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CONVERGENCE RESULTS OF ITERATIVE ALGORITHMS FOR THE SUM OF TWO MONOTONE OPERATORS IN REFLEXIVE BANACH SPACES

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Abstract. The aim of this paper is to propose two modified forward-backward splitting algorithms for zeros of the sum of a maximal monotone operator and a Bregman inverse strongly monotone operator in reflexive Banach spaces. We prove weak and strong convergence theorems of the generated sequences by the proposed methods under some suitable conditions. We apply our results to study the variational inequality problem and the equilibrium problem. Finally, a numerical example is given to illustrate the proposed methods. The results presented in this paper improve and generalize many known results in recent literature.

Keywords: maximal operator; Bregman distance; reflexive Banach space; weak convergence; strong convergence

MSC 2020: 47H09, 47H10, 47J25, 47J05

1. Introduction

Let E be a real Banach space with its dual space E^*. We study the so-called *quasi-inclusion problem*: find $z \in E$ such that

$$0 \in (A+B)z, \tag{1.1}$$

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where $A\colon E \to E^*$, $B\colon E \multimap E^*$ are single- and set-valued mappings, respectively. In particular, if $E = H$, where H is a real Hilbert space, then $E^* = H$. This problem (1.1) has received considerable attention in recent decades due to its wide theoretical value in nonlinear analysis or optimization theory and wide spectrum of applications such as in signal processing, image reconstruction and machine learning.

One of the simplest methods for solving (1.1) is the *forward-backward splitting method* (see [19], [25], [48]), which is of the following form: for any $x_1 \in H$ and $\lambda > 0$,

$$x_{n+1} = \mathrm{Res}_\lambda^B \circ A_\lambda(x_n) \quad \forall\, n \geqslant 1, \tag{1.2}$$

where $\mathrm{Res}_\lambda^B := (I + \lambda B)^{-1}$ is the resolvent of B, $A_\lambda := I - \lambda A$ and I denotes the identity mapping on H. In the context of this method, the operators Res_λ^B and A_λ are also often referred to as the backward and forward operators, respectively. Note that this method includes, as special cases, the proximal point algorithm (when $A = 0$) (see [17], [21], [41]) and the gradient method (see [8], [20]). However, from the numerical point of view, the weak convergence of this method is not enough to make it efficient.

In order to obtain the strong convergence result, Takahashi et al. [47] introduced the following *Halpern-type forward-backward splitting method* for solving (1.1) in a Hilbert space H: for any x_1, $u \in H$,

$$x_{n+1} = \alpha_n u + (1 - \alpha_n)\mathrm{Res}_{\lambda_n}^B \circ A_{\lambda_n}(x_n) \quad \forall\, n \geqslant 1, \tag{1.3}$$

where $\{\lambda_n\} \subset (0, \infty)$, $\{\alpha_n\} \subset (0, 1)$, A is an α-inverse strongly monotone mapping and B is a maximal monotone operator. It was shown that the sequence $\{x_n\}$ generated by (1.3) converges strongly to a solution of (1.1).

In 2012, López et al. [26] extended the above result to a q-uniformly smooth and uniformly convex Banach space E. They proposed the following iterative process with errors a_n and b_n: for any x_1, $u \in E$,

$$x_{n+1} = \alpha_n u + (1 - \alpha_n)(\mathrm{Res}_{\lambda_n}^B(x_n - \lambda_n(Ax_n + a_n)) + b_n) \quad \forall\, n \geqslant 1, \tag{1.4}$$

where $\{\lambda_n\} \subset (0, \infty)$ and $\{\alpha_n\} \subset (0, 1]$, B is a maximal monotone operator and A is an α-inverse strongly monotone mapping. They also proved that the sequence $\{x_n\}$ generated by (1.4) converges strongly to a solution of (1.1) under appropriate assumptions.

Very recently in 2019, Kimura and Nakajo [24] proposed the following iterative process for approximating a solution of (1.1) in a 2-uniformly convex and uniformly smooth Banach space E: for any x_1, $u \in E$,

$$x_{n+1} = \mathrm{Res}_{\lambda_n}^B \circ A_{\lambda_n} J^{-1}(\alpha_n Ju + (1 - \alpha_n)Jx_n - \lambda_n Bx_n) \quad \forall\, n \geqslant 1, \tag{1.5}$$

where $\{\lambda_n\} \subset (0,\infty)$, $\{\alpha_n\} \subset (0,1]$, A is an α-inverse strongly monotone mapping, $\operatorname{Res}^B_{\lambda_n} := (J + \lambda_n B)^{-1} J$ is the resolvent of a maximal monotone operator B, $A_\lambda := J^{-1}(J - \lambda_n A)$ and J denotes the normalized duality mapping from E into 2^{E^*}. Under appropriate assumptions, they proved that the sequence generated by (1.5) converges strongly to a solution of (1.1).

In recent years, various modified forward-backward splitting algorithms have been studied and developed by many researchers in Hilbert spaces and extended to Banach spaces (see e.g. [14], [13], [15], [16], [18], [24], [26], [43], [44], [45], [46], [49]). The following important question arises here:

Question: Can we extend the above-mentioned results to a more general class of forward-backward operators in more general Banach spaces which are not necessarily uniformly convex and uniformly smooth?

To answer the above question in this paper, we introduce two modified forward-backward splitting algorithms for solving (1.1), where A is a Bregman inverse strongly monotone mapping and B is a maximal monotone mapping in the framework of reflexive Banach spaces. The paper is organized as follows. In Section 2, we collect definitions and results which are needed for our further analysis. The weak and strong convergence theorems of the proposed algorithms are established in Section 3. Some applications of our results to the variational inequality problem and the equilibrium problem are considered in Section 4, and a numerical example is given in Section 5.

2. Preliminaries

Throughout this paper, let E be a real reflexive Banach space with its dual E^* and $f\colon E \to (-\infty,\infty]$ be a proper, lower semicontinuous and convex function. We denote by $\langle x, j\rangle$ the value of the functional $j \in E^*$ at $x \in E$. The *subdifferential* of f is defined by

$$\partial f(x) = \{x^* \in E^* \colon f(x) + \langle y - x, x^*\rangle \leqslant f(y) \ \forall\, y \in E\}, \quad x \in E.$$

The *Fenchel conjugate* of f is the function $f^*\colon E^* \to (-\infty,\infty]$ defined by

$$f^*(x^*) = \sup_{x\in E}\{\langle x, x^*\rangle - f(x)\}.$$

It is known that $x^* \in \partial f(x)$ is equivalent to $f(x) + f^*(x^*) = \langle x, x^*\rangle$ (see [39], Theorem 23.5 (d)). We denote by $\operatorname{dom} f = \{x \in E \colon f(x) < \infty\}$ the domain of f. The function f on E is said to be *cofinite* if $\operatorname{dom} f^* = E^*$ and f is said to be *strongly coercive* if $\lim\limits_{\|x\|\to\infty} f(x)/\|x\| = \infty$.

For any $x \in \operatorname{int}(\operatorname{dom} f)$ and $y \in E$, the *directional derivative* of f at x in the direction $y \in E$ is given by

$$f'(x, y) = \lim_{t \to 0^+} \frac{f(x + ty) - f(x)}{t}. \tag{2.1}$$

The function f is said to be *Gâteux differentiable* at x if the limit as $t \to 0$ in (2.1) exists for each y. In this case, the *gradient* of f at x is the linear function $\nabla f(x)\colon E \to E^*$ defined by $\langle y, \nabla f(x) \rangle = f'(x, y)$ for any $y \in E$. For more details about gradient, we recommend [7], Remark 3.32. The function f is said to be *Gâteux differentiable* if it is Gâteux differentiable for any $x \in \operatorname{int}(\operatorname{dom} f)$. It is known that if f is Gâteux differentiable at x, then $\partial f(x)$ is single-valued. Conversely, if f is continuous at x and $\partial f(x)$ is single-valued, then f is Gâteux differentiable at x and $\nabla f(x) = \partial f(x)$ (see [4], Proposition 2.40). The function f is said to be *Fréchet differentiable* at x if the limit (2.1) is attained uniformly in $\|y\| = 1$ and f is said to be *uniformly Fréchet differentiable* on a subset C of E if the limit (2.1) is attained uniformly for $x \in C$ and $\|y\| = 1$. It is well known that every Fréchet differentiable function is Gâteux differentiable and if f is Fréchet differentiable, then it is continuous, but if f is Gâteux differentiable, then it is not necessary that f is continuous (see [34], p. 142).

The function $f\colon E \to (-\infty, \infty]$ is said to be *Legendre* ([36], p. 25) if and only if it satisfies the following two conditions:

(L1) $\operatorname{int}(\operatorname{dom} f) \neq \emptyset$ and f is Gâteux differentiable with $\operatorname{dom} \nabla f = \operatorname{int}(\operatorname{dom} f)$,
(L2) $\operatorname{int}(\operatorname{dom} f^*) \neq \emptyset$ and f^* is Gâteux differentiable with $\operatorname{dom} \nabla f^* = \operatorname{int}(\operatorname{dom} f^*)$.

In a reflexive Banach space, we always obtain $(\partial f)^{-1} = \partial f^*$ (see [9], p. 83). This fact, when combined with conditions (L1) and (L2), implies the following facts:

(i) ∇f is a bijection with $\nabla f = (\nabla f^*)^{-1}$ (see [5], Theorem 5.10),
(ii) $\operatorname{ran} \nabla f = \operatorname{dom} \nabla f^* = \operatorname{int}(\operatorname{dom} f^*)$ and $\operatorname{ran} \nabla f^* = \operatorname{dom} \nabla f = \operatorname{int}(\operatorname{dom} f)$ (see [37], p. 123),

where $\operatorname{ran} \nabla f$ denotes the range of ∇f.

Also, conditions (L1) and (L2), in conjunction with [5], Theorem 5.4, imply that the functions f and f^* are strictly convex on the interior of their respective domains.

If $f\colon E \to (-\infty, \infty]$ is additionally assumed to be Gâteux differentiable, then the function $D_f\colon \operatorname{dom} f \times \operatorname{int}(\operatorname{dom} f) \to [0, \infty)$ defined by

$$D_f(x, y) = f(x) - f(y) - \langle x - y, \nabla f(y) \rangle \tag{2.2}$$

is called the *Bregman distance* with respect to f [11]. In general, D_f is not symmetric and it does not satisfy the triangle inequality. Clearly, $D_f(x, x) = 0$, but $D_f(x, y) = 0$

may not imply $x = y$. It is known that the Bregman distance is a certain useful substitute for a distance. Next, we clarify several examples of the Bregman distance which are shown in the following:

Example 2.1. Let E is a uniformly convex and uniformly smooth Banach space. Define $f(x) = \|x\|^2$ for all $x \in E$. Then $\nabla f(x) = 2Jx$, where J is the normalized duality mapping defined by $Jx = \{j \in E^*\colon \langle x, j\rangle = \|x\|^2 = \|j\|^2\}$. So, we obtain

$$D_f(x, y) = \|x\|^2 - 2\langle x, Jy\rangle + \|y\|^2 := \varphi(x, y),$$

where φ is called the Lyapunov function, which was studied in [3], [35]. Also, if E is a Hilbert space, then $\varphi(x, y) = \|x - y\|^2$, which is the Euclidean norm distance.

Example 2.2. Define $f(x) = -\sum\limits_{i=1}^{m} \ln x_i$ for all $x = (x_1, x_2, \ldots, x_m)^\top \in \mathbb{R}^m_+ := \{x \in \mathbb{R}^m\colon x_i > 0\}$. Then $\nabla f(x) = -(1/x_1, 1/x_2, \ldots, 1/x_m)^\top$. So, we obtain the Itakura-Saito distance given by

$$D_f(x, y) = \sum_{i=1}^{m} \Big(\frac{x_i}{y_i} - \ln\Big(\frac{x_i}{y_i}\Big) - 1\Big).$$

Example 2.3. Define $f(x) = x_i \ln x_i$ for all $x = (x_1, x_2, \ldots, x_m)^\top \in \mathbb{R}^m_+ := \{x \in \mathbb{R}^m\colon x_i > 0\}$. Then $\nabla f(x) = (1 + \ln(x_1), 1 + \ln(x_2), \ldots, 1 + \ln(x_m))^\top$. So, we obtain the Kullback-Leibler distance given by

$$D_f(x, y) = \sum_{i=1}^{m} \Big(x_i \ln\Big(\frac{x_i}{y_i}\Big) + y_i - x_i\Big).$$

Example 2.4. Define $f(x) = \frac{1}{2} x^\top Q x$ for all $x = (x_1, x_2, \ldots, x_m)^\top \in \mathbb{R}^m$. Then $\nabla f(x) = Qx$, where $Q = \operatorname{diag}(1, 2, \ldots, m)$. So, we obtain the squared Mahalanobis distance given by

$$D_f(x, y) = \frac{1}{2}(x - y)^\top Q(x - y).$$

For more examples of Bregman distances, we recommend [23], [31], [32].

The *modulus of total convexity* of f at $x \in \operatorname{dom} f$ is the function $v_f(x, \cdot)\colon [0, \infty) \to [0, \infty]$ defined by

$$v_f(x, t) = \inf\{D_f(y, x)\colon y \in \operatorname{dom} f,\ \|y - x\| = t\}.$$

The function f is called *totally convex at* x if $v_f(x, t) > 0$, whenever $t > 0$, and is called *totally convex* if it is totally convex at any point $x \in \operatorname{int}(\operatorname{dom} f)$. It is well known that if f is totally convex and Fréchet differentiable, then f is cofinite (see [36], Proposition 2.3, p. 39). The function f is said to be *totally convex on bounded sets* if $v_f(X,t) > 0$ for any nonempty bounded subset X of E and $t > 0$, where the *modulus of total convexity* of the function f on the set X is the function $v_f\colon \operatorname{int}(\operatorname{dom} f) \times [0,\infty) \to [0,\infty]$ defined by

$$v_f(X,t) = \inf\{v_f(x,t)\colon\ x \in X \cap \operatorname{dom} f\}.$$

Let $B_r := \{x \in E\colon\ \|x\| \leqslant r\}$ for all $r > 0$. Then a function $f\colon\ E \to \mathbb{R}$ is said to be *uniformly convex* on bounded subsets of E, if $\varrho_r(t) > 0$ for all r, $t > 0$, where $\varrho_r\colon\ [0,\infty) \to [0,\infty)$ is defined by

$$\varrho_r(t) = \inf_{x,y\in B_r, \|x-y\|=t, \alpha\in(0,1)} \frac{\alpha f(x) + (1-\alpha)f(y) - f(\alpha x + (1-\alpha)y)}{\alpha(1-\alpha)},$$

for all $t \geqslant 0$. It is well known that f is totally convex on bounded sets if and only if f is uniformly convex on bounded sets (see [12], Theorem 2.10).

Let C be a nonempty subset of E. Let $S\colon\ C \to C$ be a mapping. A point $p \in C$ is a fixed point of S if $p = Sp$ and we denote by $F(S)$ the set of fixed points of S, i.e., $F(S) = \{x \in C\colon\ x = Sx\}$. A point p in C is said to be an *asymptotic fixed point* [35] of S, if C contains a sequence $\{x_n\}$ in C such that $\{x_n\}$ converges weakly to p and $\lim\limits_{n\to\infty} \|x_n - Sx_n\| = 0$. The set of asymptotic fixed points of S will be denoted by $\widehat{F}(S)$.

Let C be a nonempty subset of $\operatorname{int}(\operatorname{dom} f)$. The mapping $S\colon\ C \to \operatorname{int}(\operatorname{dom} f)$ is said to be:

(i) *Bregman firmly nonexpansive* (BFNE) if

$$\langle Sx - Sy, \nabla f(Sx) - \nabla f(Sy)\rangle \leqslant \langle Sx - Sy, \nabla f(x) - \nabla f(y)\rangle \quad \forall\, x, y \in C.$$

(ii) *Bregman strongly nonexpansive* (BSNE) with $\widehat{F}(S) \neq \emptyset$ if

$$D_f(p, Sx) \leqslant D_f(p, x) \quad \forall\, x \in C,\ p \in \widehat{F}(S)$$

and if whenever $\{x_n\} \subset C$ is bounded, $p \in \widehat{F}(S)$ and

$$\lim_{n\to\infty} (D_f(p, x_n) - D_f(p, Sx_n)) = 0,$$

it follows that $\lim\limits_{n\to\infty} D_f(x_n, Sx_n) = 0$.

As shown in [38], Lemma 15.6, p. 308, if S is BFNE and f is a Legendre function which is bounded, uniformly Fréchet differentiable and totally convex on bounded subsets of E, then $F(S)=\widehat{F}(S)$ and $F(S)$ is closed and convex. It also follows that every BFNE mapping is BSNE with respect to $F(S)=\widehat{F}(S)$.

Recall that the *Bregman projection with respect to* f of $x\in\operatorname{int}(\operatorname{dom} f)$ onto the nonempty, closed and convex set $C\subset\operatorname{dom} f$ is the unique $P_C^f(x)\subset C$ satisfying

$$D_f(P_C^f(x),x)=\inf\{D_f(y,x)\colon\ y\in C\}. \tag{2.3}$$

It can be characterized by the following variational inequality [12], Corollary 4.4:

$$\langle y-P_C^f(x),\nabla f(x)-\nabla f(P_C^f(x))\rangle\leqslant 0\quad\forall\, y\in C. \tag{2.4}$$

Moreover,

$$D_f(y,P_C^f(x))+D_f(P_C^f(x),x)\leqslant D_f(y,x)\quad\forall\, y\in C. \tag{2.5}$$

If E is a uniformly convex and uniformly smooth Banach space and $f(x)=\|x\|^2$ for all $x\in E$, then P_C^f coincides with the generalized projection Π_C (see [2], Definition 7.2), and if E is a Hilbert space, then P_C^f coincides with the metric projection P_C.

Let $f\colon E\to\mathbb{R}$ be a Legendre function. Let $V_f\colon E\times E^*\to[0,\infty)$ associated with f be defined by

$$V_f(x,x^*)=f(x)-\langle x,x^*\rangle+f^*(x^*)\quad\forall\, x\in E,\ x^*\in E^*. \tag{2.6}$$

From [28], Proposition 1, we know the following properties:

(i) V_f is nonnegative and convex in the second variable.

(ii)

$$V_f(x,x^*)=D_f(x,\nabla f^*(x^*))\quad\forall\, x\in E,\ x^*\in E^*. \tag{2.7}$$

(iii)

$$V_f(x,x^*)+\langle\nabla f^*(x^*)-x,y^*\rangle\leqslant V_f(x,x^*+y^*)\quad\forall\, x\in E,\ x^*,y^*\in E^*. \tag{2.8}$$

Since V_f is convex in the second variable, we have for all $z\in E$,

$$D_f\biggl(z,\nabla f^*\biggl(\sum_{i=1}^N t_i\nabla f(x_i)\biggr)\biggr)\leqslant\sum_{i=1}^N t_iD_f(z,x_i), \tag{2.9}$$

where $\{x_i\}_{i=1}^N\subset E$ and $\{t_i\}_{i=1}^N$ with $\sum\limits_{i=1}^N t_i=1$.

For a set-valued operator $A\colon E \multimap E^*$, we define its domain, range and graph as follows: $\operatorname{dom} A = \{x \in E\colon Ax \neq \emptyset\}$, $\operatorname{ran} A = \bigcup\{Ax\colon x \in \operatorname{dom} A\}$ and $G(A) = \{(x, x^*) \in E \times E^*\colon x^* \in Ax\}$. An operator A is said to be *monotone*, if for each $(x, x^*), (y, y^*) \in G(A)$, we have $\langle x - y, x^* - y^*\rangle \geqslant 0$. A monotone operator A is said to be *maximal*, if its graph is not contained in the graph of any other monotone operator on E. It is known that if $f\colon E \to \mathbb{R}$ is Gâteux differentiable, strictly convex and cofinite, then A is maximal monotone if and only if $\operatorname{ran}(\nabla f + \lambda A) = E^*$ for $\lambda > 0$ (see [6], Corollary 2.4).

Let $f\colon E \to (-\infty, \infty]$ be a Fréchet differentiable function which is bounded on bounded subsets of E and let A be a maximal monotone operator. Then the *resolvent* $\operatorname{Res}^f_{\lambda A}\colon E \multimap E$ of A for $\lambda > 0$, is defined by

$$\operatorname{Res}^f_{\lambda A} = (\nabla f + \lambda A)^{-1} \circ \nabla f.$$

The mapping A satisfying $\operatorname{ran}(\nabla f - \lambda A) \subset \operatorname{ran}(\nabla f)$ is called *Bregman inverse strongly monotone* if $\operatorname{dom} A \cap \operatorname{int}(\operatorname{dom} f) \neq \emptyset$ and for any $x, y \in \operatorname{int}(\operatorname{dom} f)$ and each $u \in Ax$, $v \in Ay$, we have

$$\langle u - v, \nabla f^*(\nabla f(x) - u) - \nabla f^*(\nabla f(y) - v)\rangle \geqslant 0.$$

For any operator $A\colon E \multimap E^*$, the *anti-resolvent* $A^f_\lambda\colon E \multimap E$ of A, for $\lambda > 0$, is defined by

$$A^f_\lambda = \nabla f^* \circ (\nabla f - \lambda A).$$

Lemma 2.5. *Let $f\colon E \to \mathbb{R}$ be a Fréchet differentiable Legendre function which is bounded on bounded subsets of E. Let $A\colon E \to E^*$ be a Bregman inverse strongly monotone mapping and $B\colon E \multimap E^*$ be a maximal monotone operator. Define a mapping $T_\lambda x := \operatorname{Res}^f_{\lambda B} \circ A^f_\lambda(x)$ for $x \in E$ and $\lambda > 0$. Then $F(T_\lambda) = (A + B)^{-1}0$.*

P r o o f. Let $x \in E$ and $\lambda > 0$. We see that

$$\begin{aligned} x = T_\lambda x &\Leftrightarrow x = \operatorname{Res}^f_{\lambda B} \circ A^f_\lambda(x) \\ &\Leftrightarrow x = (\nabla f + \lambda B)^{-1} \circ \nabla f \circ (\nabla f^* \circ (\nabla f - \lambda A)x) \\ &\Leftrightarrow x = (\nabla f + \lambda B)^{-1} \circ (\nabla f - \lambda A)x) \\ &\Leftrightarrow \nabla f(x) - \lambda Ax \in \nabla f(x) + \lambda Bx \\ &\Leftrightarrow 0 \in \lambda(A + B)x \\ &\Leftrightarrow x \in (A + B)^{-1}0. \end{aligned}$$

The proof is completed. □

Lemma 2.6 ([33], Theorem 3.1). *Let $f\colon E \to \mathbb{R}$ be a strongly coercive, bounded and Fréchet differentiable Legendre function which is totally convex on bounded subsets of E. Let $A\colon E \to E^*$ be a Bregman inverse strongly monotone mapping and $B\colon E \multimap E^*$ be a maximal monotone operator. Then the following statements hold:*

(i) $D_f(z, \mathrm{Res}^f_{\lambda B} \circ A^f_\lambda(x)) + D_f(\mathrm{Res}^f_{\lambda B} \circ A^f_\lambda(x), x) \leqslant D_f(z, x)$ *for all* $z \in (A+B)^{-1}0$, $x \in E$ *and* $\lambda > 0$.

(ii) $\mathrm{Res}^f_{\lambda B} \circ A^f_\lambda$ *is a BSNE operator such that* $F(\mathrm{Res}^f_{\lambda B} \circ A^f_\lambda(x)) = \widehat{F}(\mathrm{Res}^f_{\lambda B} \circ A^f_\lambda(x))$.

We shall use the following notation in this paper:

▷ $x_n \to x$ means that $\{x_n\}$ converges strongly to x,

▷ $x_n \rightharpoonup x$ means that $\{x_n\}$ converges weakly to x.

The mapping $A\colon E \to E^*$ is called *weakly sequentially continuous* if for any sequence $\{x_n\} \subset E$, $x_n \rightharpoonup x$ implies that $Ax_n \rightharpoonup^* Ax$.

Lemma 2.7 ([28], Proposition 9). *Let $f\colon E \to \mathbb{R}$ be a Legendre function such that ∇f is weakly sequentially continuous. Suppose that the sequence $\{x_n\}$ is bounded and that $\lim_{n\to\infty} D_f(u, x_n)$ exists for any weak subsequential limit u of $\{x_n\}$. Then $\{x_n\}$ converges weakly to u.*

Lemma 2.8 ([36], Lemma 3.1). *Let $f\colon E \to \mathbb{R}$ be a Gâteux differentiable and totally convex function. Suppose that $x \in E$. If $\{D_f(x, x_n)\}$ is bounded, then the sequence $\{x_n\}$ is bounded.*

Lemma 2.9 ([30], Lemma 2.4). *Let E be a Banach space and $f\colon E \to \mathbb{R}$ be a Gâteux differentiable function which is uniformly convex on bounded subsets of E. Suppose that $\{x_n\}$ and $\{y_n\}$ are two sequences in E. Then $\lim_{n\to\infty} D_f(x_n, y_n) = 0$ if and only if $\lim_{n\to\infty} \|x_n - y_n\| = 0$.*

Lemma 2.10 ([27], Lemma 3.1). *Let $\{a_n\}$ and $\{c_n\}$ be nonnegative real sequences such that*

$$a_{n+1} \leqslant (1-\delta_n)a_n + b_n + c_n,$$

where $\{\delta_n\}$ is a sequence in $(0,1)$ and $\{b_n\}$ is a real sequence. Assume that $\sum_{n=1}^{\infty} c_n < \infty$. Then the following results hold:

(i) *If $b_n/\delta_n \leqslant M$ for some $M \geqslant 0$, then $\{a_n\}$ is a bounded sequence.*

(ii) *If $\sum_{n=1}^{\infty} \delta_n = \infty$ and $\limsup_{n\to\infty} b_n/\delta_n \leqslant 0$, then $\lim_{n\to\infty} a_n = 0$.*

Lemma 2.11 ([22], Lemma 7). *Assume that* $\{s_n\}$ *is a nonnegative real sequence such that*

$$s_{n+1} \leqslant (1-\delta_n)s_n + \delta_n \tau_n$$

and

$$s_{n+1} \leqslant s_n - \eta_n + \varrho_n,$$

where $\{\delta_n\}$ *is a sequence in* $(0,1)$, $\{\eta_n\}$ *is a nonnegative real sequence and* $\{\tau_n\}$, $\{\varrho_n\}$ *are real sequences such that*

(i) $\sum\limits_{n=1}^{\infty} \delta_n = \infty$,

(ii) $\lim\limits_{n\to\infty} \varrho_n = 0$,

(iii) $\lim\limits_{k\to\infty} \eta_{n_k} = 0$ *implies* $\limsup\limits_{k\to\infty} \tau_{n_k} \leqslant 0$ *for any subsequence* $\{n_k\}$ *of* $\{n\}$.

Then $\lim\limits_{n\to\infty} s_n = 0$.

3. Main results

In this section, we propose two modifications of a forward-backward splitting method for solving (1.1) in reflexive Banach spaces. In order to prove the convergence results, we assume the following:

Assumption 3.1.

(i) *Let* E *be a real reflexive Banach space.*

(ii) *Let* $f\colon E \to \mathbb{R}$ *be a strongly coercive, bounded and Fréchet differentiable Legendre function which is totally convex on bounded subsets of* E.

(iii) *Let* $A\colon E \to E^*$ *be a Bregman inverse strongly monotone mapping and* $B\colon E \multimap E^*$ *be a maximal monotone mapping.*

(iv) *The solution set* $(A+B)^{-1}0$ *of* (1.1) *is nonempty.*

3.1. Weak convergence theorem. In this subsection, we propose a modification of the Mann-type forward-backward splitting method and prove its weak convergence.

Algorithm 3.2. *For a given* $x_1 \in E$, *let* $\{x_n\}$ *be a sequence generated by*

$$x_{n+1} = \nabla f^*(\alpha_n \nabla f(x_n) + (1-\alpha_n)\nabla f(\mathrm{Res}^f_{\lambda B} \circ A^f_\lambda(x_n))) \quad \forall\, n \geqslant 1, \tag{3.1}$$

where $\{\alpha_n\} \subset [0,1)$ *and* $\lambda > 0$.

We now prove a weak convergence theorem for Algorithm 3.2.

Theorem 3.3. *Assume that Assumption* 3.1 *is satisfied. Let* $\{x_n\}$ *be a sequence generated by Algorithm* 3.2. *Assume that* $\limsup\limits_{n\to\infty} \alpha_n < 1$. *Suppose, in addition, that* ∇f *is weakly sequentially continuous on* E. *Then* $\{x_n\}$ *converges weakly to a point in* $(A+B)^{-1}0$.

Proof. Let $w \in (A+B)^{-1}0$. Then from (2.9) and Lemma 2.6 (i), we have

$$
\begin{aligned}
D_f(w, x_{n+1}) &= D_f(w, \nabla f^*(\alpha_n \nabla f(x_n) + (1-\alpha_n)\nabla f(\mathrm{Res}^f_{\lambda B} \circ A^f_\lambda(x_n)))) \\
&\leqslant \alpha_n D_f(w, x_n) + (1-\alpha_n) D_f(w, \mathrm{Res}^f_{\lambda B} \circ A^f_\lambda(x_n)) \\
&\leqslant \alpha_n D_f(w, x_n) + (1-\alpha_n) D_f(w, x_n) \\
&= D_f(w, x_n).
\end{aligned} \tag{3.2}
$$

This shows that $\lim\limits_{n\to\infty} D_f(w, x_n)$ exists and, consequently, $\{D_f(w, x_n)\}$ is bounded. Hence, by Lemma 2.8, we have that $\{x_n\}$ is bounded. This implies that $\{\mathrm{Res}^f_{\lambda B} \circ A^f_\lambda(x_n)\}$ is bounded. From Lemma 2.6 (i), we have

$$
\begin{aligned}
D_f(w, x_{n+1}) &\leqslant \alpha_n D_f(w, x_n) + (1-\alpha_n) D_f(w, \mathrm{Res}^f_{\lambda B} \circ A^f_\lambda(x_n)) \\
&\leqslant \alpha_n D_f(w, x_n) + (1-\alpha_n)(D_f(w, x_n) - D_f(\mathrm{Res}^f_{\lambda B} \circ A^f_\lambda(x_n), x_n)),
\end{aligned} \tag{3.3}
$$

which implies that

$$
(1-\alpha_n) D_f(\mathrm{Res}^f_{\lambda B} \circ A^f_\lambda(x_n), x_n) \leqslant D_f(w, x_n) - D_f(w, x_{n+1}). \tag{3.4}
$$

Since $\lim\limits_{n\to\infty} D_f(w, x_n)$ exists, we have

$$
\lim_{n\to\infty} D_f(\mathrm{Res}^f_{\lambda B} \circ A^f_\lambda(x_n), x_n) = 0. \tag{3.5}
$$

Moreover, from Lemma 2.9, we have

$$
\lim_{n\to\infty} \|\mathrm{Res}^f_{\lambda B} \circ A^f_\lambda(x_n) - x_n\| = 0. \tag{3.6}
$$

By the reflexivity of the Banach space E and the boundedness of $\{x_n\}$, there exists a subsequence $\{x_{n_k}\}$ of $\{x_n\}$ such that $x_{n_k} \rightharpoonup \hat{x} \in E$ as $k \to \infty$. From (3.6), we note that $\|\mathrm{Res}^f_{\lambda B} \circ A^f_\lambda(x_{n_k}) - x_{n_k}\| \to 0$ as $k \to \infty$, then we have $\hat{x} \in \widehat{F}(\mathrm{Res}^f_{\lambda B} \circ A^f_\lambda) = F(\mathrm{Res}^f_{\lambda B} \circ A^f_\lambda)$, and hence $\hat{x} \in (A+B)^{-1}0$. By Lemma 2.7, we conclude that $\{x_n\}$ converges weakly to a point in $(A+B)^{-1}0$. This completes the proof. $\square$

If $\alpha_n = 0$ for all $n \geqslant 1$, then we have the following result for the forward-backward splitting method in a reflexive Banach space.

Corollary 3.4. *Let $\{x_n\}$ be a sequence generated by $x_1 \in E$ and*

$$x_{n+1} = \operatorname{Res}^f_{\lambda B} \circ A^f_\lambda(x_n) \quad \forall\, n \geqslant 1. \tag{3.7}$$

Then $\{x_n\}$ converges weakly to a point in $(A+B)^{-1}0$.

If E is a uniformly convex Banach space which is also uniformly smooth and $f(x) = \|x\|^2$ for all $x \in E$, then we have the following result for the Mann-type forward-backward splitting method.

Corollary 3.5. *Let J be a duality mapping from E into E^* such that J is weakly sequentially continuous. Let $A\colon E \to E^*$ be a Bregman inverse strongly monotone mapping with respect to the function $f(x) = \|x\|^2$ and let $B\colon E \multimap E^*$ be a maximal monotone mapping. Let $\{x_n\}$ be a sequence generated by $x_1 \in E$ and*

$$x_{n+1} = J^{-1}(\alpha_n J x_n + (1-\alpha_n) J(\operatorname{Res}^B_\lambda \circ A_\lambda(x_n))) \quad \forall\, n \geqslant 1, \tag{3.8}$$

where $\operatorname{Res}^B_\lambda := (J+\lambda B)^{-1} J$, $A_\lambda := J^{-1}(J - \lambda A)$ for $\lambda > 0$ and $\{\alpha_n\} \subset [0,1)$. Suppose that $\limsup\limits_{n\to\infty} \alpha_n < 1$. Then $\{x_n\}$ converges weakly to a point in $(A+B)^{-1}0$.

3.2. Strong convergence theorem. In this subsection, we propose a strong convergence theorem for another modification of the forward-backward splitting method based on the Halpern-type iteration.

Algorithm 3.6. *For given $u, x_1 \in E$, let $\{x_n\}$ be a sequence generated by*

$$\begin{cases} y_n = \nabla f^*(\alpha_n \nabla f(u) + (1-\alpha_n)\nabla f(\operatorname{Res}^f_{\lambda B} \circ A^f_\lambda(x_n))), \\ x_{n+1} = \nabla f^*(\beta_n \nabla f(x_n) + (1-\beta_n)\nabla f(y_n)) \quad \forall\, n \geqslant 1, \end{cases} \tag{3.9}$$

where $\{\alpha_n\} \subset (0,1)$, $\{\beta_n\} \subset [0,1)$ and $\lambda > 0$.

Theorem 3.7. *Assume that Assumption 3.1 is satisfied. Let $\{x_n\}$ be a sequence generated by Algorithm 3.6. Suppose that $\lim\limits_{n\to\infty} \alpha_n = 0$, $\sum\limits_{n=1}^{\infty} \alpha_n = \infty$ and $\limsup\limits_{n\to\infty} \beta_n < 1$. Then $\{x_n\}$ converges strongly to $z = P^f_{(A+B)^{-1}0}(u)$, where $P^f_{(A+B)^{-1}0}$ is the Bregman projection of E onto $(A+B)^{-1}0$.*

Proof. Let $w \in (A+B)^{-1}0$. Then we have

$$\begin{aligned} D_f(w, y_n) &= D_f(w, \nabla f^*(\alpha_n \nabla f(u) + (1-\alpha_n)\nabla f(\operatorname{Res}^f_{\lambda B} \circ A^f_\lambda(x_n)))) \\ &\leqslant \alpha_n D_f(w,u) + (1-\alpha_n) D_f(w, \operatorname{Res}^f_{\lambda B} \circ A^f_\lambda(x_n)) \\ &\leqslant \alpha_n D_f(w,u) + (1-\alpha_n) D_f(w, x_n). \end{aligned} \tag{3.10}$$

It follows that

$$
\begin{aligned}
(3.11)\quad D_f(w, x_{n+1}) &= D_f(w, \nabla f^*(\beta_n \nabla f(x_n) + (1-\beta_n)\nabla f(y_n))) \\
&\leqslant \beta_n D_f(w, x_n) + (1-\beta_n) D_f(w, y_n) \\
&\leqslant \beta_n D_f(w, x_n) + (1-\beta_n)(\alpha_n D_f(w, u) + (1-\alpha_n) D_f(w, x_n)) \\
&= (1-(1-\beta_n)\alpha_n) D_f(w, x_n) + (1-\beta_n)\alpha_n D_f(w, u).
\end{aligned}
$$

By Lemma 2.10 (i), we have that $\{D_f(w, x_n)\}$ is bounded. Hence, $\{x_n\}$ is bounded by Lemma 2.8. Let $z = P^f_{(A+B)^{-1}0}(u)$. From (2.7), (2.8), (2.9), and Lemma 2.6 (i), we have

$$
\begin{aligned}
(3.12)\qquad D_f(z, y_n) &= D_f(z, \nabla f^*(\alpha_n \nabla f(u) + (1-\alpha_n)\nabla f(\mathrm{Res}^f_{\lambda B} \circ A^f_\lambda(x_n)))) \\
&= V_f(z, \alpha_n \nabla f(u) + (1-\alpha_n)\nabla f(\mathrm{Res}^f_{\lambda B} \circ A^f_\lambda(x_n))) \\
&\leqslant V_f(z, \alpha_n \nabla f(u) + (1-\alpha_n)\nabla f(\mathrm{Res}^f_{\lambda B} \circ A^f_\lambda(x_n)) \\
&\quad - \alpha_n(\nabla f(u) - \nabla f(z))) + \langle \alpha_n(y_n - z), \nabla f(u) - \nabla f(z)\rangle \\
&= V_f(z, \alpha_n \nabla f(z) + (1-\alpha_n)\nabla f(\mathrm{Res}^f_{\lambda B} \circ A^f_\lambda(x_n))) \\
&\quad + \langle \alpha_n(y_n - z), \nabla f(u) - \nabla f(z)\rangle \\
&= D_f(z, \nabla f^*(\alpha_n \nabla f(z) + (1-\alpha_n)\nabla f(\mathrm{Res}^f_{\lambda B} \circ A^f_\lambda(x_n)))) \\
&\quad + \langle \alpha_n(y_n - z), \nabla f(u) - \nabla f(z)\rangle \\
&\leqslant \alpha_n D_f(z, z) + (1-\alpha_n) D_f(z, \mathrm{Res}^f_{\lambda B} \circ A^f_\lambda(x_n)) \\
&\quad + \alpha_n \langle y_n - z, \nabla f(u) - \nabla f(z)\rangle \\
&\leqslant (1-\alpha_n)(D_f(z, x_n) - D_f(\mathrm{Res}^f_{\lambda B} \circ A^f_\lambda(x_n), x_n)) \\
&\quad + \alpha_n \langle y_n - z, \nabla f(u) - \nabla f(z)\rangle.
\end{aligned}
$$

It follows that

$$
\begin{aligned}
(3.13)\qquad D_f(z, x_{n+1}) &\leqslant \beta_n D_f(z, x_n) + (1-\beta_n) D_f(z, y_n) \\
&\leqslant (1-(1-\beta_n)\alpha_n) D_f(z, x_n) \\
&\quad - (1-\beta_n)(1-\alpha_n) D_f(\mathrm{Res}^f_{\lambda B} \circ A^f_\lambda(x_n), x_n) \\
&\quad + \alpha_n(1-\beta_n)\langle y_n - z, \nabla f(u) - \nabla f(z)\rangle.
\end{aligned}
$$

For each $n \geqslant 1$, we put $s_n = D_f(z, x_n)$, $\delta_n = (1-\beta_n)\alpha_n$, $\tau_n = \langle y_n - z, \nabla f(u) - \nabla f(z)\rangle$, $\eta_n = (1-\beta_n)(1-\alpha_n) D_f(\mathrm{Res}^f_{\lambda B} \circ A^f_\lambda(x_n), x_n)$ and $\varrho_n = \alpha_n(1-\beta_n)\langle y_n - z, \nabla f(u) - \nabla f(z)\rangle$. Then (3.13) reduces to the following formulae:

$$
(3.14)\qquad s_{n+1} \leqslant (1-\delta_n)s_n + \delta_n \tau_n, \quad n \geqslant 1,
$$

and

$$
(3.15)\qquad s_{n+1} \leqslant s_n - \eta_n + \varrho_n, \quad n \geqslant 1.
$$

Since $\sum_{n=1}^{\infty} \alpha_n = \infty$ and $\limsup_{n\to\infty} \beta_n < 1$, it follows that $\sum_{n=1}^{\infty} \delta_n = \infty$. Since $\lim_{n\to\infty} \alpha_n = 0$, we have $\lim_{n\to\infty} \varrho_n = 0$. In order to complete the proof, using Lemma 2.11, it is sufficient to show that $\lim_{k\to\infty} \eta_{n_k} = 0$ implies $\limsup_{k\to\infty} \tau_{n_k} \leqslant 0$ for any subsequence $\{n_k\}$ of $\{n\}$. Let $\{n_k\}$ be a subsequence of $\{n\}$ such that $\lim_{k\to\infty} \eta_{n_k} = 0$. Then, we have

$$\lim_{k\to\infty} D_f(\mathrm{Res}^f_{\lambda B} \circ A^f_\lambda(x_{n_k}), x_{n_k}) = 0. \tag{3.16}$$

By Lemma 2.9, we also have

$$\lim_{k\to\infty} \|\mathrm{Res}^f_{\lambda B} \circ A^f_\lambda(x_{n_k}) - x_{n_k}\| = 0. \tag{3.17}$$

Since f is uniformly Fréchet differentiable, it follows that ∇f is uniformly continuous on bounded subsets of E (see [51], Proposition 3.6.3), then

$$\|\nabla f(\mathrm{Res}^f_{\lambda B} \circ A^f_\lambda(x_{n_k})) - \nabla f(x_{n_k})\| \to 0. \tag{3.18}$$

From (3.18), we have

$$\begin{aligned} \|\nabla f(y_{n_k}) - \nabla f(x_{n_k})\| &\leqslant \|\nabla f(y_{n_k}) - \nabla f(\mathrm{Res}^f_{\lambda B} \circ A^f_\lambda(x_{n_k}))\| \\ &\quad + \|\nabla f(\mathrm{Res}^f_{\lambda B} \circ A^f_\lambda(x_{n_k})) - \nabla f(x_{n_k})\| \\ &= \alpha_{n_k}\|\nabla f(u) - \nabla f(\mathrm{Res}^f_{\lambda B} \circ A^f_\lambda(x_{n_k}))\| \\ &\quad + \|\nabla f(\mathrm{Res}^f_{\lambda B} \circ A^f_\lambda(x_{n_k})) - \nabla f(x_{n_k})\| \to 0. \end{aligned} \tag{3.19}$$

Since f is strongly coercive and uniformly convex on bounded sets, it follows that ∇f^* is uniformly continuous on bounded subsets of E^* (see [51], Theorem 3.5.10), then

$$\lim_{k\to\infty} \|y_{n_k} - x_{n_k}\| = \lim_{k\to\infty} \|\nabla f^*(\nabla f(y_{n_k})) - \nabla f^*(\nabla f(x_{n_k}))\| = 0. \tag{3.20}$$

By the reflexivity of the Banach space E and the boundedness of $\{x_{n_k}\}$, there exists a subsequence $\{x_{n_{k_i}}\}$ of $\{x_{n_k}\}$ such that $x_{n_{k_i}} \rightharpoonup \hat{x} \in E$ as $i \to \infty$ and

$$\limsup_{k\to\infty} \langle x_{n_k} - z, \nabla f(u) - \nabla f(z)\rangle = \lim_{i\to\infty} \langle x_{n_{k_i}} - z, \nabla f(u) - \nabla f(z)\rangle. \tag{3.21}$$

Since $\|\mathrm{Res}^f_{\lambda B} \circ A^f_\lambda(x_{n_{k_i}}) - x_{n_{k_i}}\| \to 0$ as $i \to \infty$, it follows that $\hat{x} \in \widehat{F}(\mathrm{Res}^f_{\lambda B} \circ A^f_\lambda) = F(\mathrm{Res}^f_{\lambda B} \circ A^f_\lambda)$. This implies that $\hat{x} \in (A+B)^{-1}0$. So we obtain

$$\limsup_{k\to\infty} \langle x_{n_k} - z, \nabla f(u) - \nabla f(z)\rangle = \langle \hat{x} - z, \nabla f(u) - \nabla f(z)\rangle \leqslant 0. \tag{3.22}$$

Furthermore, from (3.20), we also have

$$\limsup_{k\to\infty} \langle y_{n_k} - z, \nabla f(u) - \nabla f(z)\rangle \leqslant 0. \tag{3.23}$$

This means that $\limsup_{k\to\infty} \tau_{n_k} \leqslant 0$. We conclude by Lemma 2.11 that $\lim_{n\to\infty} s_n = 0$. Therefore, $x_n \to z$ as $n \to \infty$. This completes the proof. $\square$

R e m a r k 3.8. (1) Compare the results in [14], [13]. Our results are proved without the strict assumption $\lambda < (\alpha q/\kappa_q)^{1/(q-1)}$, where $\alpha > 0$, $1 < q \leqslant 2$ and κ_q is the q-uniform smoothness coefficient of E (see [50] for more detail).

(2) Compare Theorem 3.7 with other works in [29], [47], [52], [53]. The assumption $\liminf\limits_{n\to\infty} \beta_n > 0$ can be dropped.

If $\beta_n = 0$ for all $n \geqslant 1$, then we have the following result for the Halpern-type forward-backward splitting method in a reflexive Banach space.

Corollary 3.9. *Let $\{x_n\}$ be a sequence generated by $u, x_1 \in E$ and*

$$x_{n+1} = \nabla f^*(\alpha_n \nabla f(u) + (1-\alpha_n)\nabla f(\mathrm{Res}^f_{\lambda B} \circ A^f_\lambda(x_n))) \quad \forall\, n \geqslant 1, \tag{3.24}$$

where $\{\alpha_n\} \subset (0,1)$ and $\lambda > 0$. Suppose that $\lim\limits_{n\to\infty} \alpha_n = 0$ and $\sum\limits_{n=1}^{\infty} \alpha_n = \infty$. Then $\{x_n\}$ converges strongly to $z = P^f_{(A+B)^{-1}0}(u)$, where $P^f_{(A+B)^{-1}0}$ is the Bregman projection of E onto $(A+B)^{-1}0$.

If E is a uniformly convex Banach space which is also uniformly smooth and $f(x) = \|x\|^2$ for all $x \in E$, then we have the following result for the Halpern-type forward-backward splitting method.

Corollary 3.10. *Let $A\colon E \to E^*$ be a Bregman inverse strongly monotone mapping with respect to the function $f(x) = \|x\|^2$ for all $x \in E$ and let $B\colon E \multimap E^*$ be a maximal monotone mapping. Let $\{x_n\}$ be a sequence generated by $u, x_1 \in E$ and*

$$\begin{cases} y_n = J^{-1}(\alpha_n Ju + (1-\alpha_n)J(\mathrm{Res}^B_\lambda \circ A_\lambda(x_n))), \\ x_{n+1} = J^{-1}(\beta_n Jx_n + (1-\beta_n)Jy_n) \quad \forall\, n \geqslant 1, \end{cases} \tag{3.25}$$

where $\mathrm{Res}^B_\lambda := (J+\lambda B)^{-1}J$, $A_\lambda := J^{-1}(J-\lambda A)$ for $\lambda > 0$, $\{\alpha_n\} \subset (0,1)$ and $\{\beta_n\} \subset [0,1)$. Suppose that $\lim\limits_{n\to\infty} \alpha_n = 0$, $\sum\limits_{n=1}^{\infty} \alpha_n = \infty$ and $\limsup\limits_{n\to\infty} \beta_n < 1$. Then $\{x_n\}$ converges strongly to $z = \Pi_{(A+B)^{-1}0}(u)$, where $\Pi_{(A+B)^{-1}0}$ is the generalized projection of E onto $(A+B)^{-1}0$.

4. Some applications

In this section, we apply our results to the variational inequality problem and the equilibrium problem.

4.1. Variational inequality problem. Let E be a real reflexive Banach space. Let $f\colon E \to (-\infty, \infty]$ be a Legendre and totally convex function. Let $A\colon E \to E^*$ be a Bregman inverse strongly monotone mapping and C be a nonempty, closed and convex subset of $\operatorname{dom} A$. The *variational inequality problem* (VIP) is to find $z \in C$ such that

$$\langle x - z, Az\rangle \geqslant 0 \quad \forall\, x \in C.$$

The set of solutions of (VIP) is denoted by $\mathrm{VI}(C, A)$. Recall that the *indicator function* of C is given by

$$i_C(x) = \begin{cases} 0 & \text{if } x \in C, \\ \infty & \text{if } x \notin C. \end{cases}$$

It is known that i_C is a proper, lower semicontinuous and convex function and its subdifferential ∂i_C is maximal monotone (see [40], Theorem A). Moreover, from [1], Proposition 2.5.13, we know that

$$\partial i_C(x) = \begin{cases} N_C(x) & \text{if } x \in C, \\ \emptyset & \text{if } x \notin C, \end{cases}$$

where N_C is the normal cone of C given by

$$N_C(x) = \{x^* \in E^*\colon \langle y - x, x^*\rangle \leqslant 0 \quad \forall\, y \in C\}.$$

Thus, we can define the resolvent associated with ∂i_C for $\lambda > 0$ by

$$\mathrm{Res}^f_{\lambda \partial i_C}(x) = (\nabla f + \lambda \partial i_C)^{-1} \circ \nabla f(x) \quad \forall\, x \in E.$$

So we have for any $x \in E$ and $y \in C$,

$$\begin{aligned} z = \mathrm{Res}^f_{\lambda \partial i_C}(x) &\Leftrightarrow \nabla f(x) \in \nabla f(z) + \lambda \partial i_C(z) \\ &\Leftrightarrow \nabla f(x) \in \nabla f(z) + \lambda N_C(z) \\ &\Leftrightarrow \nabla f(x) - \nabla f(z) \in \lambda N_C(z) \\ &\Leftrightarrow \frac{1}{\lambda}\langle y - z, \nabla f(x) - \nabla f(z)\rangle \leqslant 0 \quad \forall\, y \in C \\ &\Leftrightarrow \langle y - z, \nabla f(x) - \nabla f(z)\rangle \leqslant 0 \quad \forall\, y \in C \\ &\Leftrightarrow z = P^f_C(x), \end{aligned}$$

where P^f_C is the Bregman projection from E onto C.

Proposition 4.1 ([37], Proposition 8). *Let $f\colon E \to (-\infty, \infty]$ be a Legendre and totally convex function. Let $A\colon E \to E^*$ be a Bregman inverse strongly monotone mapping. If C is a nonempty, closed and convex subset of* $\operatorname{dom} A \cap \operatorname{int}(\operatorname{dom} f)$*, then* $\mathrm{VI}(C, A) = F(P_C^f \circ A_\lambda^f)$ *for* $\lambda > 0$.

Setting $B = \partial i_C$ in Theorems 3.3 and 3.7, we obtain by Proposition 4.1 the following results.

Theorem 4.2. *Let $f\colon E \to \mathbb{R}$ be a strongly coercive, bounded and Fréchet differentiable Legendre function which is totally convex on bounded subsets of E such that ∇f is weakly sequentially continuous. Let $A\colon E \to E^*$ be a Bregman inverse strongly monotone mapping and C be a nonempty, closed and convex subset of* $\operatorname{dom} A \cap \operatorname{int}(\operatorname{dom} f)$*. Let $\{x_n\}$ be a sequence generated by $x_1 \in C$ and*

$$x_{n+1} = \nabla f^*(\alpha_n \nabla f(x_n) + (1-\alpha_n)\nabla f(P_C^f \circ A_\lambda^f(x_n))) \quad \forall\, n \geqslant 1, \tag{4.1}$$

where $\{\alpha_n\} \subset [0, 1)$ and $\lambda > 0$. Suppose that $\limsup_{n\to\infty} \alpha_n < 1$. If $\mathrm{VI}(C, A) \neq \emptyset$*, then $\{x_n\}$ converges weakly to a point in* $\mathrm{VI}(C, A)$.

Theorem 4.3. *Let $f\colon E \to \mathbb{R}$ be a strongly coercive, bounded and Fréchet differentiable Legendre function which is totally convex on bounded subsets of E. Let $A\colon E \to E^*$ be a Bregman inverse strongly monotone mapping and C be a nonempty, closed and convex subset of* $\operatorname{dom} A \cap \operatorname{int}(\operatorname{dom} f)$*. Let $\{x_n\}$ be a sequence generated by $u, x_1 \in C$ and*

$$\begin{cases} y_n = \nabla f^*(\alpha_n \nabla f(u) + (1-\alpha_n)\nabla f(P_C^f \circ A_\lambda^f(x_n))), \\ x_{n+1} = \nabla f^*(\beta_n \nabla f(x_n) + (1-\beta_n)\nabla f(y_n)) \quad \forall\, n \geqslant 1, \end{cases} \tag{4.2}$$

where $\{\alpha_n\} \subset (0, 1)$, $\{\beta_n\} \subset [0, 1)$ and $\lambda > 0$. Suppose that $\lim_{n\to\infty} \alpha_n = 0$, $\sum_{n=1}^{\infty} \alpha_n = \infty$ and $\limsup_{n\to\infty} \beta_n < 1$. If $\mathrm{VI}(C, A) \neq \emptyset$*, then $\{x_n\}$ converges strongly to $z = P^f_{\mathrm{VI}(C,A)}(u)$.*

4.2. Equilibrium problem. Let C be a nonempty, closed and convex subset of a real reflexive Banach space E. Let $G\colon C \times C \to \mathbb{R}$ be a bifunction. The *equilibrium problem* (EP) is to find $z \in C$ such that

$$G(z, x) \geqslant 0 \quad \forall\, x \in C.$$

The set of solutions of (EP) is denoted by $\mathrm{EP}(G)$. For solving this problem, let us assume that the bifunction satisfies the following conditions:

(A1) $G(x,x)=0$ for all $x\in C$,

(A2) G is monotone, i.e., $G(x,y)+G(y,x)\leqslant 0$ for all $x,y\in C$,

(A3) for each $x,y,z\in C$, $\limsup\limits_{t\to 0} G(tz+(1-t)x,y)\leqslant G(x,y)$,

(A4) for each $x\in C$, the function $y\mapsto G(x,y)$ is convex and lower semicontinuous.

For $\lambda>0$ and $x\in E$, we define the resolvent operator $\operatorname{Res}^f_{\lambda G}\colon E\multimap C$ by

$$\operatorname{Res}^f_{\lambda G}(x)=\Big\{z\in C\colon\ G(z,x)+\frac{1}{\lambda}\langle y-z,\nabla f(z)-\nabla f(x)\rangle\geqslant 0\ \forall\, y\in C\Big\}.$$

Proposition 4.4 ([37], Lemma 2)**.** *Let $f\colon E\to(-\infty,\infty]$ be a Legendre function. Let C be a closed and convex subset of E. Let $G\colon C\times C\to\mathbb{R}$ be a bifunction which satisfies conditions* (A1)–(A4)*. Then the following hold:*

(i) $\operatorname{Res}^f_{\lambda G}$ *is single-valued,*

(ii) $\operatorname{Res}^f_{\lambda G}$ *is BFNE,*

(iii) $F(\operatorname{Res}^f_{\lambda G})=\operatorname{EP}(G)$,

(iv) $\operatorname{EP}(G)$ *is closed and convex,*

(v) *for all $x\in E$ and $z\in F(\operatorname{Res}^f_{\lambda G})$, we have*

$$D_f(z,\operatorname{Res}^f_{\lambda G}(x))+D_f(\operatorname{Res}^f_{\lambda G}(x),x)\leqslant D_f(z,x).$$

Remark 4.5. If f is uniformly Fréchet differentiable and bounded on bounded subsets of E, then from Proposition 4.4, we have $F(\operatorname{Res}^f_{\lambda G})=\widehat{F}(\operatorname{Res}^f_{\lambda G})$.

Proposition 4.6 ([42], Proposition 4.2)**.** *Let $f\colon E\to(-\infty,\infty]$ be a strongly coercive and Fréchet differentiable Legendre function which is totally convex on bounded subsets of E. Let C be a closed and convex subset of E. Let $G\colon C\times C\to\mathbb{R}$ be a bifunction which satisfies conditions* (A1)–(A4)*. Let $A_G\colon E\multimap E^*$ be defined by*

$$A_G(x)=\begin{cases}\{z\in E^*\colon\ G(x,y)\geqslant\langle y-x,z\rangle\ \forall\, y\in C\} & \text{if } x\in C,\\ \emptyset & \text{if } x\notin C.\end{cases}$$

Then the following hold:

(i) *A_G is maximal monotone with* $\operatorname{EP}(G)=A_G^{-1}0$,

(ii) $\operatorname{Res}^f_{\lambda G}$ *is the resolvent of A_G, i.e.,* $\operatorname{Res}^f_{\lambda G}=(\nabla f+\lambda A_G)^{-1}$ *for* $\lambda>0$.

If we set $B=A_G$ and $A=0$ in Theorems 3.3 and 3.7, then by Proposition 4.6, we obtain the following results.

Theorem 4.7. *Let $f\colon E\to\mathbb{R}$ be a strongly coercive, bounded and Fréchet differentiable Legendre function which is totally convex on bounded subsets of E such that ∇f is weakly sequentially continuous. Let C be a closed and convex subset of E.*

Let $G\colon C\times C\to\mathbb{R}$ be a bifunction which satisfies conditions (A1)–(A4). *Let $\{x_n\}$ be a sequence generated by $x_1\in C$ and*

$$x_{n+1}=\nabla f^*(\alpha_n\nabla f(x_n)+(1-\alpha_n)\nabla f(\mathrm{Res}^f_{\lambda G}(x_n)))\quad\forall\, n\geqslant 1,\tag{4.3}$$

where $\{\alpha_n\}\subset[0,1)$ and $\lambda>0$. Suppose that $\limsup\limits_{n\to\infty}\alpha_n<1$. If $\mathrm{EP}(G)\neq\emptyset$, then $\{x_n\}$ converges weakly to a point in $\mathrm{EP}(G)$.

Theorem 4.8. *Let $f\colon E\to\mathbb{R}$ be a strongly coercive, bounded and Fréchet differentiable Legendre function which is totally convex on bounded subsets of E. Let C be a nonempty, closed and convex subset of E. Let $G\colon C\times C\to\mathbb{R}$ be a bifunction which satisfies conditions* (A1)–(A4). *Let $\{x_n\}$ be a sequence generated by $u,x_1\in C$ and*

$$\begin{cases} y_n=\nabla f^*(\alpha_n\nabla f(u)+(1-\alpha_n)\nabla f(\mathrm{Res}^f_{\lambda G}(x_n))),\\ x_{n+1}=\nabla f^*(\beta_n\nabla f(x_n)+(1-\beta_n)\nabla f(y_n))\quad\forall\, n\geqslant 1,\end{cases}\tag{4.4}$$

where $\{\alpha_n\}\subset(0,1)$, $\{\beta_n\}\subset[0,1)$ and $\lambda>0$. Suppose that $\lim\limits_{n\to\infty}\alpha_n=0$, $\sum\limits_{n=1}^{\infty}\alpha_n=\infty$ and $\limsup\limits_{n\to\infty}\beta_n<1$. If $\mathrm{EP}(G)\neq\emptyset$, then $\{x_n\}$ converges strongly to $z=P^f_{\mathrm{EP}(G)}(u)$.

5. Numerical example

In this section, we provide a numerical example to illustrate the behaviour of our Algorithms 3.2 and 3.6.

Example 5.1. Let $E=\mathbb{R}^3$ with Euclidean norm. Let $f(x)=\|x\|^2$ for all $x=(y_1,y_2,y_3)^\top\in\mathbb{R}^3$, then f satisfies Assumption 3.1 (see [10]). Thus we have $\nabla f(x)=2x$ and $\nabla f^*(x^*)=\frac{1}{2}x^*$. Let $A\colon\mathbb{R}^3\to\mathbb{R}^3$ be defined by $Ax=\frac{1}{2}x+(-1,2,0)^\top$ and $B\colon\mathbb{R}^3\to\mathbb{R}^3$ be defined by $Bx=4x$ for all $x\in\mathbb{R}^3$. We see that A is a Bregman inverse strongly monotone mapping and B is a maximal monotone mapping. Indeed, let $x,y\in\mathbb{R}^3$, then

$$\begin{aligned}&\langle Ax-Ay,\nabla f^*(\nabla f(x)-Ax)-\nabla f^*(\nabla f(y)-Ay)\rangle\\ &\quad=\Big\langle\frac{1}{2}x-\frac{1}{2}y,\nabla f^*\Big(2x-\frac{1}{2}x-(-1,2,0)\Big)-\nabla f^*\Big(2y-\frac{1}{2}y-(-1,2,0)\Big)\Big\rangle\\ &\quad=\Big\langle\frac{1}{2}x-\frac{1}{2}y,\frac{3}{4}x-\frac{3}{4}y\Big\rangle=\frac{3}{8}\|x-y\|^2\geqslant 0.\end{aligned}$$

We also have

$$\langle Bx-By,x-y\rangle=4\|x-y\|^2\geqslant 0$$

and $\operatorname{ran}(\nabla f + \lambda B) = \mathbb{R}^3$ for all $\lambda > 0$. Then the explicit form of the resolvent operator of B corresponding to f can be written by

$$\begin{aligned}\operatorname{Res}^f_{\lambda B} \circ A^f_\lambda(x) &= (\nabla f + \lambda B)^{-1} \circ \nabla f \circ (\nabla f^* \circ (\nabla f - \lambda A)x)\\ &= \frac{4-\lambda}{2(2+4\lambda)}x - \frac{\lambda}{2+4\lambda}(-1,2,0),\end{aligned}$$

where $\lambda > 0$. It is not hard to check that $(A+B)^{-1}0 = \{(\frac{2}{9}, -\frac{4}{9}, 0)^\top\}$. Let us denote $z = (\frac{2}{9}, -\frac{4}{9}, 0)^\top$. We choose $\alpha_n = 1/(100n+1)$, $\beta_n = n/(3n+1)$ and $u = x_1 = (5,-4,9)^\top$ and use the stopping rule $\|x_n - z\| < \varepsilon$ to terminate the iterative processes. We consider three different cases of λ ($\lambda = 0.01$, $\lambda = 0.05$ and $\lambda = 0.25$). The numerical results are reported in Table 1.

		Algorithm 3.2		Algorithm 3.6	
λ	ε	No. of Iter.	Time (sec.)	No. of Iter.	Time (sec.)
0.01	10^{-6}	558	2.805×10^{-3}	810	5.390×10^{-3}
	10^{-9}	867	4.358×10^{-3}	1276	8.510×10^{-3}
	10^{-15}	1487	7.569×10^{-3}	2209	1.518×10^{-2}
0.05	10^{-6}	132	6.630×10^{-4}	193	1.276×10^{-3}
	10^{-9}	196	9.737×10^{-4}	291	1.923×10^{-3}
	10^{-15}	324	1.863×10^{-3}	486	3.246×10^{-3}
0.25	10^{-6}	35	1.788×10^{-4}	53	3.526×10^{-4}
	10^{-9}	50	2.575×10^{-4}	77	4.884×10^{-4}
	10^{-15}	79	4.481×10^{-4}	125	8.284×10^{-4}

Table 1. The numerical experiments for Example 5.1 in each given λ and ε.

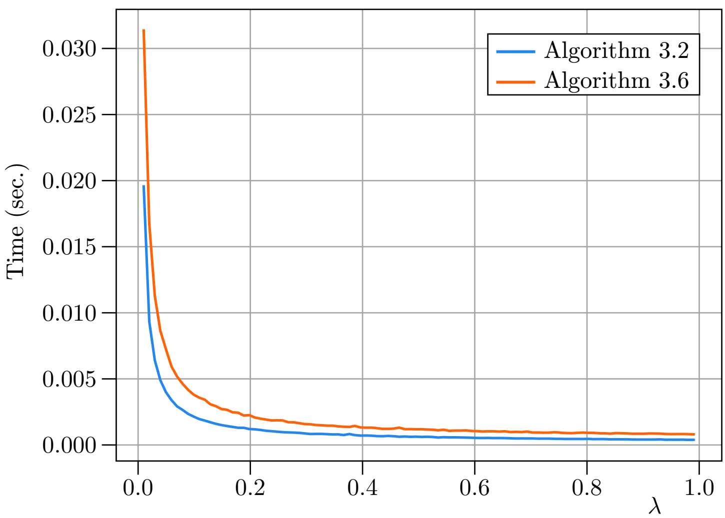

Figure 1. The relation between $\lambda \in (0,1)$ and the average running time with $\varepsilon = 10^{-15}$.

According to the experiments, we see that the average running times are significantly dependent on the values of λ. One observes that when λ increases, less iterations are required. We illustrate the relation between $\lambda \in (0,1)$ and the average running time in Figure 1 with $\varepsilon = 10^{-15}$.

6. Conclusions

In this paper, we introduced two modified forward-backward splitting algorithms for the problem of finding zeros of the sum of a maximal monotone operator and a Bregman inverse strongly monotone operator in reflexive Banach spaces. Our algorithms are based on two well-known methods, which are the Mann-type iteration and the Halpern-type iteration. We study weak and strong convergence results of the proposed algorithms for solving such a problem. Some applications related to the obtained results are presented. A numerical example is performed to illustrate the convergence results.

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