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L^p-IMPROVING PROPERTIES OF CERTAIN SINGULAR MEASURES ON THE HEISENBERG GROUP

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Abstract. Let μ_A be the singular measure on the Heisenberg group $\mathbb{H}^n$ supported on the graph of the quadratic function $\varphi(y) = y^t Ay$, where A is a $2n \times 2n$ real symmetric matrix. If $\det(2A \pm J) \neq 0$, we prove that the operator of convolution by μ_A on the right is bounded from $L^{(2n+2)/(2n+1)}(\mathbb{H}^n)$ to $L^{2n+2}(\mathbb{H}^n)$. We also study the type set of the measures $\mathrm{d}\nu_\gamma(y,s) = \eta(y)|y|^{-\gamma}\mathrm{d}\mu_A(y,s)$, for $0 \leqslant \gamma < 2n$, where η is a cut-off function around the origin on $\mathbb{R}^{2n}$. Moreover, for $\gamma = 0$ we characterize the type set of ν_0.

Keywords: Heisenberg group; singular Borel measure; L^p-improving property

MSC 2020: 43A80, 42A38

1. Introduction

Let I_n be the $n \times n$ identity matrix and J be the $2n \times 2n$ skew-symmetric matrix given by

$$J = \begin{pmatrix} 0 & I_n \\ -I_n & 0 \end{pmatrix}. \tag{1}$$

The Heisenberg group is $\mathbb{H}^n = \mathbb{R}^{2n} \times \mathbb{R}$ endowed with the group law (non-commutative)

$$(x,t) \cdot (y,s) = (x + y, t + s + \langle x, y \rangle),$$

where $\langle x, y \rangle$ is the standard symplectic form on $\mathbb{R}^{2n}$, i.e. $\langle x, y \rangle = x^t Jy$ with neutral element $(0,0)$ and with inverse $(x,t)^{-1} = (-x,-t)$. The topology in $\mathbb{H}^n$ is induced by $\mathbb{R}^{2n+1}$, so the borelian sets of $\mathbb{H}^n$ are identified with those of $\mathbb{R}^{2n+1}$. The Haar measure in $\mathbb{H}^n$ is the Lebesgue measure of $\mathbb{R}^{2n+1}$, thus $L^p(\mathbb{H}^n) \equiv L^p(\mathbb{R}^{2n+1})$. Given

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a borelian function $f\colon \mathbb{H}^n \to \mathbb{C}$ and a Borel measure μ on $\mathbb{H}^n$, define the convolution by μ on the right by

$$(2)\qquad (f * \mu)(x,t) = \int_{\mathbb{H}^n} f((x,t)\cdot(y,s)^{-1})\,\mathrm{d}\mu(y,s),$$

provided the integral exists.

A Borel measure μ on the Heisenberg group $\mathbb{H}^n$ is said to be *L^p-improving* if the operator $T_\mu : f \mapsto f*\mu$ is bounded from $L^p(\mathbb{H}^n)$ into $L^q(\mathbb{H}^n)$ for some $1 \leqslant p < q < \infty$. A remarkable fact is that singular measures can be L^p-improving. If in (2) we replace the Heisenberg group $\mathbb{H}^n$ by $\mathbb{R}^n$ with the ordinary convolution in $\mathbb{R}^n$ and considering there $\mu = \eta\sigma_M$, where σ_M is the surface measure on a given manifold M (in $\mathbb{R}^n$) and η is a smooth cut-off function, then the L^p-improving properties of a measure of this type are closely related to the existence of a certain amount of curvature of the manifold M (see [5], [6], [7]). A similar result holds on general Lie groups (see Theorem 1.1, page 362 in [9]).

A more delicate problem consists in determining the exact range of pairs (p,q) for which $L^p * \mu \subseteq L^q$ embeds continuously. Given a manifold M (in $\mathbb{H}^n$), define the *type set* $E_{\eta\sigma_M}$ by

$$E_{\eta\sigma_M} = \Big\{\Big(\frac{1}{p},\frac{1}{q}\Big) \in [0,1]\times[0,1]\colon\ \|T_{\eta\sigma_M}\|_{p,q} < \infty\Big\}.$$

A very interesting survey of results concerning the type sets for convolution operators with singular measures in $\mathbb{R}^n$ can be found in [8].

In the $\mathbb{H}^n$ setting, Secco in [10] and [11] obtained L^p-improving properties of measures supported on curves in $\mathbb{H}^1$, under certain assumptions. In [9], Ricci and Stein showed that the type set of the measure given by (3) for the case $\varphi \equiv 0$, $\gamma = 0$ and $n = 1$ is the triangle with vertices $(0,0)$, $(1,1)$ and $(\frac{3}{4},\frac{1}{4})$. In [3] and [4], the author jointly with Godoy generalized the work of Ricci and Stein for the case $\varphi(w) = w^tAw = \sum\limits_{j=1}^{n}\alpha_j|w_j|^2$, where A is a $2n\times 2n$ real diagonal matrix such that $a_{ii} = a_{(i+1)(i+1)}$ for $i = 2j-1$ with $j = 1,2,\ldots,n$, $\alpha_j = a_{(2j-1)(2j-1)}$, $w_j \in \mathbb{R}^2$, $0 \leqslant \gamma < 2n$ and $n \in \mathbb{N}$. There we also gave some examples of surfaces with degenerate curvature at the origin.

Let $\varphi\colon \mathbb{R}^{2n} \to \mathbb{R}$ be the function defined by $\varphi(y) = y^tAy$, where A is a $2n\times 2n$ real symmetric matrix. It is well known that if A is an arbitrary matrix, then there exists a symmetric matrix $\tilde{A}$ such that $y^tAy = y^t\tilde{A}y$ for all y. We consider two borelian measures on $\mathbb{H}^n$ supported on the graph of φ, μ_A and ν_γ, $0 \leqslant \gamma < 2n$, given by

$$\mu_A(E) = \int_{\mathbb{R}^{2n}} \chi_E(y,\varphi(y))\,\mathrm{d}y$$

and

$$\nu_\gamma(E) = \int_{\mathbb{R}^{2n}} \chi_E(y, \varphi(y))\eta(y)|y|^{-\gamma}\, \mathrm{d}y, \tag{3}$$

where $\eta : \mathbb{R}^{2n} \to [0,1]$ is a smooth cut-off function such that $\eta(y) = 1$ if $|y| \leqslant 1$, $\eta(y) = 0$ if $|y| \geqslant 2$, and E is a borelian set of $\mathbb{H}^n$. Let $T_{\mu_A} f = f * \mu_A$ and $T_{\nu_\gamma} f = f * \nu_\gamma$ be the operators of convolution by μ_A and ν_γ on the right, respectively.

We are interested in studying the L^p-improving properties of the operator T_{μ_A} and in the characterization of the type set E_{ν_γ}. We point out that our measure μ_A is not the surface measure on the graph $\mathrm{gr}(\varphi)$ of φ, however the measures $\eta\mu_A$ and $\eta\sigma_{\mathrm{gr}(\varphi)}$ are equivalent, see Proposition 2 below, so $E_{\eta\mu_A} = E_{\eta\sigma_{\mathrm{gr}(\varphi)}}$.

The following restrictions for the type sets E_{ν_γ}, $0 \leqslant \gamma < 2n$, were proved in [3] and [4] for the case $\varphi(w_1, \ldots, w_n) = \sum_{j=1}^{n} \alpha_j |w_j|^2$ with $w_j \in \mathbb{R}^2$. It is easy to see that such an argument works as well for our function $\varphi(y) = y^t A y$. Thus, if $(1/p, 1/q) \in E_{\nu_\gamma}$, $0 \leqslant \gamma < 2n$, then

$$p \leqslant q, \quad \frac{1}{q} \geqslant \frac{2n+1}{p} - 2n, \quad \frac{1}{q} \geqslant \frac{1}{(2n+1)p}. \tag{4}$$

Another necessary condition for the pair $(1/p, 1/q)$ to be in E_{ν_γ} is the following:

$$\frac{1}{q} \geqslant \frac{1}{p} - \frac{2n-\gamma}{2n+2}. \tag{5}$$

This last condition is relevant only for the case $0 < \gamma < 2n$. Let D be the point of intersection, in the $(1/p, 1/q)$ plane, of the lines $1/q = (2n+1)/p - 2n$, $1/q = 1/p - (2n-\gamma)/(2n+2)$, and let D' be its symmetric image with respect to the symmetry axis $1/q = 1 - 1/p$. So

$$D = \Big(\frac{4n^2 + 2n + \gamma}{2n(2n+2)}, \frac{2n + (2n+1)\gamma}{2n(2n+2)}\Big) = \Big(\frac{1}{p_D}, \frac{1}{q_D}\Big) \quad \text{and} \quad D' = \Big(1 - \frac{1}{q_D}, 1 - \frac{1}{p_D}\Big).$$

Since $0 \leqslant \gamma < 2n$, it is clear that $\|T_{\nu_\gamma} f\|_p \leqslant c\|f\|_p$ for all Borel functions $f \in L^p(\mathbb{H}^n)$ and all $1 \leqslant p \leqslant \infty$, so $(1/p, 1/p) \in E_{\mu_\gamma}$. Thus, for $0 < \gamma < 2n$ the set E_{ν_γ} is contained in the closed trapezoid with vertices $(0,0)$, $(1,1)$, D and D', and the set E_{ν_0} is contained in the closed triangle with vertices $(0,0)$, $(1,1)$ and $((2n+1)/(2n+2), 1/(2n+2))$.

In Section 3, our main result appears. There we prove that the operator T_{μ_A} is bounded from $L^{(2n+2)/(2n+1)}(\mathbb{H}^n)$ to $L^{2n+2}(\mathbb{H}^n)$, see Theorem 3 below. This result allows us to characterize the type set E_{ν_0} as well as the interior of E_{ν_γ} for $0 < \gamma < 2n$.

More precisely, we show that E_{ν_0} is the closed triangle with vertices $(0,0)$, $(1,1)$ and $((2n+1)/(2n+2), 1/(2n+2))$ and the interior of E_{ν_γ} coincides with the interior of the closed trapezoid with vertices $(0,0)$, $(1,1)$, D and D', see Theorem 4 and Theorem 6 below.

Throughout this paper, c will denote a positive real constant not necessarily the same at each occurrence. The symbol $A \lesssim B$ stands for the inequality $A \leqslant cB$ for a constant c. We use the following convention for the Fourier transform in $\mathbb{R}^n$ $\hat{f}(\xi) = \int f(x) \mathrm{e}^{-\mathrm{i}\xi\cdot x}\,\mathrm{d}x$. The Fourier transform $\widehat{u}$ of a distribution u on $\mathbb{R}^n$ is the distribution defined by $(\widehat{u}, \varphi) = (u, \widehat{\varphi})$ for all rapidly decreasing functions φ on $\mathbb{R}^n$.

2. Preliminaries

In the sequel J will denote the $2n \times 2n$ skew-symmetric matrix defined in (1). It is easy to check that

(a) $J^2 = -I$,
(b) $J^t = -J$,
(c) $x^t J x = 0$ for all $x \in \mathbb{R}^{2n}$,
(d) $x^t J y = -y^t J x$ for all $x, y \in \mathbb{R}^{2n}$.

Lemma 1. *Let A be a $2n \times 2n$ real diagonal matrix. Then*

$$\det(A \pm J) = (a_{11}a_{(n+1)(n+1)} + 1) \cdot (a_{22}a_{(n+2)(n+2)} + 1) \ldots (a_{nn}a_{(2n)(2n)} + 1),$$

where the a_{ii}'s are the diagonal entries of A.

Proof. Since $\det(A+J) = \det((A+J)^t) = \det(A-J)$, it is sufficient to prove the statement of the lemma for $\det(A+J)$. Applying induction on n, the lemma follows. □

Proposition 2. *Let A be a $2n \times 2n$ real symmetric matrix. Then the graph of the function $\varphi(y) = y^t A y$ generates all the group $\mathbb{H}^n$. Moreover, the measure $\nu_0 = \eta\mu_A$ is equivalent to the measure $\eta\sigma$, where η is a cut-off function and σ is the surface measure on the graph of φ.*

Proof. The first statement will follow if we prove that $(x,0)$ and $(0,t)$ belong to the set $G_{\mathrm{gr}(\varphi)}$ generated by the graph $\mathrm{gr}(\varphi)$ of φ, since $(x,t) = (x,0)\cdot(0,t)$. It is clear that $(x,\varphi(x)) \in G_{\mathrm{gr}(\varphi)}$, so $(-t^{1/2}x, \varphi(t^{1/2}x)) = (-t^{1/2}x, \varphi(-t^{1/2}x)) \in G_{\mathrm{gr}(\varphi)}$ for all $x \in \mathbb{R}^{2n}$ and all $t > 0$. From that it follows that $(0, t\varphi(x)) \in G_{\mathrm{gr}(\varphi)}$ for all $t > 0$ and all x. If A is a non-null matrix, then $(0,-t) = (0,t)^{-1} \in G_{\mathrm{gr}(\varphi)}$ and $(x,0) = (x,\varphi(x)) \cdot (0, -\varphi(x)) \in G_{\mathrm{gr}(\varphi)}$. If A is the null matrix, it is sufficient to prove that $(0,t) \in G_{\mathrm{gr}(\varphi)}$ for all t. Indeed, for x and y such that $\langle x, y\rangle \neq 0$ we have $(0,t) = (x,0)\cdot(ty/\langle x,y\rangle, 0)\cdot(-x-ty/\langle x,y\rangle,0) \in G_{\mathrm{gr}(\varphi)}$. So $G_{\mathrm{gr}(\varphi)} = \mathbb{H}^n$.

For the second part of the proposition, we have that the surface measure on the graph of φ is given by

$$\sigma(E) = \int_{\varphi^{-1}(E)} \sqrt{\det[(\partial_{x_i}\varphi, \partial_{x_j}\varphi)_x]}\,\mathrm{d}x,$$

where $\varphi(x) = (x, \varphi(x))$ and E is a borelian set of $\mathbb{R}^{2n+1}$ (see pages 43–45 in [1]). A computation gives

$$\det[(\partial_{x_i}\varphi, \partial_{x_j}\varphi)_x] = 1 + \sum_{j=1}^{2n} (\partial_{x_j}\varphi(x))^2 \quad \forall\, x.$$

So

$$\int_{\mathbb{R}^{2n}} \chi_E(\varphi(x))\eta(x)\,\mathrm{d}x \leqslant \int_{\varphi^{-1}(E)} \sqrt{\det[(\partial_{x_i}\varphi, \partial_{x_j}\varphi)_x]}\eta(x)\,\mathrm{d}x \lesssim \int_{\mathbb{R}^{2n}} \chi_E(\varphi(x))\eta(x)\,\mathrm{d}x.$$

Then ν_0 is equivalent to $\eta\sigma$. □

The λ-twisted convolution is defined by

$$(f \times_\lambda g)(x) = \int_{\mathbb{R}^{2n}} f(x-y)g(y)\mathrm{e}^{-\mathrm{i}\lambda x^t J y}\,\mathrm{d}y.$$

Given a $2n\times 2n$ real symmetric matrix A, we put

$$e_A(x) = \mathrm{e}^{\mathrm{i}x^t A x}.$$

It is easy to check, using the properties (b) and (c) of the matrix J, that

$$(f \times_\lambda e_{\lambda A})(x) = e_{\lambda A}(x)(e_{\lambda A}(\cdot)f(\cdot))\widehat{\ }(\lambda(2A+J)x),$$

where $\hat{f}(\xi) = \int_{\mathbb{R}^{2n}} f(x)\mathrm{e}^{-\mathrm{i}x\cdot\xi}\,\mathrm{d}x$ is the Fourier transform of f. Thus, for each $f \in L^1(\mathbb{R}^{2n}) \cap L^2(\mathbb{R}^{2n})$ we have

$$\|f \times_\lambda e_{\lambda A}\|_{L^2(\mathbb{R}^{2n})} = (2\pi)^n |\lambda|^{-n} |\det(2A \pm J)|^{-1/2} \|f\|_{L^2(\mathbb{R}^{2n})} \tag{6}$$

if $\det(2A \pm J) \neq 0$.

3. Main result

To prove the $L^{(2n+2)/(2n+1)}(\mathbb{H}^n) - L^{2n+2}(\mathbb{H}^n)$ boundedness of the operator T_{μ_A} we embed our operator in an analytic family $\{T_z\}$ of operators on the strip $-n \leqslant \Re(z) \leqslant 1$, and then we apply the complex interpolation theorem.

Theorem 3. *If* $\det(2A \pm J) \neq 0$, *then the operator* T_{μ_A} *is bounded from* $L^{(2n+2)/(2n+1)}(\mathbb{H}^n)$ *to* $L^{2n+2}(\mathbb{H}^n)$.

P r o o f. To prove the statement of the theorem we consider the family $\{|s|^{z-1}\}$ of functions initially defined when $\Re(z) > 0$ and $s \in \mathbb{R}\backslash\{0\}$. This family of functions can be extended in the z variable to an analytic family of distributions on $\mathbb{C} \setminus \{-2k\colon k \in \mathbb{N} \cup \{0\}\}$. By abuse of notation, we denote this extension by $|s|^{z-1}$. The family $\{|s|^{z-1}\}$ has simple poles in $z = -2k$ for $k \in \mathbb{N} \cup \{0\}$. Since the meromorphic continuation of the function $\Gamma(\frac{1}{2}z)$ (we keep the notation for his continuation) has simple poles at the same points (i.e. $z = -2k$), the family $\{I_z\}$ of distributions defined by

$$I_z(s) = \frac{2^{-z/2}}{\Gamma\left(\frac{1}{2}z\right)}|s|^{z-1} \tag{7}$$

results in an entire family of distributions (see pages 55–56 in [2]).

From this construction and by taking the ratios of the corresponding residues at $z = 0$, we have $I_0 = \delta$, where δ is the Dirac distribution at the origin on $\mathbb{R}$ (see equation (3), page 57 in [2]), also $\widehat{I_z} = cI_{1-z}$ for a real constant c independent of z (see equation (12′), page 173 in [2]).

For $z \in \mathbb{C}$, we also define U_z as the distribution on $\mathbb{H}^n$ given by the tensor product

$$U_z = \delta_{\mathbb{R}^{2n}} \otimes I_z,$$

where $\delta_{\mathbb{R}^{2n}}$ is the Dirac distribution at the origin on $\mathbb{R}^{2n}$ and I_z is given by (7). Let $\{T_z\}$ be the analytic family of operators on the strip $-n \leqslant \Re(z) \leqslant 1$, given by

$$T_z f = f * \mu_A * U_z.$$

It is clear that $T_0 = T_{\mu_A}$. For $\Re(z) = 1$ we have

$$\|T_z f\|_\infty = \|f * \mu_A * U_z\|_\infty \leqslant \|f\|_1 \|\mu_A * U_z\|_\infty.$$

Since $\mu_A * U_{1+ib}(x,t) = I_{1+ib}(t - \varphi(x)) = (2^{-(1+\mathrm{i}b)/2}/\Gamma(\frac{1}{2}(1+\mathrm{i}b)))|t - \varphi(x)|^{\mathrm{i}b}$, it follows that

$$\|T_{1+\mathrm{i}b}\|_{1,\infty} \leqslant \left|\frac{2^{-(1+\mathrm{i}b)/2}}{\Gamma(\frac{1}{2}(1+\mathrm{i}b))}\right| \quad \forall\, b \in \mathbb{R}.$$

For $\Re(z) = -n$ we will prove that the operator T_z is bounded on $L^2(\mathbb{H}^n)$. This is equivalent to showing that

$$\int_{\mathbb{R}^{2n}} |(T_z f)^\lambda(x)|^2 \,\mathrm{d}x \leqslant c \int_{\mathbb{R}^{2n}} |f^\lambda(x)|^2 \,\mathrm{d}x,$$

where $h^\lambda(x) := \int_{\mathbb{R}} h(x,t)\mathrm{e}^{-\mathrm{i}\lambda t}\,\mathrm{d}t$. A computation gives

$$\begin{aligned}(T_{-n+\mathrm{i}b} f)^\lambda(x) &= \widehat{I}_{-n+\mathrm{i}b}(\lambda) \int_{\mathbb{R}^{2n}} f^\lambda(x-y) e_{\lambda A}(y) \mathrm{e}^{-\mathrm{i}\lambda x^t J y}\,\mathrm{d}y \\ &= \widehat{I}_{-n+\mathrm{i}b}(\lambda)(f^\lambda \times_\lambda e_{\lambda A})(x).\end{aligned}$$

From the identity in (6) and since $\widehat{I}_z = cI_{1-z}$, we get

$$\|(T_{-n+\mathrm{i}b} f)^\lambda\|_{L^2(\mathbb{R}^{2n})} = \Bigl|\frac{c2^{-(1+n-\mathrm{i}b)/2}}{\Gamma\bigl(\frac{1}{2}(1+n-\mathrm{i}b)\bigr)}\Bigr|(2\pi)^n |\det(2A \pm J)|^{-1/2} \|f^\lambda\|_{L^2(\mathbb{R}^{2n})}$$

for each $b \in \mathbb{R}$. So T_{-n+ib} is bounded on $L^2(\mathbb{H}^n)$ if $\det(2A \pm J) \neq 0$. Finally, it is easy to see, with the aid of the Stirling formula (see e.g. [12]), that the family $\{T_z\}$ satisfies, on the strip $-n \leqslant \Re(z) \leqslant 1$, the hypothesis of the complex interpolation theorem (see [13], page 205) and so $T_0 = T_{\mu_A}$ is bounded from $L^{(2n+2)/(2n+1)}(\mathbb{H}^n)$ into $L^{2n+2}(\mathbb{H}^n)$. $\square$

Theorem 4. *Let ν_0 be the measure defined by* (3) *with* $\gamma = 0$. *If* $\det(2A \pm J) \neq 0$, *then the type set E_{ν_0} is the closed triangle with vertices* $(0,0)$, $(1,1)$ *and* $((2n+1)/(2n+2), 1/(2n+2))$.

P r o o f. Since the inequality $T_{\nu_0} f \leqslant T_{\mu_A} f$ holds for each borelian function $f \geqslant 0$, the theorem follows from the restrictions that appear in (4), Theorem 3 and the Riesz convexity theorem. $\square$

Corollary 5. *If* $\det(2A \pm J) \neq 0$, *then the operator T_{μ_A} is bounded from $L^p(\mathbb{H}^n)$ into $L^p(\mathbb{H}^n)$ if and only if* $p = (2n+2)/(2n+1)$ *and* $q = 2n+2$.

P r o o f. The "*if*" part of the corollary is Theorem 3. To see the reciprocal we introduce the action of the dilation group $\mathbb{R}^{>0}$ on $\mathbb{H}^n$, i.e. $\delta \cdot (x,t) = (\delta x, \delta^2 t)$, $\delta > 0$. For a function f defined on $\mathbb{H}^n$ we put $f_\delta(x,t) = f(\delta \cdot (x,t))$. It is easy to check that

$$(T_{\mu_A} f)_\delta = \delta^{2n}\, T_{\mu_A}(f_\delta).$$

If $\|T_{\mu_A} f\|_q \leqslant c_{p,q}\|f\|_p$, then

$$\delta^{-(2n+2)/q}\|T_{\mu_A} f\|_q = \|(T_{\mu_A} f)_\delta\|_q = \delta^{2n}\|T_{\mu_A}(f_\delta)\|_q \leqslant \delta^{2n} c\|f_\delta\|_p = \delta^{2n-(2n+2)/p} c\|f\|_p$$

for all $\delta > 0$. So $1/q = 1/p - 2n/(2n+2)$. Since $T_{\nu_0} f \leqslant T_{\mu_A} f$ for $f \geqslant 0$, from Theorem 4 it follows that $p = (2n+2)/(2n+1)$ and $q = 2n+2$. $\square$

Theorem 6. *Let ν_γ be the measure defined by equation* (3) *with* $0 < \gamma < 2n$. *If* $\det(2A \pm J) \neq 0$, *then the type set E_{ν_γ} is contained in the closed trapezoid with vertices* $(0,0)$, $(1,1)$, D *and* D', *where*

$$D = \Big(\frac{4n^2+2n+\gamma}{2n(2n+2)}, \frac{2n+(2n+1)\gamma}{2n(2n+2)}\Big) = \Big(\frac{1}{p_D}, \frac{1}{q_D}\Big) \quad \text{and} \quad D' = \Big(1-\frac{1}{q_D}, 1-\frac{1}{p_D}\Big)$$

and with the only possible exception of the closed segment joining the two points D and D'.

P r o o f. For each $k \in \mathbb{N} \cup \{0\}$ we define the sets $A_k \subset \mathbb{R}^{2n}$ by

$$A_k = \{y \in \mathbb{R}^{2n} \colon\ 2^{-k} < |y| \leqslant 2^{-k+1}\}.$$

Let $\nu_{\gamma,k}$ be the fractional Borel measure given by

$$\nu_{\gamma,k}(E) = \int_{A_k} \chi_E(y, \varphi(y))\eta(y)|y|^{-\gamma}\,\mathrm{d}y$$

and let $T_{\nu_{\gamma,k}}$ be its corresponding convolution operator, i.e. $T_{\nu_{\gamma,k}} f = f * \nu_{\gamma,k}$. Now, it is clear that $\nu_\gamma = \sum\limits_k \nu_{\gamma,k}$ and $\|T_{\nu_\gamma}\|_{p,q} \leqslant \sum\limits_k \|T_{\nu_{\gamma,k}}\|_{p,q}$. For $f \geqslant 0$ we have that

$$\int_{\mathbb{H}^n} f(y,s)\,\mathrm{d}\nu_{\gamma,k}(y,s) \leqslant 2^{k\gamma} \int_{\mathbb{R}^{2n}} f(y,\varphi(y))\eta(y)\,\mathrm{d}y.$$

Thus $\|T_{\nu_{\gamma,k}}\|_{p,q} \leqslant c2^{k\gamma}\|T_{\nu_0}\|_{p,q}$, from Theorem 4 it follows that

$$\|T_{\nu_{\gamma,k}}\|_{(2n+2)/(2n+1),2n+2} \leqslant c2^{k\gamma}.$$

It is easy to check that $\|T_{\nu_{\gamma,k}}\|_{1,1} \leqslant |\nu_{\gamma,k}(\mathbb{R}^{2n+1})| \sim \int_{A_k} |y|^{-\gamma}\,\mathrm{d}y = c2^{-k(2n-\gamma)}$. For $0 < \theta < 1$ we define

$$\Big(\frac{1}{p_\theta}, \frac{1}{q_\theta}\Big) = \Big(\frac{2n+1}{2n+2}, \frac{1}{2n+2}\Big)(1-\theta) + (1,1)\theta.$$

By the Riesz convexity theorem we have

$$\|T_{\nu_{\gamma,k}}\|_{p_\theta,q_\theta} \leqslant c2^{k\gamma(1-\theta)-k(2n-\gamma)\theta}.$$

Choosing θ such that $k\gamma(1-\theta) - k(2n-\gamma)\theta = 0$ yields $\sup\limits_{k\in\mathbb{N}} \|T_{\nu_{\gamma,k}}\|_{p_\theta,q_\theta} \leqslant c < \infty$. A simple computation gives $\theta = (2n-\gamma)/(2n)$, then $(1/p_\theta, 1/q_\theta) = (1/p_D, 1/q_D)$, so

$\|T_{\nu_{\gamma,k}}\|_{p_D,q_D} \leqslant c$, where c is independent of k. Interpolating once again, but now between the points $(1/p_D, 1/q_D)$ and $(1,1)$ we obtain for each $0 < \tau < 1$ fixed

$$\|T_{\nu_{\gamma,k}}\|_{p_\tau,q_\tau} \leqslant c2^{-k(2n-\gamma)\tau}.$$

Since $\|T_{\nu_\gamma}\|_{p,q} \leqslant \sum\limits_k \|T_{\nu_{\gamma,k}}\|_{p,q}$ and $0 < \gamma < 2n$, it follows that

$$\|T_{\nu_\gamma}\|_{p_\tau,q_\tau} \leqslant c \sum_{k\in\mathbb{N}} 2^{-k(2n-\gamma)\tau} < \infty.$$

By duality we also have

$$\|T_{\nu_\gamma}\|_{q_\tau/(q_\tau-1),p_\tau/(p_\tau-1)} \leqslant c_\tau < \infty.$$

Finally, the theorem follows from the Riesz convexity theorem, and the restrictions that appear in (4) and (5). □

We conclude this note with the following remarks.

Remark 7. Let ν_0 be the measure of compact support defined by (3), but now with $\det(2A \pm J) = 0$. In this case, by Theorem 1.1 in [9] and Proposition 2, we can be sure that the type set E_{ν_0} has a nonempty interior.

Remark 8. Lemma 1 provides us with examples of diagonal matrices A such that $\det(2A \pm J) = 0$. By the above remark we know that the interior of the type set of measure $\nu_0 = \eta\mu_A$ is nonempty. If $n \geqslant 2$ and A also satisfies that $\varphi(y) = y^t A y = \sum\limits_{j=1}^{n} \alpha_j |y_j|^2$ ($\alpha_j \in \mathbb{R}$ and $y_j \in \mathbb{R}^2$), then the type set of ν_0 is the closed triangle with vertices $(0,0)$, $(1,1)$ and $((2n+1)/(2n+2), 1/(2n+2))$. This result is independent of the value of $\det(2A \pm J)$ (see Theorem 1, page 102 in [3]).

These final comments illustrate the limits of the techniques used in this note as well as of those developed in the works [3] and [4].

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