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EXTRAPOLATED POSITIVE DEFINITE AND POSITIVE SEMI-DEFINITE SPLITTING METHODS FOR SOLVING NON-HERMITIAN POSITIVE DEFINITE LINEAR SYSTEMS

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Abstract. Recently, Na Huang and Changfeng Ma in (2016) proposed two kinds of typical practical choices of the PPS method. In this paper, we extrapolate two versions of the PPS iterative method, and we introduce the extrapolated Hermitian and skew-Hermitian positive definite and positive semi-definite splitting (EHPPS) iterative method and extrapolated triangular positive definite and positive semi-definite splitting (ETPPS) iterative method. We also investigate convergence analysis and consistency of the proposed iterative methods. Then, we study upper bounds for the spectral radius of iteration matrices and give upper bounds for the extrapolation parameter of the methods. Moreover, the optimal parameters which minimize upper bounds of the spectral radius are obtained. Finally, several numerical examples are given to show the efficiency of the presented method.

Keywords: extrapolated; non-Hermitian; positive definite; skew-Hermitian; splitting; HSS iteration method

MSC 2020: 65F10, 65B05

1. Introduction

In scientific computing, many problems require solving the system of linear equations

$$\mathcal{A}u = b, \tag{1.1}$$

where $\mathcal{A} \in \mathbb{C}^{n\times n}$ is a large sparse non-Hermitian and positive definite matrix, and $u, b \in \mathbb{C}^n$.

For solving system (1.1), researchers introduced a few kinds of splitting iteration methods [3], [5], [7], [16], [18], [26] and Krylov subspace methods [9], [10]. Bai et al. [5]

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obtained the Hermitian and skew-Hermitian splitting (HSS) iteration method. They also proved that the HSS method converges unconditionally to the unique solution of (1.1). Then they obtained the normal and skew-Hermitian splitting iteration method with generalizing the HSS method [6]. Based on block triangular and skew-Hermitian splitting, a class of iteration method for solving positive definite linear systems were explored in [4]. The authors suggested generalized skew-Hermitian triangular splitting iteration methods for saddle-point problems in [17]. Recently, Na Huang and Chang-feng Ma in [15] introduced new splitting iterative method for solving (1.1) based on the positive definite and positive semi-definite splitting (PPS) of the coefficient matrix $\mathcal{A}$.

It is well known that the first degree methods to solve (1.1) can be written as

$$u^{(p+1)} = \mathcal{T}u^{(p)} + \mathcal{C}, \quad p = 1, 2, \dots, \tag{1.2}$$

where $\mathcal{T}$ is the iteration matrix. To accelerate the convergence rate of method (1.2), variant procedures and modifications can be used. One of them is the extrapolation method.

The extrapolated iterative method. The extrapolated iteration method [1] based on the iteration method (1.2) is defined by

$$u^{(p+1)} = \beta(\mathcal{T}u^{(p)} + \mathcal{C}) + (1-\beta)u^{(p)}, \quad p = 0, 1, 2, \dots, \tag{1.3}$$

where β is a real parameter different from zero, called the extrapolation parameter. The extrapolated method converges independently of whether the original iteration method is convergent or not. For $\beta = 1$, method (1.3) coincides with (1.2). We denote the iteration matrix of method (1.3) with $\mathcal{T}_\beta = \beta\mathcal{T} + (1-\beta)I$, where I is the identity matrix of order n. Thus, we can write (1.3) in the form

$$u^{(p+1)} = \mathcal{T}_\beta u^{(p)} + \mathcal{C}_\beta, \quad p = 0, 1, 2, \dots, \tag{1.4}$$

where $\mathcal{C}_\beta = \beta\mathcal{C}$. The parameter β must be chosen so that $\varrho(\mathcal{T}_\beta) < \varrho(\mathcal{T})$ with $\varrho(\mathcal{T}_\beta) < 1$. Also, we find optimum value for β, say β_{opt}, such that

$$\varrho(\mathcal{T}_{\beta_{\text{opt}}}) = \min_\beta \varrho(\mathcal{T}_\beta) \leqslant \varrho(\mathcal{T}), \quad \varrho(\mathcal{T}_{\beta_{\text{opt}}}) < 1.$$

Missirlis and Evans [19] discussed an extrapolated Gauss-Seidel method (EGS) and the successive overrelaxation (ESOR) method for the solution of linear systems. In [13], extrapolated generalized Jacobi (EGJ) and extrapolated generalized

Gauss-Seidel (EGGS) method were studied. Extrapolated accelerated overrelaxation (EAOR) was introduced in [11]. Hadjidimos and Yeyios [14] used the extrapolation principle to extend some of the theoretical results concerning AOR method. Also Hadjidimos [12] studied determining an optimum value of the extrapolation parameter for complex systems. Yeyios [24], [25] derived ranges for the extrapolation parameter and Cao [8] obtained convergence conditions for extrapolation methods. Song and Wang [21], [22], [23] presented the sufficient and necessary conditions for semi-convergence of the extrapolated iterative methods for singular problems, also they obtained the upper bounds and optimum extrapolation parameter. Recently, in order to improve the efficiency of the MHSS iteration method, Zeng and Zhang [27] presented complex-extrapolated MHSS iteration method (CMHSS) for solving both singular and nonsingular complex symmetric system of linear equations.

In this work, we extrapolate two versions of the PPS iterative method, called EHPPS and ETPPS iterative methods. These new methods accelerate the convergence of the HPPS and TPPS iterative methods. We discuss the convergence theorems for the extrapolated iterative methods and the optimal parameters of the EHPPS and ETPPS methods. We find the upper bound for extrapolation parameter β, such that the spectral radius of the iteration matrices of our new methods is smaller than spectral radius of the HPPS and TPPS methods. Finally, we compare the EHPPS and ETPPS methods with the HPPS and TPPS methods. By numerical experiments and theoretic analysis, we conclude that the proposed methods are superior to some existing methods.

We write $\mathcal{A}^{-1}$, $\|\mathcal{A}\|_2$, $\Lambda(\mathcal{A})$, and $\varrho(\mathcal{A})$ to denote the inverse, 2-norm, the spectrum, and the spectral radius of the matrix $\mathcal{A}$, respectively. We denote by $\mathrm{i} = \sqrt{-1}$ the imaginary unit and by $\otimes$ Kronecker product. I_n denotes the identity matrix of size $n \times n$. We call a matrix M positive definite (positive semi-definite) if for all $0 \neq u \in \mathbb{C}^n$, $u^{\mathrm{H}}(M + M^{\mathrm{H}})u > 0$ $(u^{\mathrm{H}}(M + M^{\mathrm{H}})u = 0)$.

The outline of the paper is organized as follows. In Section 2, we describe the PPS iterative method. In Section 3, we present the EHPPS and ETPPS iterative methods and their convergence. In Section 4, numerical results are discussed. Finally, some conclusions are given in Section 5.

2. Preliminaries

In this section, we review the PPS iterative method. This method is based on the splitting

$$\mathcal{A} = M + N,$$

where M is a positive definite matrix and N is a positive semi-definite matrix.

The PPS iterative method. Given an initial guess $u^{(0)}$ for $p = 0, 1, 2, \ldots$ until $\{u^{(p)}\}$ converges, compute

$$\begin{cases} (\alpha I + M)u^{(p+1/2)} = (\alpha I - N)u^{(p)} + b, \\ (\alpha I + N)u^{(p+1)} = (\alpha I - M)u^{(p+1/2)} + b, \end{cases} \tag{2.1}$$

where α is a given positive constant. The PPS iterative scheme (2.1) can be reformulated as (1.2) with

$$\mathcal{T}_\alpha = (\alpha I + N)^{-1}(\alpha I - M)(\alpha I + M)^{-1}(\alpha I - N)$$

and

$$\mathcal{C}_\alpha = 2\alpha b[(\alpha I + M)(\alpha I + N)]^{-1},$$

where $\mathcal{T}_\alpha$ is the iteration matrix of the PPS iterative method. Formal manipulation reduces this to

$$\mathcal{T}_\alpha = I - 2\alpha(Q + \alpha I)^{-1}, \tag{2.2}$$

where

$$Q = (M + N)^{-1}(MN + \alpha^2 I). \tag{2.3}$$

For typical practical choices of the PPS iterative method, HPPS and TPPS methods were presented in [15].

The HPPS iterative method. Let $H(\mathcal{A}) = \frac{1}{2}(\mathcal{A} + \mathcal{A}^{\mathrm{H}})$, $S(\mathcal{A}) = \frac{1}{2}(\mathcal{A} - \mathcal{A}^{\mathrm{H}})$ be the Hermitian and skew-Hermitian parts of the matrix $\mathcal{A}$, respectively. Consider

$$M = H(\mathcal{A}) + \mathrm{i}\eta I, \quad N = S(\mathcal{A}) - \mathrm{i}\eta I, \tag{2.4}$$

where η is any real constant. It is obvious that M is positive definite and N is positive semi-definite.

Given an initial guess $u^{(0)}$ for $p = 0, 1, 2, \ldots$ until $\{u^{(p)}\}$ converges, compute

$$\begin{cases} (\alpha I + H(\mathcal{A}) + \mathrm{i}\eta I)u^{(p+1/2)} = (\alpha I - S(\mathcal{A}) + \mathrm{i}\eta I)u^{(p)} + b, \\ (\alpha I + S(\mathcal{A}) - \mathrm{i}\eta I)u^{(p+1)} = (\alpha I - H(\mathcal{A}) - \mathrm{i}\eta I)u^{(p+1/2)} + b, \end{cases} \tag{2.5}$$

where α is a given positive constant. If $\eta = 0$, the iteration method (2.5) reduces to the HSS iteration method. Thus, the HSS iterative method is a special case of the PPS iterative method.

The TPPS iteration method. Consider the coefficient matrix $\mathcal{A}$ has a splitting of the form $\mathcal{A} = E + D + F$, where D is the diagonal matrix, E and F are the strict lower and upper triangular parts of $\mathcal{A}$. For any $\omega \in \mathbb{R}$, let

$$M = D + E + F^{\mathrm{H}} + \mathrm{i}\omega I, \quad N = F - F^{\mathrm{H}} - \mathrm{i}\omega I. \tag{2.6}$$

Let $u^{(0)}$ be an arbitrary initial guess. For $p = 0, 1, 2, \ldots$ until sequence of iterates $\{u^{(p)}\}$ converges, compute the next iterate $\{u^{(p+1)}\}$ according to the following procedure:

$$\begin{cases} (\alpha I + D + F^{\mathrm{H}} + E + \mathrm{i}\omega I)u^{(p+1/2)} = (\alpha I - F + F^{\mathrm{H}} + \mathrm{i}\omega I)u^{(p)} + b, \\ (\alpha I + F - F^{\mathrm{H}} - \mathrm{i}\omega I)u^{(p+1)} = (\alpha I - D - F^{\mathrm{H}} - E - \mathrm{i}\omega I)u^{(p+1/2)} + b, \end{cases} \tag{2.7}$$

or

$$\begin{cases} (\alpha I + D + E^{\mathrm{H}} + F + \mathrm{i}\omega I)u^{(p+1/2)} = (\alpha I - E + E^{\mathrm{H}} + \mathrm{i}\omega I)u^{(p)} + b, \\ (\alpha I + E - E^{\mathrm{H}} - \mathrm{i}\omega I)u^{(p+1)} = (\alpha I - D - E^{\mathrm{H}} - F - \mathrm{i}\omega I)u^{(p+1/2)} + b. \end{cases} \tag{2.8}$$

If $\omega = 0$, then the TPPS iterative method reduces to the triangular and skew-Hermitian splitting (TSS) iteration method proposed in [4]. This shows that the TSS iterative method is a special case of the TPPS iterative method.

3. The EHPPS and ETPPS iteration methods

In this section, we extrapolate two versions of PPS iterative method and obtain the EHPPS and ETPPS iterative methods.

3.1. The EHPPS iteration method and its convergence.

Given an initial guess $u^{(0)}$ for $p = 0, 1, 2, \ldots$ until $\{u^{(p)}\}$ converges, compute the next iterate $u^{(p+1)}$ according to the following procedure:

$$\begin{cases} (\alpha I + H(\mathcal{A}) + \mathrm{i}\eta I)u^{(p+1/2)} = (\alpha I - S(\mathcal{A}) + \mathrm{i}\eta I)u^{(p)} + b, \\ (\alpha I + S(\mathcal{A}) - \mathrm{i}\eta I)\tilde{u}^{(p+1)} = (\alpha I - H(\mathcal{A}) - \mathrm{i}\eta I)u^{(p+1/2)} + b, \\ u^{(p+1)} = (1 - \beta)u^{(p)} + \beta\tilde{u}^{(p+1)}, \end{cases} \tag{3.1}$$

where α is a given positive constant, $\eta \in \mathbb{R}$ and $\beta \in \mathbb{R} \setminus \{0\}$ (called the extrapolation parameter).

We can rewrite EHPPS as the standard iterative method

$$u^{(p+1)} = \mathcal{T}_{\alpha,\beta,\eta} u^{(p)} + \mathcal{C}_{\alpha,\beta,\eta} b, \tag{3.2}$$

where

$$\mathcal{T}_{\alpha,\beta,\eta} = ((1-\beta)I + \beta \mathcal{T}_{\alpha,\eta}), \quad \mathcal{C}_{\alpha,\beta,\eta} = \beta \mathcal{C}_{\alpha,\eta}, \tag{3.3}$$

that

$$\begin{aligned} \mathcal{T}_{\alpha,\eta} &= (\alpha I + S(\mathcal{A}) - \mathrm{i}\eta I)^{-1}(\alpha I + H(\mathcal{A}) + \mathrm{i}\eta I)^{-1} \\ &\quad \times (\alpha I - H(\mathcal{A}) - \mathrm{i}\eta I)(\alpha I - S(\mathcal{A}) + \mathrm{i}\eta I), \\ \mathcal{C}_{\alpha,\eta} &= \frac{2\alpha}{(\alpha I + H(\mathcal{A}) + \mathrm{i}\eta I)(\alpha I + S(\mathcal{A}) - \mathrm{i}\eta I)}, \end{aligned}$$

where $\mathcal{T}_{\alpha,\beta,\eta}$ is the iteration matrix of the EHPPS and $\mathcal{T}_{\alpha,\eta}$ is the iteration matrix of the HPPS iterative method.

Now, we discuss the consistency of the EHPPS iterative method.

Theorem 3.1 ([26]). *If $\mathcal{A}$ is a nonsingular matrix, then the iteration method* (1.2) *is consistent with* (1.1) *if and only if $\mathcal{C} = (I - \mathcal{T})\mathcal{A}^{-1}b$.*

Theorem 3.2 ([26]). *If $\mathcal{A}$ is a nonsingular matrix, then the iteration method* (1.2) *is completely consistent with* (1.1) *if and only if it is consistent and $(I - \mathcal{T})$ is nonsingular.*

Since

$$(I - \mathcal{T}_{\alpha,\beta,\eta})\mathcal{A}^{-1}b = (I - (1-\beta)I - \beta\mathcal{T}_{\alpha,\eta})\mathcal{A}^{-1}b = \beta(I - \mathcal{T}_{\alpha,\eta})\mathcal{A}^{-1}b = \beta\mathcal{C}_{\alpha,\eta} = \mathcal{C}_{\alpha,\beta,\eta}$$

and

$$\det(I - \mathcal{T}_{\alpha,\beta,\eta}) = \det(I - (1-\beta)I - \beta\mathcal{T}_{\alpha,\eta}) = \beta^n \det(I - \mathcal{T}_{\alpha,\eta}) \neq 0,$$

it follows that the EHPPS iterative method is completely consistent with system (1.1).

Also, the EHPPS iterative method can be obtained from the splitting of the coefficient matrix $\mathcal{A}$ as follows:

$$\mathcal{A} = B(\alpha,\beta,\eta) - S(\alpha,\beta,\eta),$$

where

$$\begin{aligned} B(\alpha,\beta,\eta) &= \frac{1}{2\alpha\beta}(\alpha I + H(\mathcal{A}) + \mathrm{i}\eta I)(\alpha I + S(\mathcal{A}) - \mathrm{i}\eta I), \\ S(\alpha,\beta,\eta) &= \frac{1}{2\alpha\beta}[(1-\beta)(\alpha I + H(\mathcal{A}) + \mathrm{i}\eta I)(\alpha I + S(\mathcal{A}) - \mathrm{i}\eta I) \\ &\qquad + \beta(\alpha I - H(\mathcal{A}) - \mathrm{i}\eta I)(\alpha I - S(\mathcal{A}) + \mathrm{i}\eta I)]. \end{aligned}$$

Theorem 3.3. *Iteration scheme* (2.1) *is convergent for constant $\alpha > 0$ if and only if the real parts of the eigenvalues of matrix Q defined in* (2.3) *are all greater than zero.*

Proof. Let $\gamma = \Re(\gamma) + \mathrm{i}\Im(\gamma)$ and ν be eigenvalues of Q and $\mathcal{T}_\alpha$, respectively. From (2.2) we have

$$\nu = 1 - \frac{2\alpha}{\gamma+\alpha} = \frac{\gamma-\alpha}{\gamma+\alpha} = \frac{(\Re(\gamma)-\alpha)+\mathrm{i}\Im(\gamma)}{(\Re(\gamma)+\alpha)+\mathrm{i}\Im(\gamma)},$$

then

$$|\nu|^2 = \frac{(\Re(\gamma)-\alpha)^2+(\Im(\gamma))^2}{(\Re(\gamma)+\alpha)^2+(\Im(\gamma))^2} < 1$$

if and only if $\Re(\gamma) > 0$. □

Theorem 3.4 ([15])**.** *Let $\mathcal{A} = M + N$ be a positive definite and positive semi-definite splitting of $\mathcal{A}$. Then for any positive constant α, $\varrho(\mathcal{T}_{\alpha,\eta}) < 1$. Therefore the PPS iterative method is convergent to the exact solution $u^{(*)} \in \mathbb{C}^n$ of the linear system* (1.1)*.*

Theorem 3.5 ([15])**.** *Let $\mathcal{A} \in \mathbb{C}^n$ be a positive definite matrix, $M = H(\mathcal{A}) + \mathrm{i}\eta I$ be a positive definite matrix and $N = S(A) - \mathrm{i}\eta I$ be a positive semi-definite matrix, and $\lambda_{\max}$ and $\lambda_{\min}$ be the maximum and minimum eigenvalues of $H(\mathcal{A})$. Then for any $\eta \in \mathbb{R}$ and $\alpha > 0$, the spectral radius $\varrho(\mathcal{T}_{\alpha,\eta})$ of the iteration matrix $\mathcal{T}_{\alpha,\eta}$ is bounded by*

$$\varrho(\mathcal{T}_{\alpha,\eta}) \leqslant \sqrt{1-\varphi(\alpha,\eta)}, \tag{3.4}$$

where

$$\varphi(\alpha,\eta) = \min\Big\{\frac{4\alpha\lambda_{\max}}{(\alpha+\lambda_{\max})^2+\eta^2}, \frac{4\alpha\lambda_{\min}}{(\alpha+\lambda_{\min})^2+\eta^2}\Big\}. \tag{3.5}$$

Theorem 3.6. *Let all the assumptions of Theorem* 3.4 *hold and $M = H(\mathcal{A}) + \mathrm{i}\eta I$, $N = S(\mathcal{A}) - \mathrm{i}\eta I$. Let α be positive constant and ν be the eigenvalue of $\mathcal{T}_{\alpha,\eta}$. The EHPPS iterative method is convergent to the exact solution of the linear system* (1.1) *or equivalently, $\varrho(\mathcal{T}_{\alpha,\beta,\eta}) < 1$ if and only if*

$$0 < \beta < \frac{\Re_{\min}(\gamma)+\alpha}{\alpha},$$

where $\gamma_{\min} = \Re_{\min}(\gamma) + \mathrm{i}\Im_{\min}(\gamma)$, and $\Re_{\min}(\gamma) = \min\limits_{\gamma\in\Lambda(Q)}\{\Re(\gamma)\}$. Moreover,

$$\beta_{\mathrm{opt}} = \frac{\Re_{\min}(\gamma)+\alpha}{2\alpha} \quad \text{and} \quad \varrho(\mathcal{T}_{\alpha,\beta_{\mathrm{opt}},\eta}) = \frac{\Im_{\min}(\gamma)}{(|\gamma_{\min}|^2+\alpha^2+2\alpha\Re_{\min}(\gamma))^{1/2}}.$$

Proof. Let μ be the eigenvalue of $\mathcal{T}_{\alpha,\beta,\eta}$. It follows from Eq. (3.3) that

$$\mu = 1 - \beta + \beta\nu = 1 - \beta + \beta\frac{(\Re(\gamma) - \alpha) + \mathrm{i}\Im(\gamma)}{(\Re(\gamma) + \alpha) + \mathrm{i}\Im(\gamma)}, \tag{3.6}$$

by simple computations it follows that

$$\mu = \frac{(\Re(\gamma) + \alpha - 2\alpha\beta) + \mathrm{i}\Im(\gamma)}{(\Re(\gamma) + \alpha) + \mathrm{i}\Im(\gamma)}.$$

We have

$$|\mu|^2 = \frac{(\Re(\gamma) + \alpha - 2\alpha\beta)^2 + (\Im(\gamma))^2}{(\Re(\gamma) + \alpha)^2 + (\Im(\gamma))^2}.$$

To get $\varrho(\mathcal{T}_{\alpha,\beta,\eta}) < 1$, it is enough to have $|\mu| < 1$, therefore,

$$4\alpha^2\beta^2 - 4\alpha\beta(\Re(\gamma) + \alpha) < 0. \tag{3.7}$$

Since $\beta > 0$, then (3.7) implies $\beta < (\Re(\gamma) + \alpha)/\alpha$, which gives

$$0 < \beta < \min_{\gamma\in(Q)}\left\{\frac{\Re(\gamma) + \alpha}{\alpha}\right\}.$$

Thus, $\varrho(\mathcal{T}_{\alpha,\beta,\eta}) < 1$ if and only if

$$0 < \beta < \frac{\Re_{\min}(\gamma) + \alpha}{\alpha}.$$

To compute the value of β_{opt}, let

$$f(\beta) = \frac{(\Re(\gamma) + \alpha - 2\alpha\beta)^2 + (\Im(\gamma))^2}{(\Re(\gamma) + \alpha)^2 + (\Im(\gamma))^2}.$$

We have

$$\frac{\mathrm{d}f}{\mathrm{d}\beta} = -4\alpha(\Re(\gamma) + \alpha - 2\alpha\beta) = 0,$$

so

$$\beta_{\mathrm{opt}} = \frac{\Re_{\min}(\gamma) + \alpha}{2\alpha}.$$

We can conclude that

$$\begin{aligned}\min_{\beta} f(\beta) = f(\beta_{\mathrm{opt}}) &= \frac{(\Re_{\min}(\gamma) + \alpha - 2\alpha\beta_{\mathrm{opt}})^2 + (\Im_{\min}(\gamma))^2}{(\Re_{\min}(\gamma) + \alpha)^2 + (\Im_{\min}(\gamma))^2}\\ &= \frac{(\Im_{\min}(\gamma))^2}{|\gamma_{\min}|^2 + \alpha^2 + 2\alpha\Re_{\min}(\gamma)}.\end{aligned}$$

Thus, obtain

$$\varrho(\mathcal{T}_{\alpha,\beta_{\mathrm{opt}},\eta}) = \frac{\Im_{\min}(\gamma)}{(|\gamma_{\min}|^2 + \alpha^2 + 2\alpha\Re_{\min}(\gamma))^{1/2}}.$$

$\square$

Theorem 3.7. *Let all the assumptions of Theorem 3.5 hold and*

$$\varrho(\mathcal{T}_{\alpha,\eta}) < 1, \quad 0 < \beta < \frac{\Re_{\min}(\gamma) + \alpha}{\alpha}, \quad \alpha > 0,$$

and $\eta \in \mathbb{R}$. Then the spectral radius $\varrho(\mathcal{T}_{\alpha,\beta,\eta})$ of the EHPPS iterative is bounded by

$$\delta(\alpha, \beta, \eta) = |1 - \beta| + \beta\sqrt{1 - \varphi(\alpha, \eta)}, \tag{3.8}$$

where

$$\varphi(\alpha, \eta) = \min\Big\{\frac{4\alpha\lambda_{\max}}{(\alpha + \lambda_{\max})^2 + \eta^2}, \frac{4\alpha\lambda_{\min}}{(\alpha + \lambda_{\min})^2 + \eta^2}\Big\}. \tag{3.9}$$

Moreover, the optimal point $(\alpha^*, \beta^*, \eta^*)$ of $\varrho(\mathcal{T}_{\alpha,\beta,\eta})$ is

$$(\alpha^*, \beta^*, \eta^*) = \arg \max_{\alpha>0,\eta\in\mathbb{R}} \delta(\alpha, \beta, \eta) = \Big(\sqrt{\lambda_{\min}\lambda_{\max}}, \frac{\Re_{\min}(\gamma) + \alpha}{2\alpha}, 0\Big).$$

Proof. Let

$$R(\alpha, \eta) = \frac{\alpha I - H(\mathcal{A}) - \mathrm{i}\eta I}{\alpha I + H(\mathcal{A}) + \mathrm{i}\eta I} \quad \text{and} \quad Z(\alpha, \eta) = \frac{\alpha I - S(\mathcal{A}) + \mathrm{i}\eta I}{\alpha I + S(\mathcal{A}) - \mathrm{i}\eta I}$$

It is obvious that $(\alpha I - S(\mathcal{A}) + \mathrm{i}\eta I)^{-1}S(A) = S(\mathcal{A})(\alpha I - S(\mathcal{A}) + \mathrm{i}\eta I)^{-1}$.
Since

$$Z(\alpha, \eta)^{\mathrm{H}} Z(\alpha, \eta) = \frac{(\alpha I + S(\mathcal{A}) - \mathrm{i}\eta I)(\alpha I - S(\mathcal{A}) + \mathrm{i}\eta I)}{(\alpha I + S(\mathcal{A}) - \mathrm{i}\eta I)(\alpha I - S(\mathcal{A}) + \mathrm{i}\eta I)} = I.$$

We can similarly get $Z(\alpha, \eta)Z(\alpha, \eta)^{\mathrm{H}} = 1$. This shows that $Z(\alpha, \eta)$ is a unitary matrix, therefore $\|Z(\alpha, \eta)\|_2 = 1$. Then

$$\varrho(\mathcal{T}_{\alpha,\eta}) = \varrho(R(\alpha, \eta)Z(\alpha, \eta)) \leqslant \|R(\alpha, \eta)\|_2\|Z(\alpha, \eta)\|_2 = \|R(\alpha, \eta)\|_2.$$

It follows that

$$\begin{aligned}
\|R(\alpha, \eta)\|_2^2 &= \varrho(R(\alpha, \eta)^{\mathrm{H}} R(\alpha, \eta)) \\
&= \varrho\Big(\frac{(\alpha I - H(\mathcal{A}) + \mathrm{i}\eta I)(\alpha I - H(\mathcal{A}) - \mathrm{i}\eta I)}{(\alpha I + H(\mathcal{A}) - \mathrm{i}\eta I)(\alpha I + H(\mathcal{A}) + \mathrm{i}\eta I)}\Big) \\
&= \max_{\lambda\in\Lambda(H(\mathcal{A}))} \Big\{\frac{(\alpha - \lambda + \mathrm{i}\eta)(\alpha - \lambda - \mathrm{i}\eta)}{(\alpha + \lambda - \mathrm{i}\eta)(\alpha + \lambda + \mathrm{i}\eta)}\Big\} \\
&= \max_{\lambda\in\Lambda(H(\mathcal{A}))} \Big\{\frac{(\alpha - \lambda)^2 + \eta^2}{(\alpha + \lambda)^2 + \eta^2}\Big\},
\end{aligned}$$

where $\Lambda(H(\mathcal{A}))$ is the spectrum of matrix $H(\mathcal{A})$. We can write

$$\begin{aligned}\frac{(\alpha-\lambda)^2+\eta^2}{(\alpha+\lambda)^2+\eta^2} &= \frac{\alpha^2+2\lambda\alpha+\lambda^2+\eta^2-4\lambda\alpha}{\alpha^2+2\lambda\alpha+\lambda^2+\eta^2} = 1-\frac{4\lambda\alpha}{\alpha^2+2\lambda\alpha+\lambda^2+\eta^2}\\ &= 1-\frac{4\alpha}{(\alpha^2+\eta^2)/\lambda+\lambda+2\alpha}.\end{aligned}$$

Hence

$$\begin{aligned}\|R(\alpha,\eta)\|_2^2 &= \max_{\lambda\in\Lambda(H(\mathcal{A}))}\Big\{1-\frac{4\alpha}{(\alpha^2+\eta^2)/\lambda+\lambda+2\alpha}\Big\}\\ &= 1-\min_{\lambda\in\Lambda H(\mathcal{A})}\Big\{\frac{4\alpha}{(\alpha^2+\eta^2)/\lambda+\lambda+2\alpha}\Big\}\\ &= 1-\min\Big\{\frac{4\alpha}{(\alpha^2+\eta^2)/\lambda_{\min}+\lambda_{\min}+2\alpha},\frac{4\alpha}{(\alpha^2+\eta^2)/\lambda_{\max}+\lambda_{\max}+2\alpha}\Big\}\\ &= 1-\min\Big\{\frac{4\alpha\lambda_{\max}}{(\alpha+\lambda_{\max})^2+\eta^2},\frac{4\alpha\lambda_{\min}}{(\alpha+\lambda_{\min})^2+\eta^2}\Big\}.\end{aligned}$$

Thus

$$\|R(\alpha,\eta)\|_2=\sqrt{1-\varphi(\alpha,\eta)},$$

where

$$\varphi(\alpha,\eta)=\min\Big\{\frac{4\alpha\lambda_{\max}}{(\alpha+\lambda_{\max})^2+\eta^2},\frac{4\alpha\lambda_{\min}}{(\alpha+\lambda_{\min})^2+\eta^2}\Big\}.$$

We have

$$\begin{aligned}\varrho(\mathcal{T}_{\alpha,\beta,\eta}) \leqslant \|\mathcal{T}_{\alpha,\beta,\eta}\|_2 &= \|(1-\beta)I+\beta\mathcal{T}_{\alpha,\eta}\|_2\\ &\leqslant |1-\beta|+|\beta|\|\mathcal{T}_{\alpha,\eta}\|_2\\ &= |1-\beta|+|\beta|\|R(\alpha,\eta)Z(\alpha,\eta)\|_2\\ &\leqslant |1-\beta|+\beta\sqrt{1-\varphi(\alpha,\eta)}.\end{aligned}$$

This completes the first result. Let $\delta(\alpha,\beta,\eta)=|1-\beta|+\beta\sqrt{1-\varphi(\alpha,\eta)}$.

Noticing that function $\varphi(\alpha,\eta)$ decreases monotonically with respect to η^2, we give

$$\eta^*\equiv\arg\max_{\alpha>0}\varphi(\alpha,\eta)=0,$$

so $\delta(\alpha,\beta,0)=|1-\beta|+\beta\sqrt{1-\varphi(\alpha,0)}$ gets

$$\begin{aligned}\eta^* &= \arg\max\delta(\alpha,\beta,0)\\ &= \max\left\{|1-\beta|+\beta\sqrt{1-\frac{4\alpha\lambda_{\max}}{(\alpha+\lambda_{\max})^2}},\ |1-\beta|+\beta\sqrt{1-\frac{4\alpha\lambda_{\min}}{(\alpha+\lambda_{\min})^2}}\right\}.\end{aligned}$$

If α^* is a minimum point of $\delta(\alpha,\beta,0)$ and β^* is optimum extrapolated parameter, then they must satisfy

$$|1-\beta^*|+\beta^*\sqrt{1-\frac{4\alpha^*\lambda_{\max}}{(\alpha^*+\lambda_{\max})^2}}=|1-\beta^*|+\beta^*\sqrt{1-\frac{4\alpha^*\lambda_{\min}}{(\alpha^*+\lambda_{\min})^2}},$$
$$1-\frac{4\alpha^*\lambda_{\max}}{(\alpha^*+\lambda_{\max})^2}=1-\frac{4\alpha^*\lambda_{\min}}{(\alpha^*+\lambda_{\min})^2},$$
$$\frac{\lambda_{\max}}{(\alpha^*+\lambda_{\max})^2}=\frac{\lambda_{\min}}{(\alpha^*+\lambda_{\min})^2}\Rightarrow\alpha^*=\sqrt{\lambda_{\min}\lambda_{\max}}.$$

□

We emphasize that the optimal parameter α^* only minimizes the upper bound $\delta(\alpha,\beta,\eta)$ of the spectral radius of the iteration matrix, but does not minimize the spectral radius itself.

3.2. The ETPPS iteration method and its convergence. Given an initial guess $u^{(0)}$ for $p=0,1,2,\dots$ until $\{u^{(p)}\}$ converges, compute the next iterate $u^{(p+1)}$ according to the following procedure:

$$\begin{cases}(\alpha I+D+E+F^{\mathrm{H}}+\mathrm{i}\omega I)u^{(p+1/2)}=(\alpha I-F+F^{\mathrm{H}}+\mathrm{i}\omega I)u^{(p)}+b,\\(\alpha I+F-F^{\mathrm{H}}-\mathrm{i}\omega I)\tilde{u}^{(p+1)}=(\alpha I-D-E-F^{\mathrm{H}}-\mathrm{i}\omega I)u^{(p+1/2)}+b,\\u^{(p+1)}=(1-\beta)u^{(p)}+\beta\tilde{u}^{(p+1)},\end{cases}\tag{3.10}$$

where β is a real parameter different from zero, called the extrapolation parameter, α is a given positive constant and $\omega\in\mathbb{R}$. From (2.6) we have $M=D+E+F^{\mathrm{H}}+\mathrm{i}\omega I$ and $N=F-F^{\mathrm{H}}-\mathrm{i}\omega I$. At each step of iteration $u^{(p)}$, for solving two systems with the matrices $\alpha I+M$ and $\alpha I+N$ in (3.10), we use the LU factorization method.

In matrix-vector form, the ETPPS iterative method can be rewritten as

$$u^{(p+1)}=\overline{\mathcal{T}}_{\alpha,\beta,\omega}u^{(p)}+\overline{\mathcal{C}}_{\alpha,\beta,\omega}b\tag{3.11}$$

with iteration matrix $\overline{\mathcal{T}}_{\alpha,\beta,\omega}=(1-\beta)I+\beta\overline{\mathcal{T}}_{\alpha,\omega}$ and $\overline{\mathcal{C}}_{\alpha,\beta,\omega}=\beta\overline{\mathcal{C}}_{\alpha,\omega}$, where

$$\begin{aligned}\overline{\mathcal{T}}_{\alpha,\omega}&=(\alpha I+E-E^{\mathrm{H}}-\mathrm{i}\omega I)^{-1}(\alpha I-D+E^{\mathrm{H}}+F+\mathrm{i}\omega I)\\&\quad\times(\alpha I+D+E^{\mathrm{H}}+F+\mathrm{i}\omega I)^{-1}(\alpha I-E-E^{\mathrm{H}}-\mathrm{i}\omega I)\end{aligned}$$

and

$$\overline{\mathcal{C}}_{\alpha,\omega}=\frac{2\alpha}{(\alpha I+D+E^{\mathrm{H}}+F+\mathrm{i}\omega I)(\alpha I+E-E^{\mathrm{H}}-\mathrm{i}\omega I)}.$$

Note that $\overline{\mathcal{T}}_{\alpha,\omega}$ is the iteration matrix of the TPPS iterative method.

Since

$$(I-\overline{\mathcal{T}}_{\alpha,\beta,\omega})\mathcal{A}^{-1}b = (I-(1-\beta)I+\beta\overline{\mathcal{T}}_{\alpha,\omega})\mathcal{A}^{-1}b = \beta(I-\overline{\mathcal{T}}_{\alpha,\omega})\mathcal{A}^{-1}b = \beta\overline{\mathcal{C}}_{\alpha,\omega} = \overline{\mathcal{C}}_{\alpha,\beta,\omega}$$

and

$$\det(I-\overline{\mathcal{T}}_{\alpha,\beta,\omega}) = \det(I-(1-\beta)I+\beta\overline{\mathcal{T}}_{\alpha,\omega}) = \beta^n \det(I-\overline{\mathcal{T}}_{\alpha,\omega}) \neq 0,$$

from Theorems 3.1 and 3.2, the ETPPS iterative method is also completely consistent with system (1.1). Iteration (3.11) may be considered as a splitting iteration induced from the splitting

$$A = \overline{B}(\alpha,\beta,\omega) - \overline{S}(\alpha,\beta,\omega)$$

of the matrix $\mathcal{A}$, where

$$\begin{aligned}
\overline{B}(\alpha,\beta,\omega) &= \frac{1}{2\alpha\beta}(\alpha I + D + E^{\mathrm{H}} + F + \mathrm{i}\omega I)(\alpha I + E - E^{\mathrm{H}} - \mathrm{i}\omega I),\\
\overline{S}(\alpha,\beta,\omega) &= \frac{1}{2\alpha\beta}[(1-\beta)(\alpha I + D + E^{\mathrm{H}} + F + \mathrm{i}\omega I)(\alpha I + E - E^{\mathrm{H}} - \mathrm{i}\omega I)\\
&\qquad + \beta(\alpha I - D + E^{\mathrm{H}} + F + \mathrm{i}\omega I)(\alpha I - E - E^{\mathrm{H}} - \mathrm{i}\omega I)].
\end{aligned}$$

Theorem 3.8. *Let $\mathcal{A} = M + N$, where $M = D + E^{\mathrm{H}} + F + \mathrm{i}\omega I$ and $N = E - E^{\mathrm{H}} - \mathrm{i}\omega I$ is a positive definite and positive semi-definite matrix, respectively. Suppose ν be the eigenvalue of $\overline{\mathcal{T}}_{\alpha,\omega}$. For any positive constant α and $\omega \in \mathbb{R}$, the ETPPS iterative method is convergent or equivalently, $\varrho(\overline{\mathcal{T}}_{\alpha,\beta,\omega}) < 1$ if and only if*

$$0 < \beta < \frac{\Re_{\min}(\gamma) + \alpha}{\alpha},$$

where $\gamma_{\min} = \Re_{\min}(\gamma) + \mathrm{i}\Im_{\min}(\gamma)$, and $\Re_{\min}(\gamma) = \min\limits_{\gamma\in\Lambda(Q)}\{\Re(\gamma)\}$. Moreover,

$$\beta_{\mathrm{opt}} = \frac{\Re_{\min}(\gamma) + \alpha}{2\alpha} \quad \text{and} \quad \varrho(\mathcal{T}_{\alpha,\beta_{\mathrm{opt}},\omega}) = \frac{\Im_{\min}(\gamma)}{(|\gamma_{\min}|^2 + \alpha^2 + 2\alpha\Re_{\min}(\gamma))^{1/2}}.$$

P r o o f. Because N is a positive semi-definite matrix and M is a positive definite matrix, by using Theorem 3.4, we have $\varrho(\overline{\mathcal{T}}_{\alpha,\omega}) < 1$. From the proof of Theorem 3.6, we can immediately obtain the result. $\square$

Theorem 3.9. *Let $\mathcal{A} \in \mathbb{C}^n$ be a positive definite matrix suppose that $D = \operatorname{diag}(a_{11}, \ldots, a_{nn})$, and E and F are the strictly lower-triangular and the strictly upper-triangular matrices of the matrix $\mathcal{A}$, respectively. Then for any*

$$\alpha > 0, \quad 0 < \beta < \frac{\Re_{\min}(\gamma) + \alpha}{\alpha}, \quad \omega \in \mathbb{R},$$

and $\varrho(\overline{\mathcal{T}}_{\alpha,\omega}) < 1$, we have:

(1) *The spectral radius $\varrho(\overline{\mathcal{T}}_{\alpha,\beta,\omega})$ of the iteration matrix $\overline{\mathcal{T}}_{\alpha,\beta,\omega}$ is bounded by*

$$\sigma(\alpha, \beta, \omega) = |1 - \beta| + \beta \max_{1 \leqslant i \leqslant n} \frac{|\alpha - \mathrm{i}\omega - a_{ii}|}{|\alpha + \mathrm{i}\omega + a_{ii}|}. \tag{3.12}$$

(2) *If a_{ii}, $i = 1, \ldots, n$ are all real numbers with $a_{\min}$ and $a_{\max}$ being the minimum and the maximum elements of a_{ii}, $i = 1, \ldots, n$, respectively, then it holds that*

$$(\alpha^*, \beta^*, \omega^*) \equiv \arg \min_{\alpha > 0, \omega \in \mathbb{R}} \sigma(\alpha, \beta, \omega) = \Big(\sqrt{a_{\min} a_{\max}}, \frac{\Re_{\min}(\gamma) + \alpha}{2\alpha}, 0\Big). \tag{3.13}$$

Proof. Let

$$R(\alpha, \omega) = \frac{\alpha I - F + F^{\mathrm{H}} + \mathrm{i}\omega I}{\alpha I + F - F^{\mathrm{H}} - \mathrm{i}\omega I} \quad \text{and} \quad Z(\alpha, \omega) = \frac{\alpha I - D - E - F^{\mathrm{H}} - \mathrm{i}\omega I}{\alpha I + D + E + F^{\mathrm{H}} + \mathrm{i}\omega I}$$

Then

$$\mathcal{T}(\alpha, \omega) = Z(\alpha, \omega) \frac{\alpha I - F + F^{\mathrm{H}} + \mathrm{i}\omega I}{\alpha I + F - F^{\mathrm{H}} - \mathrm{i}\omega I}.$$

It is obvious that $(\alpha I - F + F^{\mathrm{H}} + \mathrm{i}\omega I)^{-1}(F - F^{\mathrm{H}}) = (F - F^{\mathrm{H}})(\alpha I - F + F^{\mathrm{H}} + \mathrm{i}\omega I)^{-1}$. By direct computations, we have $\|R(\alpha, \omega)\|_2 = R(\alpha, \omega)^{\mathrm{H}} R(\alpha, \omega) = R(\alpha, \omega) R(\alpha, \omega)^{\mathrm{H}} = I$. It holds that $R(\alpha, \omega)$ is a unitary matrix. From Lemma 3.1 in [15] we have

$$\|Z(\alpha, \omega)\|_2 = \max_{1 \leqslant i \leqslant n} \left| \frac{\alpha - \mathrm{i}\omega - a_{ii}}{\alpha + \mathrm{i}\omega + a_{ii}} \right|.$$

Thus

$$\begin{aligned}
\varrho(\overline{\mathcal{T}}_{\alpha,\beta,\omega}) \leqslant \|\overline{\mathcal{T}}_{\alpha,\beta,\omega}\|_2 &= \|(1 - \beta) I + \beta \overline{\mathcal{T}}_{\alpha,\omega}\|_2 \leqslant |1 - \beta| + |\beta| \|\overline{\mathcal{T}}_{\alpha,\omega}\|_2 \\
&= |1 - \beta| + |\beta| \|Z(\alpha, \omega) R(\alpha, \omega)\|_2 \\
&\leqslant |1 - \beta| + |\beta| \|Z(\alpha, \omega)\|_2 \\
&\leqslant |1 - \beta| + \beta \max_{1 \leqslant i \leqslant n} \left| \frac{\alpha - \mathrm{i}\omega - a_{ii}}{\alpha + \mathrm{i}\omega + a_{ii}} \right|.
\end{aligned}$$

This completes the first result. Define

$$\sigma(\alpha,\beta,\omega) = |1-\beta| + \beta \max_{1\leqslant i\leqslant n}\left|\frac{\alpha - \mathrm{i}\omega - a_{ii}}{\alpha + \mathrm{i}\omega + a_{ii}}\right|,$$

where function

$$\max_{1\leqslant i\leqslant n}\left|\frac{\alpha - \mathrm{i}\omega - a_{ii}}{\alpha + \mathrm{i}\omega + a_{ii}}\right|$$

decreases monotonically with respect to ω^2, thus

$$\omega^* = \arg \min_{\alpha>0,\omega\in\mathbb{R}} \left\{ \max_{1\leqslant i\leqslant n}\left|\frac{\alpha - \mathrm{i}\omega - a_{ii}}{\alpha + \mathrm{i}\omega + a_{ii}}\right|\right\} = 0$$

and

$$\sigma(\alpha,\beta,0) = |1-\beta| + \beta \max_{1\leqslant i\leqslant n}\left|\frac{\alpha - a_{ii}}{\alpha + a_{ii}}\right|.$$

So

$$\sigma(\alpha,\beta,0) = \left\{|1-\beta| + \beta\max\left|\frac{\alpha - a_{\min}}{\alpha + a_{\min}}\right|, |1-\beta| + \beta\max\left|\frac{\alpha - a_{\max}}{\alpha + a_{\max}}\right|\right\}.$$

If α^* is a minimum point of $\sigma(\alpha,\beta,0)$ and β^* is optimum extrapolated parameter, then they must satisfy

$$|1-\beta^*| + \beta^*\frac{\alpha^* - a_{\min}}{\alpha^* + a_{\min}} = |1-\beta^*| + \beta^*\frac{a_{\max} - \alpha^*}{a_{\max} + \alpha^*},$$
$$\frac{\alpha^* - a_{\min}}{\alpha^* + a_{\min}} = \frac{a_{\max} - \alpha^*}{a_{\max} + \alpha^*} \Rightarrow \alpha^* = \sqrt{a_{\max}a_{\min}}.$$

□

Finally, in the next section some numerical examples are reported to illustrate the effectiveness of the proposed methods.

4. Numerical experiments

In this section, we use the test problems to illustrate the effectiveness of the EHPPS and ETPPS iterative methods. We use parameter η for EHPPS and HPPS iterative methods and we also use parameter ω for ETPPS and TPPS iterative methods. We compare the performance of the EHPPS and ETPPS iterative methods with those of the HPPS and TPPS methods from the point of view of the iteration counts (denoted as "IT"), CPU times (denoted as "CPU") and the spectral radius (denoted as "ϱ"). The optimal value of α (denoted as "α_{opt}") and the value of β (denoted as "$\beta_{\exp}$") were found experimentally. In each iteration of the EHPPS and ETPPS methods, we use the LU factorization of a coefficient matrices to solve the subsystems. The numerical experiments were computed in double precision and the algorithms were implemented in Matlab R2016b on a PC computer with Intel(R) Core (TM) i7-7700k CPU 4.20GHz, and 8.00 GB memory with machine precision and Windows 10 operating system. In our implementations, the initial guess $u(0)$ is chosen to be a zero vector and the stopping criteria for all the methods is

$$\frac{\|b - \mathcal{A}u^{(p)}\|_2}{\|b\|_2} < 10^{-5}.$$

In our tests, we take $h = 1/(m+1)$ and $n = m^2$.

Example 4.1 ([2], [3], [20]). Consider the complex symmetric $(W + \mathrm{i}T)u = b$ is given by

$$W = \Bigl(B_n + \frac{3-\sqrt{3}}{m+1} I_n\Bigr) \in \mathbb{R}^{n\times n}, \quad T = B_n + \frac{3+\sqrt{3}}{m+1} I_n \in \mathbb{R}^{n\times n},$$

where

$$B_n = I_m \otimes V_m + V_m \otimes I_m \in \mathbb{R}^{n\times n} \quad \text{and} \quad V_m = \operatorname{tridiag}(-1, 2, -1) \in \mathbb{R}^{m\times m}.$$

The right-hand side vector b with its jth entry b_s is given by

$$b_s = \frac{(1-i)s}{(m+1)(s+1)^2}, \quad s = 1, 2, \ldots, n.$$

This complex symmetric system of linear equations arises in five-point centered difference discretizations of the R22–Padé approximations in the time integration of parabolic partial differential equations. The numerical results are given in Tables 1–4.

$m = 16$			HPPS				EHPPS	
η	$\alpha_{\rm opt}$	IT	CPU	$\varrho(\mathcal{T}_{\alpha,\beta})$	$\beta_{\rm exp}$	IT	CPU	$\varrho(\mathcal{T}_{\alpha,\beta,\eta})$
-8	7.60	576	0.0280	0.9823	3.48	187	0.0101	0.9465
-6	5.75	432	0.0229	0.9765	3.36	151	0.0082	0.9341
-4	3.75	289	0.0156	0.9650	3.30	112	0.0074	0.9120
-2	2	144	0.0076	0.9312	2.90	78	0.0041	0.8753
0	1	36	0.0021	0.7779	1.12	36	0.0019	0.7497
2	2	144	0.0076	0.9312	1.34	118	0.0065	0.9163
4	3.75	289	0.0156	0.9650	1.31	228	0.0115	0.9559
6	5.75	432	0.0229	0.9765	1.37	323	0.0169	0.9547
8	7.60	576	0.0280	0.9823	1.44	409	0.0197	0.9752

Table 1. Numerical results of HPPS and EHPPS for Example 4.1 when $m = 16$.

$m=16$			TPPS				ETPPS	
ω	α_{opt}	IT	CPU	$\varrho(\overline{\mathcal{T}}_{\alpha,\beta})$	β_{exp}	IT	CPU	$\varrho(\overline{\mathcal{T}}_{\alpha,\beta,\omega})$
-8	3.90	266	0.0118	0.9626	2.10	142	0.0072	0.9300
-6	1.85	123	0.0064	0.9216	1.60	83	0.0052	0.8852
-4	1.10	54	0.0044	0.8249	1.20	51	0.0039	0.8146
-2	2.55	171	0.0081	0.9416	1.37	134	0.0069	0.9260
0	4.20	312	0.0126	0.9675	1.37	237	0.0106	0.9577
2	2.55	655	0.0248	0.9844	1.23	538	0.0208	0.9810
4	1.10	2299	0.1025	0.9955	1.08	2131	0.0875	0.9952
6	1.85	2133	0.0854	0.9952	1.13	1891	0.0741	0.9878
8	3.90	1537	0.0621	0.9933	1.35	1146	0.0448	0.9768

Table 2. Numerical results of TPPS and ETPPS for Example 4.1 when $m=16$.

$m=32$			HPPS				EHPPS	
η	α_{opt}	IT	CPU	$\varrho(\mathcal{T}_{\alpha,\beta})$	β_{exp}	IT	CPU	$\varrho(\mathcal{T}_{\alpha,\beta,\eta})$
-8	8.0	1353	0.3848	0.9930	3.45	419	0.0104	0.9774
-6	5.9	1015	0.2588	0.9906	3.40	326	0.0821	0.9745
-4	4	677	0.4490	0.9860	3.30	233	0.0589	0.9598
-2	1.95	339	0.2384	0.9721	3.21	138	0.0355	0.9328
0	0.67	57	0.0441	0.8458	1.18	55	0.0123	0.8401
2	1.95	339	0.2384	0.9721	1.3	270	0.0682	0.9652
4	4	677	0.4490	0.9860	1.25	549	0.1502	0.9827
6	5.9	1015	0.2588	0.9906	1.40	736	0.2024	0.9856
8	8.0	1353	0.3848	0.9930	1.48	926	0.2457	0.9897

Table 3. Numerical results of HPPS and EHPPS for Example 4.1 when $m=32$.

$m=32$			TPPS				ETPPS	
ω	α_{opt}	IT	CPU	$\varrho(\overline{\mathcal{T}}_{\alpha,\beta})$	β_{exp}	IT	CPU	$\varrho(\overline{\mathcal{T}}_{\alpha,\beta,\omega})$
-8	3.7	642	0.1103	0.9854	2.10	324	0.0546	0.9712
-6	2.25	309	0.0519	0.9702	1.74	191	0.0333	0.9520
-4	0.75	74	0.0144	0.8657	1.14	70	0.0123	0.8550
-2	2.2	364	0.0607	0.9737	1.34	280	0.0476	0.9662
0	4.0	704	0.1202	0.9864	1.42	507	0.0845	0.9813
2	2.2	1640	0.2998	0.9942	1.21	1362	0.2517	0.9930
4	0.75	7558	1.2381	0.9987	1.055	7166	1.1614	0.9971
6	2.25	4068	0.6872	0.9976	1.162	3506	0.5719	0.9962
8	3.7	3693	0.5978	0.9974	1.36	2725	0.4427	0.9946

Table 4. Numerical results of TPPS and ETPPS for Example 4.1 when $m=32$.

E x a m p l e 4.2 ([2], [3], [20]). Consider the complex symmetric system (1.1) with the coefficient matrix $\mathcal{A} = W + \mathrm{i}T$ given by

$$W = \Big(\frac{-\pi^2 I_n}{(m+1)^2} + B_n\Big) \in \mathbb{R}^{n\times n}, \quad T = (m+1)^{-2}(10\pi I_n + 0.02 B_n) \in \mathbb{R}^{n\times n},$$

where

$$B_n = I \otimes V_m + V_m \otimes I \in \mathbb{R}^{n\times n} \quad \text{and} \quad V_m = \operatorname{tridiag}(-1, 2, -1) \in \mathbb{R}^{m\times m}.$$

The right-hand side vector is $b = (1+\mathrm{i})\mathcal{A}1/(m+1)^2$, with 1 being the vector of all entries equal to 1. This complex symmetric system of linear equations arises in direct frequency domain analysis of an n-degree-of-freedom (n-DOF) linear system. The numerical results are listed in Tables 5–8.

$m = 16$			HPPS				EHPPS	
η	$\alpha_{\rm opt}$	IT	CPU	$\varrho(\mathcal{T}_{\alpha,\beta})$	$\beta_{\rm exp}$	IT	CPU	$\varrho(\mathcal{T}_{\alpha,\beta,\eta})$
-8	8	2318	0.1092	0.9958	3.09	784	0.0405	0.9875
-6	6	1738	0.0777	0.9944	2.6	699	0.0343	0.9860
-4	4	1159	0.0616	0.9915	2	604	0.0302	0.9838
-2	2	580	0.0332	0.9832	1.56	389	0.0198	0.9750
0	0.42	67	0.0040	0.8990	0.96	62	0.0031	0.8515
2	2	580	0.0332	0.9832	1.43	422	0.0232	0.9769
4	4	1159	0.0616	0.9915	1.95	620	0.0393	0.9842
6	6	1738	0.0777	0.9944	2.46	737	0.0372	0.9867
8	8	2318	0.1092	0.9958	2.95	820	0.0465	0.9881

Table 5. Numerical results of HPPS and EHPPS for Example 4.2 when $m = 16$.

$m = 16$			TPPS				ETPPS	
ω	$\alpha_{\rm opt}$	IT	CPU	$\varrho(\overline{\mathcal{T}}_{\alpha,\beta})$	$\beta_{\rm exp}$	IT	CPU	$\varrho(\overline{\mathcal{T}}_{\alpha,\beta,\omega})$
-8	7.8	2246	0.1157	0.9956	2.7	862	0.0315	0.9886
-6	5.6	1661	0.0778	0.9941	2.1	815	0.0305	0.9880
-4	3.6	1070	0.0405	0.9908	1.75	630	0.0239	0.9840
-2	1.5	461	0.0184	0.9785	1.4	355	0.0141	0.9707
0	0.48	110	0.0110	0.9050	1.062	105	0.0104	0.8964
2	1.5	581	0.0225	0.9829	1.38	429	0.0171	0.9769
4	3.6	1185	0.0494	0.9917	1.7	716	0.0270	0.9863
6	5.6	1775	0.0688	0.9945	2.15	852	0.0306	0.9885
8	7.8	2357	0.0991	0.9958	2.64	923	0.0338	0.9894

Table 6. Numerical results of TPPS and ETPPS for Example 4.2 when $m = 16$.

$m = 32$		HPPS				EHPPS		
η	$\alpha_{\rm opt}$	IT	CPU	$\varrho(\mathcal{T}_{\alpha,\beta})$	$\beta_{\rm exp}$	IT	CPU	$\varrho(\mathcal{T}_{\alpha,\beta,\eta})$
-8	7.50	7848	2.0105	0.9989	3.06	2594	0.6923	0.9966
-6	5.15	5943	1.5187	0.9985	2.52	2384	0.6284	0.9963
-4	3.92	3917	1.0116	0.9977	2.04	1942	0.4912	0.9954
-2	2.40	1991	0.5254	0.9956	1.53	1316	0.3496	0.9933
0	0.23	120	0.0323	0.9439	0.972	115	0.0272	0.9255
2	2.40	1991	0.5254	0.9956	1.47	1368	0.3511	0.9935
4	3.92	3917	1.0116	0.9977	1.95	2030	0.5505	0.9956
6	5.15	5943	1.5187	0.9985	2.47	2432	0.6510	0.9964
8	7.50	7848	2.0105	0.9989	2.96	2680	0.7190	0.9967

Table 7. Numerical results of HPPS and EHPPS for Example 4.2 when $m = 32$.

$m = 32$		TPPS				ETPPS		
ω	$\alpha_{\rm opt}$	IT	CPU	$\varrho(\overline{\mathcal{T}}_{\alpha,\beta})$	$\beta_{\rm exp}$	IT	CPU	$\varrho(\overline{\mathcal{T}}_{\alpha,\beta,\omega})$
-8	7.8	7710	1.2260	0.9989	2.697	2886	0.4702	0.9969
-6	5.8	5747	0.9159	0.9985	2.19	2647	0.4349	0.9967
-4	3.8	3779	0.6168	0.9977	1.77	2153	0.3558	0.9959
-2	1.8	1790	0.3347	0.9950	1.46	1238	0.2347	0.9928
0	0.25	201	0.1166	0.9501	1.035	195	0.1043	0.9486
2	1.8	2016	0.3408	0.9956	1.453	1400	0.2423	0.9937
4	3.8	4001	0.9772	0.9978	1.75	2304	0.4115	0.9962
6	5.8	5967	0.9784	0.9985	2.16	2785	0.4801	0.9968
8	7.8	7928	1.2988	0.9989	2.67	2996	0.4870	0.9970

Table 8. Numerical results of TPPS and ETPPS for Example 4.2 when $m = 32$.

Example 4.3 ([2], [3], [20]). Consider the system of linear equations (1.1), where $\mathcal{A} = W + \mathrm{i}T$ with

$$W = 10(I_m \otimes B_m + B_m \otimes I_m) + 9(e_1 e_m^\top + e_m e_1^\top) \otimes I_m \in \mathbb{R}^{n\times n},$$
$$T = I_m \otimes V_m + V_m \otimes I_m \in \mathbb{R}^{n\times n},$$

where

$$V_m = \mathrm{tridiag}(-1, 2, -1) \in \mathbb{R}^{m\times m}, \quad B_m = V - e_1 e_m^\top - e_m e_1^\top \in \mathbb{R}^{m\times m},$$

and e_1 and e_m are the first and the last unit vectors in $\in \mathbb{R}^m$, respectively. The right-hand side vector b is defined as $b = (1 + \mathrm{i})\mathcal{A}1$ with 1 being the vector of all entries equal to 1. In Tables 9–12, the numerical results are shown.

$m = 16$			HPPS				EHPPS	
η	α_{opt}	IT	CPU	$\varrho(\mathcal{T}_{\alpha,\beta})$	$\beta_{\exp}$	IT	CPU	$\varrho(\mathcal{T}_{\alpha,\beta,\eta})$
-8	7.7	210	0.0142	0.9563	1.37	152	0.0114	0.9403
-6	6.3	158	0.0119	0.9422	1.27	124	0.0095	0.9269
-4	3.5	107	0.0087	0.9158	1.01	106	0.0079	0.9149
-2	4.0	73	0.0066	0.9041	0.95	76	0.0062	0.8957
0	5.4	71	0.0061	0.8762	1.01	70	0.0058	0.8750
2	4.0	73	0.0066	0.9041	0.95	70	0.0062	0.8788
4	3.5	106	0.0087	0.9155	1.002	105	0.0080	0.9152
6	6.3	158	0.0119	0.9422	1.05	150	0.0113	0.9394
8	7.7	210	0.0142	0.9563	1.072	195	0.0131	0.9532

Table 9. Numerical results of HPPS and EHPPS for Example 4.3 when $m = 16$.

$m = 16$			TPPS				ETPPS	
ω	α_{opt}	IT	CPU	$\varrho(\overline{\mathcal{T}}_{\alpha,\beta})$	$\beta_{\exp}$	IT	CPU	$\varrho(\overline{\mathcal{T}}_{\alpha,\beta,\omega})$
-8	10.25	262	0.0139	0.9628	1.17	230	0.0127	0.9565
-6	9.75	247	0.0137	0.9603	1.16	213	0.0124	0.9540
-4	9.75	242	0.0130	0.9595	1.15	210	0.0116	0.9535
-2	10	248	0.0136	0.9606	1.142	217	0.0121	0.9551
0	4.4	387	0.0499	0.9738	1.078	359	0.0459	0.9717
2	10	292	0.0162	0.9667	1.16	251	0.0141	0.9614
4	9.75	331	0.0172	0.9707	1.16	286	0.0153	0.9660
6	9.75	379	0.0193	0.9746	1.16	327	0.0172	0.9705
8	10.25	429	0.0223	0.9777	1.17	366	0.0193	0.9739

Table 10. Numerical results of TPPS and ETPPS for Example 4.3 when $m = 16$.

$m = 32$			HPPS				EHPPS	
η	α_{opt}	IT	CPU	$\varrho(\mathcal{T}_{\alpha,\beta})$	$\beta_{\exp}$	IT	CPU	$\varrho(\mathcal{T}_{\alpha,\beta,\eta})$
-8	8.5	604	0.2118	0.9858	1.44	418	0.1541	0.9757
-6	6	452	0.1616	0.9811	1.40	323	0.1228	0.9735
-4	4.5	304	0.1132	0.9720	1.33	229	0.0863	0.9632
-2	2.8	161	0.0662	0.9475	1.10	146	0.0543	0.9433
0	3.5	133	0.0445	0.9368	1.06	125	0.0393	0.9331
2	2.8	160	0.0620	0.9475	1.03	156	0.0596	0.9460
4	4.5	304	0.1132	0.9720	1.07	290	0.1062	0.9701
6	6	452	0.1616	0.9811	1.07	422	0.1535	0.9798
8	8.5	603	0.2117	0.9858	1.09	553	0.2034	0.9846

Table 11. Numerical results of HPPS and EHPPS for Example 4.3 when $m = 32$.

$m=32$		TPPS				ETPPS		
ω	α_{opt}	IT	CPU	$\varrho(\overline{\mathcal{T}}_{\alpha,\beta})$	$\beta_{\exp}$	IT	CPU	$\varrho(\overline{\mathcal{T}}_{\alpha,\beta,\omega})$
-8	8	591	0.1774	0.9842	1.13	523	0.1459	0.9821
-6	9	543	0.1440	0.9828	1.155	470	0.1199	0.9802
-4	9.5	532	0.1530	0.9826	1.17	457	0.1233	0.9796
-2	8.12	542	0.4137	0.9827	1.14	476	0.3666	0.9803
0	8.5	589	0.4430	0.9842	1.15	512	0.3844	0.9819
2	8.12	675	0.5094	0.9863	1.14	592	0.4498	0.9844
4	9.5	770	0.2003	0.9883	1.17	659	0.1849	0.9863
6	9	921	0.2301	0.9903	1.155	797	0.2021	0.9888
8	8	1162	0.3287	0.9923	1.13	1028	0.2681	0.9913

Table 12. Numerical results of TPPS and ETPPS for Example 4.3 when $m=32$.

In three examples, we compare the EPPS iterative method with the PPS iteration method. Numerical results for Examples 1–3 are listed in Tables 1–12. In Tables 1, 3, 5, 7, 9 and 11 we compare the EHPPS iterative method with the HPPS iterative method and we observe that the performance of EHPPS iterative method is better than the HPPS iterative method from the point of view of spectral radius, iteration numbers and CPU time. By comparing the results in Tables 2, 4, 6, 8, 10 and 12, it can be seen that the convergence of the ETPPS iterative method is faster than the TPPS iterative method. Besides, we see that the iteration numbers and the CPU time ETPPS iterative method are less than in the case of TPPS iterative method.

5. Conclusions

We construct two new methods, called EHPPS and ETPPS iterative methods which were obtained from the combination of PPS splitting iteration method and the extrapolated method, for solving sparse non-Hermitian positive definite linear system. We demonstrate that our methods converge to the unique solution of (1.1). An upper bound for the spectral radius of the iteration matrix is derived and also, we obtain an upper bound for the extrapolation parameter. We compare the numerical results of the EPPS iterative method with the PPS iterative method. Numerical results show that the EPPS method is superior to the other methods in terms of the iteration counts, the CPU time and the spectral radius.

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