following example shows the failure of the statement of assertion (2) (or (3)) of Theorem 3.9 in the case the assumption "for all $u \in U(A)$ and $j \in J$, $f(u) + j \in U(B)$" (or "for all $p \in \mathrm{Per}(A)$ and $j \in J$, $f(p) + j \in \mathrm{Per}(B)$", respectively) is not satisfied.

Example 3.10. Let A be any semiclean ring (for instance, take $A := \mathbb{Z}_{11}[(\mathbb{Z}_3, +)]$, a group ring), $B := A[X]$ be the polynomial ring with coefficients in A, $J := XA[X]$ be an ideal of B and $f\colon A \hookrightarrow B$ be the natural injection. Then:

(1) $1 \in U(A)$ and $f(1) + X = 1 + X \notin U(B)$ (or $0 \in \mathrm{Per}(A)$ and $f(0) + X = X \notin \mathrm{Per}(B)$, respectively),

(2) A is semiclean,

(3) $A \bowtie^f J$ is not semiclean.

Proof. (1) Straightforward.

(2) By [23], Theorem 3.1 A is semiclean.

(3) We claim that $A \bowtie^f J$ is not semiclean. Indeed, $f(A)+J = A+XA[X] = A[X]$ is not semiclean by [23], Example 3.2. Then in virtue of Theorem 3.9 (1), $A \bowtie^f J$ is not semiclean. □

Theorem 3.9 (2) recovers a known result for trivial ring extensions which is Proposition 2.10 of [19].

Corollary 3.11. *Let A be a ring, E be an A-module and $R := A \propto E$ be the trivial ring extension of A by E. Then R is semiclean if and only if so is A.*

Proof. Consider the injective ring homomorphism $f\colon A \hookrightarrow R$ defined by $f(a) = (a, 0)$ for every $a \in A$, $J := 0 \propto E$ is the ideal of R. Clearly, $f^{-1}(J) = 0$. Therefore, from [9], Proposition 5.1 (3) $f(A) + J = A \propto 0 + 0 \propto E = A \propto E = R \simeq A \bowtie^f J$. On the other hand, one can easily check that for all $a \in U(A)$ and for all $j \in J$, $f(a) + j = (a, 0) + (0, e) = (a, e) \in U(R) = U(f(A) + J)$. Hence, by application to the assertion (2) of Theorem 3.9, the conclusion is straightforward. □

The following corollaries are consequences of Theorem 3.9.

Corollary 3.12. *Let $A \subset B$ be an extension of commutative rings and $X := \{X_1, X_2, \ldots, X_n\}$ be a finite set of indeterminates over B. Set $A + XB[[X]] := \{P \in B[[X]]\colon P(0) \in A\}$, the subring of the ring of formal power series $B[[X]]$. Then $A + XB[[X]]$ is semiclean if and only if A is semiclean.*

Proof. By [9], Example 2.5 $A + XB[[X]] \simeq A \bowtie^\sigma J$, where $\sigma\colon A \hookrightarrow B[[X]]$ is the natural embedding and $J = XB[[X]]$ is an ideal of $B[[X]]$. It is well known that $\mathrm{Jac}(B[[X]]) = \{h \in B[[X]]\colon h(0) \in \mathrm{Jac}(A)\}$. Clearly, $J \subseteq \mathrm{Jac}(B[[X]])$. Hence, by Theorem 3.9 (2), it follows that $A \bowtie^\sigma J$ is semiclean if and only if so is A. □

Corollary 3.13. *Let (A, B) be a pair of rings, $f\colon A \to B$ be a ring homomorphism and J be an ideal of B such that $J \subseteq \mathrm{Jac}(B)$. Then A is semiclean if and only if so is $A \bowtie^f J$.*

Proof. Assume that $J \subseteq \operatorname{Jac}(B)$ and let $u \in U(A)$ and $j \in J$. Then $f(u) + j = f(u)(1 + f(u^{-1})j) \in U(B)$. Therefore, by Theorem 3.9 (2), we have the desired result. $\square$

Now, we show how one may use Theorem 3.9 to construct a new original class of semiclean rings that are not clean.

Example 3.14. Let $M_{\mathbb{N}}(F)$ be the ring of $\mathbb{N} \times \mathbb{N}$ infinite matrices over a field F in which each column has finitely many nonzero entries. Let $A_1 = \{A = (a_{ij})_{i,j} \in M_{\mathbb{N}}(F)\colon$ there exists $n_A \in \mathbb{N}$ such that $a_{ij} = a_{i+1j+1}$ for every $i \geqslant n_A$, $j \geqslant 1\}$ with $F = \mathbb{F}_2(X)$ which is the field fraction of the set of all polynomials over $\mathbb{F}_2$, a field with two elements. Take $A := \{N \in A_1\colon N^4 = N$ and $NC = CN$ for all $C \in A_1\}$. Let $B := A \propto E$ be the trivial ring extension of A by E, $f\colon A \to B$ be the injective ring homomorphism defined by $f(a) = (a, 0)$ and $J = \operatorname{Jac}(B) = \operatorname{Jac}(A) \propto E$ be the Jacobson radical of B. Then:

(1) $A \bowtie^f J$ is semiclean,
(2) $A \bowtie^f J$ is not clean.

Proof. (1) First notice that for every $E \in A$, $E^4 = E$. So, for every $E \in A$, $E = I + (E - I)$ with $(E - I)^4 = (E - I)$. Therefore, A is semiclean. Hence, by Theorem 3.9 (2), $A \bowtie^f J$ is semiclean.

(2) $A \bowtie^f J$ is not clean. Indeed, using similar argument as in [22], Example 3.1 it follows that A_1 is not clean. Therefore, A is not clean. Hence, by [8], Proposition 2.1 (1) $A \bowtie^f J$ is not clean. $\square$

The next example provides a new original class of semiclean rings that are neither weakly clean nor Noetherian. Recall that weakly clean rings are closed under homomorphic images.

Example 3.15. Let $A := C(X, \mathbb{Z}_3[\sqrt{3}])$ be the ring of continuous functions, where X is P-space. From [2], A is semiclean that is not weakly clean. Let $E = A^{\infty}$ be an A-module and $R := A \propto E$ be the trivial ring extension of A by E. Then:

(1) R is semiclean,
(2) R is not weakly clean,
(3) R is not Noetherian.

Proof. (1) By Corollary 3.11, R is semiclean since A is semiclean.

(2) R is not weakly clean, since $A \simeq A \propto E/0 \propto E$ is not weakly clean, as the weakly clean property is stable under factor ring.

(3) R is not Notherian since E is not a finitely generated A-module. $\square$

4. Pullbacks

In this section we examine the transfer of periodic, UU and semiclean rings to different context of pullbacks. First, we recall the following definition.

Definition 4.1 ([9], Definition 4.1)**.** If $\alpha\colon A \to C$ and $\beta\colon B \to C$ are ring homomorphisms, the subring $D := \alpha \times_C \beta := \{(a,b) \in A \times B\colon \alpha(a) = \beta(b)\}$ of $A \times B$ is called the *pullback* (or *fiber product*) of α and β.

The fact that D is pullback can also be described by saying that the triplet (D, p_A, p_B) is a solution of the universal problem of rendering commutative the diagram built on α and β,

$$\begin{tikzcd} D \arrow[r, "p_A"] \arrow[d, "p_B"'] & A \arrow[d, "\alpha"] \\ B \arrow[r, "\beta"] & C \end{tikzcd}$$

where p_A (or p_B) is the restriction to $D := \alpha \times_C \beta$ of the projection of $A \times B$ onto A (or B, respectively).

Our first theorem examines the transfer of periodic and UU properties to pullback, defined above.

Theorem 4.2. *Under the above notation, the following assertions hold:*

(1) *D is periodic if and only if so are $p_A(D)$ and $p_B(D)$.*

(2) *Consider the following assertions:*

   (a) *$p_A(D)$ is periodic and* $\ker(\beta) = \{0\}$,

   (b) *$p_B(D)$ is periodic and* $\ker(\alpha) = \{0\}$.

If (a) *or* (b) *holds, then D is periodic.*

(3) *If $p_A(D)$ and $p_B(D)$ are* UU, *then so is D.*

P r o o f. (1) If D is periodic, then so are $P_A(D)$ and $P_B(D)$ as homomorphic images of D. Conversely, assume that $p_A(D)$ and $p_B(D)$ are periodic and let $(a,b) \in D$. Then $a \in \mathrm{Per}(A)$ and $b \in \mathrm{Per}(B)$. So, by Lemma 2.7 (2), $(a,b) \in \mathrm{Per}(A \times B)$. Since $D \subseteq A \times B$, then by Lemma 2.7 (1), D is periodic.

(2) Assume that (a) holds. We claim that D is periodic. Indeed, suppose that $p_A(D)$ is periodic and $\ker(\beta) = \{0\}$, and let $(a,b) \in D$. Then $\alpha(a) = \beta(b)$ and $a^m = a^n$ for some distinct positive integers m, n. So, $\alpha^m(a) = \beta^m(b) = \alpha^n(a) = \beta^n(b)$. Consequently, $\beta(b^m) = \beta(b^n)$. Hence, $b^m = b^n$ since $\ker(\beta) = \{0\}$. Finally, $(a,b)^m = (a,b)^n$, making D a periodic ring. Same argument for (b) holds.

(3) Assume that $p_A(D)$ and $p_B(D)$ are UU and let $(u,v) \in U(D)$. Then $\alpha(u) = \beta(v)$, $u = 1 + m$ and $b = 1 + n$ for some $m \in \mathrm{Nilp}(A)$ and $n \in \mathrm{Nilp}(B)$. Thus $1 + \alpha(m) = 1 + \beta(n)$ and so $\alpha(m) = \beta(n)$. Hence, $(u,v) = (1,1) + (m,n)$ is unipotent in D. Finally, D is UU, as desired. □

Recall that pullback can be defined as follows: Let T be a ring, M be a nonzero ideal of T, π be the natural surjection $\pi\colon T \to T/M$ and D be a subring of T/M. Then $R := \pi^{-1}(D)$ is a subring of T and M is a common ideal of R and T and $D = R/M$, R is called a *pullback ring* associated to the pullback diagram

$$\begin{tikzcd} R = \pi^{-1}(D) \arrow[r, "\pi/R"] \arrow[d, "i"'] & D = R/M \arrow[d, "j"] \\ T \arrow[r, "\pi"] & T/M \end{tikzcd}$$

where i and j are the natural injections.

We assume that $R \subset T$ and we refer to this as a diagram of type $\triangle$. Our next result investigates the transfer of semiclean property to the pullback of type $\triangle$.

Theorem 4.3. *For a diagram of type $\triangle$:*

(1) *If R is semiclean, then so is D.*

(2) *Assume that D is a field and $\mathrm{Per}(R) = \mathrm{Per}(T)$. If T is semiclean, then so is R.*

Proof. (1) This follows from [23], Proposition 2.1 as D is a homomorphic image of R.

(2) Suppose D is a field, $\mathrm{Per}(R) = \mathrm{Per}(T)$ and T is semiclean. We claim that R is semiclean. Indeed, consider an element $r \in R$. Clearly $r \in T$ which is semiclean. So, $r = u + x$ with $u \in U(T)$ and $x \in \mathrm{Per}(T)$. Notice that $u \notin M$ as $u \in U(T)$. On the other hand, $u = r - x \in R$ since $x \in \mathrm{Per}(R)$ and so $\pi(u) = v$ which is a unit of D. Consequently, $\pi^{-1}(u) = v^{-1}$. It follows that u is a unit of R. Hence, T is semiclean, as desired. $\square$

The next example illustrates Theorem 4.3.

Example 4.4. Let $R = \mathbb{Q}(\sqrt{3}) + X\mathbb{Q}(\sqrt{3})[[X]]$ be the power series ring with coefficients in the field $\mathbb{Q}(\sqrt{3})$ and X an indeterminate. Consider $T := R[[Y]]$ being the power series ring with coefficients in R. By [1], Proposition 2.2 (2) we have $\mathrm{Per}(T) = \mathrm{Per}(R)$ since $\mathrm{Per}(R) = \mathrm{Per}(\mathbb{Q}(\sqrt{3}))$. Consider the pullback

$$\begin{tikzcd} R = \pi^{-1}(D) \arrow[r, "\pi/R"] \arrow[d, "i"'] & D = \mathbb{Q}(\sqrt{3}) \arrow[d, "j"] \\ T = \mathbb{Q}(\sqrt{3})[[X]][[Y]] \arrow[r, "\pi"] & \mathbb{Q}(\sqrt{3})[[X]] \end{tikzcd}$$

By [23], Proposition 3.3 T is semiclean. Hence, by Theorem 4.3 (2), it follows that R is semiclean, as desired.

Recall that periodics of a ring R can be lifted modulo an ideal M of R if, for each $r \in R$ with $r^m - r^n \in M$, there exists $s \in R$ such that $s^m = s^n$ in R and $r - s \in M$, see [23], definition on page 5610. Now, we study the stability of the periodic and semiclean properties in the following special case of pullback.

Proposition 4.5. *Let $\varphi\colon R \to T$ be an injective ring homomorphism and Q be an ideal of R such that $QT = Q$.*

(1) *If R is semiclean then so is R/Q.*

(2) *Assume that $Q \subset \operatorname{Jac}(R)$ and periodic elements of R can be lifted modulo Q. Then R is semiclean if and only if so is R/Q.*

Proof. (1) Assume that R is semiclean. Then by [23], Proposition 2.1 R/Q is semiclean.

(2) Assume that R/Q is semiclean. From [23], Proposition 2.2 R is semiclean. The converse is straightforward by assertion (1) above. $\square$

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