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Further generalized versions of Ilmanen’s lemma on insertion of $C^{1,\omega}$ or $C_{\text{loc}}^{1,\omega}$ functions

VÁCLAV KRYŠTOF

Abstract. The author proved in 2018 that if G is an open subset of a Hilbert space, $f_1, f_2: G \rightarrow \mathbb{R}$ continuous functions and ω a nontrivial modulus such that $f_1 \leq f_2$, f_1 is locally semiconvex with modulus ω and f_2 is locally semiconcave with modulus ω , then there exists $f \in C_{\text{loc}}^{1,\omega}(G)$ such that $f_1 \leq f \leq f_2$. This is a generalization of Ilmanen’s lemma (which deals with linear modulus and functions on an open subset of \mathbb{R}^n). Here we extend the mentioned result from Hilbert spaces to some superreflexive spaces, in particular to L^p spaces, $p \in [2, \infty)$. We also prove a “global” version of Ilmanen’s lemma (where a $C^{1,\omega}$ function is inserted between functions on an interval $I \subset \mathbb{R}$).

Keywords: Ilmanen’s lemma; $C^{1,\omega}$ function; semiconvex function with general modulus

Classification: 26B25

1. Introduction

Let $A \subset \mathbb{R}^n$ be a convex set. We say that $f: A \rightarrow \mathbb{R}$ is classically semiconvex if there exists $C > 0$ such that the function $x \mapsto f(x) + C|x|^2$, $x \in A$, is convex. We say that $f: A \rightarrow \mathbb{R}$ is classically semiconcave if $-f$ is classically semiconvex. T. Ilmanen proved the following result [8, proof of 4F from 4G, page 199]:

Ilmanen’s lemma. *Let $G \subset \mathbb{R}^n$ be an open set and $f_1, f_2: G \rightarrow \mathbb{R}$. Suppose that $f_1 \leq f_2$, f_1 is locally classically semiconvex and f_2 is locally classically semiconcave. Then there exists $f \in C_{\text{loc}}^{1,1}(G)$ such that $f_1 \leq f \leq f_2$.*

We will work with semiconvex (or semiconcave) functions with general modulus, see Definition 2.2 below and cf. [3, Definition 2.1.1]. Note that the classically semiconvex functions coincide with semiconvex functions with modulus $\omega(t) = Ct$ where $C > 0$, cf. [3, Proposition 1.1.3].

The following generalization of Ilmanen’s lemma was proved in [9]:

Theorem ([9, Theorem 4.5]). *Let X be a Hilbert space, $G \subset X$ an open set, $f_1, f_2: G \rightarrow \mathbb{R}$ continuous functions and ω a modulus. Suppose that*

$$\liminf_{t \rightarrow 0^+} \frac{\omega(t)}{t} > 0, \quad f_1 \leq f_2$$

and that for every $x \in G$ there exist $C, r > 0$ such that $B(x, r) \subset G$, $f_1 \upharpoonright_{B(x, r)}$ is semiconvex with modulus $C\omega$ and $f_2 \downharpoonright_{B(x, r)}$ is semiconcave with modulus $C\omega$. Then there exists $f \in C_{loc}^{1, \omega}(G)$ such that $f_1 \leq f \leq f_2$.

We generalize this result to some superreflexive spaces, see Theorem 3.9 below, in particular to Lebesgue spaces L^p , $p \in (2, \infty)$, see Corollary 3.10 below.

In [9], we also proved a result ([9, Corollary 3.2]) on insertion of $C^{1, \omega}$ functions which generalizes [2, Theorem 2] (which works with linear modulus and functions on Hilbert space). [9, Corollary 3.2] can be reformulated as follows:

Theorem. *Let X be a normed linear space, $f_1, f_2: X \rightarrow \mathbb{R}$ continuous functions and ω a modulus. Suppose that f_1 is semiconvex with modulus ω , f_2 is semiconcave with modulus ω and $f_1 \leq f_2$. Then there exists $f \in C^{1, \omega}(X)$ such that $f_1 \leq f \leq f_2$.*

In the last section we prove Corollary 4.4 which is a precise analogue of the previous theorem for functions on an interval $I \subset \mathbb{R}$.

2. Preliminaries

Throughout this article, all normed linear spaces (Banach spaces, respectively) are real. By $B(x, r)$ we denote the open ball with center x and radius r . If P is a metric space, then we denote by $C(P)$ the set of all continuous functions $f: P \rightarrow \mathbb{R}$.

Notation 2.1. We denote by \mathcal{M} the set of all $\omega: [0, \infty) \rightarrow [0, \infty)$ which are non-decreasing and satisfy $\lim_{t \rightarrow 0^+} \omega(t) = 0$.

Definition 2.2. Let X be a normed linear space, $A \subset X$ a convex set and $\omega \in \mathcal{M}$.

- We say that $f: A \rightarrow \mathbb{R}$ is semiconvex with modulus ω if

$$f(\lambda x + (1 - \lambda)y) \leq \lambda f(x) + (1 - \lambda)f(y) + \lambda(1 - \lambda)\|x - y\|\omega(\|x - y\|)$$

for every $x, y \in A$ and $\lambda \in [0, 1]$.

- We say that $f: A \rightarrow \mathbb{R}$ is semiconcave with modulus ω if $-f$ is semiconvex with modulus ω .

- We denote by $SC^\omega(A)$ the set of all $f: A \rightarrow \mathbb{R}$ which are semiconvex with modulus $C\omega$ for some $C > 0$. We denote by $-SC^\omega(A)$ the set of all $f: A \rightarrow \mathbb{R}$ such that $-f \in SC^\omega(A)$.

Let G be an open subset of a normed linear space, $\alpha \in (0, 1]$ and $\omega \in \mathcal{M}$. We denote by $C^{1,\omega}(G)$ the set of all Fréchet differentiable $f: G \rightarrow \mathbb{R}$ such that f' is uniformly continuous with modulus $C\omega$ for some $C > 0$, and we denote by $C_{loc}^{1,\omega}(G)$ the set of all $f: G \rightarrow \mathbb{R}$ which are locally $C^{1,\omega}$. If $\omega(t) = t^\alpha$ for every $t \in [0, \infty)$, then we sometimes write $C^{1,\alpha}(G)$ instead of $C^{1,\omega}(G)$.

Let X be a normed linear space, $G \subset X$ an open convex set and $\omega \in \mathcal{M}$. Then (cf. [5, Corollary 3.6])

$$\left(f \in C(G) \cap SC^\omega(G), \liminf_{t \rightarrow 0^+} \frac{\omega(t)}{t} = 0 \right) \Rightarrow f \text{ is convex}$$

and, see [9, Proposition 2.5],

$$(1) \quad (f \in C(G) \cap SC^\omega(G)) \Rightarrow f \text{ is locally Lipschitz}$$

and (see [9, Theorem 2.6] or for the case $X = \mathbb{R}^d$ see [3, Proposition 2.1.2])

$$(2) \quad C^{1,\omega}(G) \subset C(G) \cap SC^\omega(G) \cap (-SC^\omega(G)).$$

If, moreover, G is bounded or $G = X$, then, see [9, Theorem 2.6],

$$C^{1,\omega}(G) = C(G) \cap SC^\omega(G) \cap (-SC^\omega(G)).$$

Let $\alpha \in (0, 1]$. We say that a Banach space X admits an equivalent norm with modulus of smoothness of power type $1 + \alpha$ if there exists an equivalent norm $\|\cdot\|$ on X and $C > 0$ such that

$$\frac{\|x + ty\| + \|x - ty\|}{2} - 1 \leq Ct^{1+\alpha}, \quad t > 0, x, y \in X, \|x\| = \|y\| = 1.$$

For a Banach space X we have the following (Pisier's) result, see [1, Theorem A.6]:

$$(3) \quad X \text{ is superreflexive if and only if } X \text{ admits an equivalent norm with modulus of smoothness of power type } 1 + \alpha \text{ for some } \alpha \in (0, 1].$$

3. Insertion of a $C_{loc}^{1,\omega}$ function

In this section we prove a generalization of Ilmanen's lemma which improves [9, Theorem 4.5] (which works with functions on Hilbert spaces) to some superreflexive spaces, in particular to L^p spaces, $p \in [2, \infty)$ (see Theorem 3.9 or

Corollary 3.10 below). We use essentially the same methods as in [9]. However, besides these methods, we need Proposition 3.7 below. This result is implicitly contained in [7] for $G = X$. We reduce the case of an arbitrary G to the case $G = X$ using the notion of a partition ring and [7, Lemma 7.49], see the proof of Lemma 3.6 below. So, we need to recall some definitions from [7].

Definition 3.1 (cf. [7, page 411]). Let P be a metric space. We say that a family $\{A_\gamma\}_{\gamma \in \Gamma}$ of subsets of P is

- locally finite if for every $x \in P$ there exists a neighbourhood U of x such that $\{\gamma \in \Gamma : A_\gamma \cap U \neq \emptyset\}$ is finite;
- uniformly discrete if there exists $\delta > 0$ such that $\text{dist}(A_{\gamma_1}, A_{\gamma_2}) \geq \delta$ whenever $\gamma_1, \gamma_2 \in \Gamma$, $\gamma_1 \neq \gamma_2$;
- σ -uniformly discrete if there are $\Gamma_n \subset \Gamma$, $n \in \mathbb{N}$, such that $\Gamma = \bigcup_{n \in \mathbb{N}} \Gamma_n$ and $\{A_\gamma\}_{\gamma \in \Gamma_n}$ is uniformly discrete for every $n \in \mathbb{N}$.

Remark 3.2. If \mathcal{A} is a (nonindexed) family of subsets of a metric space, then we can regard it as an (indexed) family $\{A\}_{A \in \mathcal{A}}$. Hence the previous definition can be applied to \mathcal{A} too.

Notation 3.3. Let P be a metric space and $f : P \rightarrow \mathbb{R}$. Then we set $\text{supp}_o f := \{x \in P : f(x) \neq 0\}$.

Definition 3.4 (see [7, page 411]). Let P be a metric space and $S \subset C(P)$. We say that P admits locally finite and σ -uniformly discrete S -partitions of unity if for every open cover \mathcal{U} of P there exists a system $\{\psi_\gamma\}_{\gamma \in \Gamma}$ of functions from S such that the following hold:

- $\{\text{supp}_o \psi_\gamma\}_{\gamma \in \Gamma}$ is locally finite and σ -uniformly discrete.
- For every $\gamma \in \Gamma$ there exists $U \in \mathcal{U}$ such that $\text{supp}_o \psi_\gamma \subset U$.
- $0 \leq \psi_\gamma \leq 1$ for every $\gamma \in \Gamma$, and $\sum_{\gamma \in \Gamma} \psi_\gamma(x) = 1$ for every $x \in P$.

Definition 3.5 (see [7, page 411]). Let P be a metric space and $S \subset C(P)$ a ring of functions. We say that S is a partition ring on P if the following hold:

- For every $S_0 \subset S$ such that $\text{supp}_o f$ is bounded for every $f \in S_0$ and $\{\text{supp}_o f : f \in S_0\}$ is uniformly discrete, there exists $g \in S$ such that

$$\text{supp}_o g = \bigcup_{f \in S_0} \text{supp}_o f.$$

- For every $f \in S$ and every open $U_1, U_2 \subset P$ such that $\text{dist}(U_1, U_2) > 0$ and $\text{supp}_o f = U_1 \cup U_2$, we have $\chi_{U_1} \cdot f \in S$.
- For every $f \in S$ bounded below and every $\varepsilon > 0$ there exists $g \in S$ such that $0 \leq g \leq 1$, $f^{-1}((-\infty, \varepsilon]) \subset g^{-1}(\{0\})$ and $f^{-1}([2\varepsilon, \infty)) \subset g^{-1}(\{1\})$.

Lemma 3.6. *Let P be a metric space and S a partition ring on P . Then the following are equivalent:*

- (i) *Space P admits locally finite and σ -uniformly discrete S -partitions of unity.*
- (ii) *Every open $G \subset P$ admits locally finite and σ -uniformly discrete S_G -partitions of unity (where $S_G := \{f \upharpoonright_G : f \in S, \text{supp}_o f \subset G\}$).*
- (iii) *$\{\text{supp}_o f : f \in S\}$ contains a σ -uniformly discrete basis for the topology of P .*

PROOF: (ii) trivially implies (i) and (i) is equivalent to (iii) by [7, Lemma 7.49].

(iii) \Rightarrow (ii): Let $G \subset P$ be an open set. It follows easily from the definitions that S_G is a partition ring on G . By the assumption, there exists $S^* \subset S$ such that $\{\text{supp}_o f : f \in S^*\}$ is a σ -uniformly discrete basis for the topology of P . Set $S_G^* := \{f \upharpoonright_G : f \in S^*, \text{supp}_o f \subset G\}$. Then $S_G^* \subset S_G$ and $\{\text{supp}_o f : f \in S_G^*\}$ is a σ -uniformly discrete basis for the topology of G . Thus $\{\text{supp}_o f : f \in S_G\}$ contains a σ -uniformly discrete basis for the topology of G . If we apply the implication (iii) \Rightarrow (i) to the case $P := G$ and $S := S_G$, we obtain that G admits locally finite and σ -uniformly discrete S_G -partitions of unity. □

Proposition 3.7. *Let X be a Banach space, $G \subset X$ an open set and $\alpha \in (0, 1]$. Suppose that X admits an equivalent norm with modulus of smoothness of power type $1 + \alpha$. Then G admits locally finite and σ -uniformly discrete $C^{1,\alpha}(G)$ -partitions of unity.*

PROOF: Firstly, X is superreflexive, see (3). Further, X admits an equivalent norm which is uniformly rotund and has modulus of smoothness of power type $1 + \alpha$ (this fact is contained in the proof of [4, Proposition IV.5.2], see also the note after that proposition). Note also that the Fréchet derivative of such a norm is α -Hölder on the corresponding unit sphere, see [4, Proposition IV.5.1] or [5, Lemma 2.6] for more details. So, all the statements of [7, Theorem 7.56] and [7, Proposition 7.58] holds with our α (for this, see the beginning of the proof of [7, Proposition 7.58]).

Denote by S the set of all $f \in C^{1,\alpha}(X)$ which are bounded and have bounded derivative, and set $S_G := \{f \upharpoonright_G : f \in S, \text{supp}_o f \subset G\}$. It is proved in the proof of [7, Theorem 7.56] that S is a partition ring on X and that the condition (ii) of [7, Lemma 7.49] holds. Thus, by [7, Lemma 7.49], X admits locally finite and σ -uniformly discrete S -partitions of unity. Hence G admits locally finite and σ -uniformly discrete S_G -partitions of unity by Lemma 3.6. Now the assertion of the proposition follows. □

Note that the converse of the previous proposition also holds as is shown in the following remark:

Remark 3.8. Let X be a Banach space and $\alpha \in (0, 1]$. Suppose that some nonempty open subset G of X admits locally finite and σ -uniformly discrete $C^{1,\alpha}(G)$ -partitions of unity. Then we can easily find $f \in C^{1,\alpha}(X)$ such that $\text{supp}_o f$ is nonempty and bounded (i.e. X admits a $C^{1,\alpha}(X)$ -smooth bump). Hence X admits an equivalent norm with modulus of smoothness of power type $1 + \alpha$ by [7, Theorem 5.50].

Theorem 3.9. Let X be a Banach space, $G \subset X$ an open set, $f_1, f_2 : G \rightarrow \mathbb{R}$, $\omega \in \mathcal{M}$ and $\alpha \in (0, 1]$. Suppose that the following hold:

- (a) Space X admits an equivalent norm with modulus of smoothness of power type $1 + \alpha$.
- (b) $\liminf_{t \rightarrow 0^+} \omega(t)/t^\alpha > 0$.
- (c) Functions f_1 and f_2 are continuous, $f_1 \leq f_2$ and for every $x \in G$ there exist $C, r > 0$ such that $B(x, r) \subset G$, $f_1 \upharpoonright_{B(x,r)}$ is semiconvex with modulus $C\omega$ and $f_2 \upharpoonright_{B(x,r)}$ is semiconcave with modulus $C\omega$.

Then there exists $f \in C_{\text{loc}}^{1,\omega}(G)$ such that $f_1 \leq f \leq f_2$.

PROOF: By (1) and condition (c), for every $x \in G$ we can find $r_x \in (0, 1)$ such that $B(x, 2r_x) \subset G$,

$$f_1 \upharpoonright_{B(x,2r_x)} \in SC^\omega(B(x, 2r_x)), \quad f_2 \upharpoonright_{B(x,2r_x)} \in -SC^\omega(B(x, 2r_x))$$

and f_1 and f_2 are Lipschitz on $B(x, 2r_x)$. Since $\{B(x, r_x) : x \in G\}$ is an open cover of G , we can find, by Proposition 3.7, a system $\{\psi_\gamma\}_{\gamma \in \Gamma}$ of functions from $C^{1,\alpha}(G)$ such that the following hold:

- $\{\text{supp}_o \psi_\gamma\}_{\gamma \in \Gamma}$ is locally finite;
- for every $\gamma \in \Gamma$ there exists $x_\gamma \in G$ such that $\text{supp}_o \psi_\gamma \subset B(x_\gamma, r_{x_\gamma})$;
- $0 \leq \psi_\gamma \leq 1$ for every $\gamma \in \Gamma$, and $\sum_{\gamma \in \Gamma} \psi_\gamma(x) = 1$ for every $x \in G$.

We will show that for every $\gamma \in \Gamma$ there exists $F_\gamma \in C^{1,\omega}(G)$ such that

$$(4) \quad \psi_\gamma(x)f_1(x) \leq F_\gamma(x) \leq \psi_\gamma(x)f_2(x), \quad x \in G.$$

Let $\gamma \in \Gamma$. We put $B := B(x_\gamma, 2r_{x_\gamma})$ and for $i \in \{1, 2\}$ define a function h_i by

$$h_i(x) := \begin{cases} \psi_\gamma(x)f_i(x), & x \in G, \\ 0, & x \in X \setminus G. \end{cases}$$

Then clearly f_1 and f_2 are Lipschitz on B , $f_1 \upharpoonright_B \in SC^\omega(B)$, $f_2 \upharpoonright_B \in -SC^\omega(B)$, $h_1 \leq h_2$ and

$$(5) \quad \text{supp } h_i := \overline{\text{supp}_o h_i} \subset \overline{B(x_\gamma, r_{x_\gamma})} \subset B \subset G, \quad i \in \{1, 2\}.$$

It follows from condition (b) that there exists $c > 0$ such that

$$(6) \quad t^\alpha \leq c\omega(t), \quad t \in [0, 4].$$

Noting that $\text{diam } B = 4r_{x_\gamma} < 4$, (6) implies $\psi_\gamma \upharpoonright_B \in C^{1,\omega}(B)$. Further, it also follows from condition (b) that $\liminf_{t \rightarrow 0^+} \omega(t)/t > 0$. Thus

$$h_1 \upharpoonright_B \in SC^\omega(B) \quad \text{and} \quad h_2 \upharpoonright_B \in -SC^\omega(B)$$

by [9, Lemma 4.2] (applied with $A := B$, $g_1 := \psi_\gamma \upharpoonright_B$ and $g_2 := f_1 \upharpoonright_B$ or $g_2 := -f_2 \upharpoonright_B$). Hence $h_1 \in SC^\omega(X)$ and $h_2 \in -SC^\omega(X)$ by [9, Lemma 4.3]. Note that it follows from (5) (and the continuity of ψ_γ, f_1 and f_2) that h_1 and h_2 are continuous. Thus, by [9, Corollary 3.2], there exists $h \in C^{1,\omega}(X)$ such that $h_1 \leq h \leq h_2$. Then (4) holds with $F_\gamma := h \upharpoonright_G$ and we are done.

Set

$$f(x) := \sum_{\gamma \in \Gamma} F_\gamma(x), \quad x \in G.$$

It follows from (4) that $\{\text{supp}_o F_\gamma\}_{\gamma \in \Gamma}$ is locally finite. Hence f is well defined and $f \in C^{1,\omega}_{\text{loc}}(G)$. Summing (4) over $\gamma \in \Gamma$ we obtain that $f_1 \leq f \leq f_2$. \square

Corollary 3.10. *Let μ be a nonnegative measure (on an arbitrary σ -algebra) and $p \in [2, \infty)$. Denote by X the Lebesgue space $L^p(\mu)$. Let $G \subset X$ be an open set, $f_1, f_2: G \rightarrow \mathbb{R}$ and $\omega \in \mathcal{M}$. Suppose further that $\liminf_{t \rightarrow 0^+} \omega(t)/t > 0$ and that condition (c) of Theorem 3.9 holds. Then there exists $f \in C^{1,\omega}_{\text{loc}}(G)$ such that $f_1 \leq f \leq f_2$.*

PROOF: The canonical norm on X has modulus of smoothness of power type 2 by [4, Corollary V.1.2]. The rest now follows from Theorem 3.9. \square

4. Insertion of a $C^{1,\omega}$ function on an interval $I \subset \mathbb{R}$

The main results of this section are Theorem 4.3 and Corollary 4.4 below. We begin with two facts concerning the set $SC^\omega(I) \cap (-SC^\omega(I))$.

Proposition 4.1. *Let $I \subset \mathbb{R}$ be an interval and $\omega \in \mathcal{M}$. Then the following hold:*

- (i) $SC^\omega(I) \cap (-SC^\omega(I)) \subset C(I)$.
- (ii) *If I is open, then $C^{1,\omega}(I) = SC^\omega(I) \cap (-SC^\omega(I))$.*

PROOF: (i): Let $f \in SC^\omega(I) \cap (-SC^\omega(I))$. Then there exists $C > 0$ such that

$$|f(\lambda x + (1 - \lambda)y) - \lambda f(x) - (1 - \lambda)f(y)| \leq \lambda(1 - \lambda)|x - y|C\omega(|x - y|)$$

for every $x, y \in I$ and $\lambda \in [0, 1]$. Thus

$$\begin{aligned} \limsup_{\lambda \rightarrow 1^-} |f(\lambda x + (1 - \lambda)y) - f(x)| \\ &= \limsup_{\lambda \rightarrow 1^-} |f(\lambda x + (1 - \lambda)y) - \lambda f(x) - (1 - \lambda)f(y)| \\ &\leq \limsup_{\lambda \rightarrow 1^-} \lambda(1 - \lambda)|x - y|C\omega(|x - y|) = 0, \quad x, y \in I, \end{aligned}$$

and hence $\lim_{\lambda \rightarrow 1^-} |f(\lambda x + (1 - \lambda)y) - f(x)| = 0, x, y \in I$. This implies that f is continuous.

(ii): $C^{1,\omega}(I) \subset SC^\omega(I) \cap (-SC^\omega(I))$ by (2). The reverse inclusion follows from part (i), [6, Lemma 2.5 (i)] and [6, Proposition 2.8 (i)]. \square

Lemma 4.2. *Let $I \subset \mathbb{R}$ be an interval, $u, v \in I, u < v$, and $\omega \in \mathcal{M}$. Let $f: I \rightarrow \mathbb{R}$ be semiconvex with modulus ω and let $q: [u, v] \rightarrow \mathbb{R}$ be convex. Suppose that $q(u) \leq f(u), q(v) \leq f(v)$, and define a function s by*

$$s(x) := \begin{cases} \max\{q(x), f(x)\}, & x \in [u, v], \\ f(x), & x \in I \setminus [u, v]. \end{cases}$$

Then s is semiconvex with modulus ω .

PROOF: Let $x_1, x_2 \in I, x_1 < x_2$, and $\lambda \in (0, 1)$. We want to show that

$$s(\lambda x_1 + (1 - \lambda)x_2) \leq \lambda s(x_1) + (1 - \lambda)s(x_2) + \lambda(1 - \lambda)(x_2 - x_1)\omega(x_2 - x_1).$$

Set $z := \lambda x_1 + (1 - \lambda)x_2$. If $s(z) = f(z)$, then (using the semiconvexity of f)

$$\begin{aligned} s(z) = f(z) &\leq \lambda f(x_1) + (1 - \lambda)f(x_2) + \lambda(1 - \lambda)(x_2 - x_1)\omega(x_2 - x_1) \\ &\leq \lambda s(x_1) + (1 - \lambda)s(x_2) + \lambda(1 - \lambda)(x_2 - x_1)\omega(x_2 - x_1). \end{aligned}$$

So, we will further suppose that $s(z) = q(z)$ and $z \in (u, v)$. Define a function p by

$$\begin{aligned} p(x) &:= \frac{x_2 - x}{x_2 - x_1}s(x_1) + \frac{x - x_1}{x_2 - x_1}s(x_2) \\ &\quad + \frac{x_2 - x}{x_2 - x_1} \frac{x - x_1}{x_2 - x_1} (x_2 - x_1)\omega(x_2 - x_1), \quad x \in [x_1, x_2]. \end{aligned}$$

Then p is concave and

$$\begin{aligned} p(lx_1 + (1 - l)x_2) &= ls(x_1) + (1 - l)s(x_2) \\ &\quad + l(1 - l)(x_2 - x_1)\omega(x_2 - x_1), \quad l \in [0, 1]. \end{aligned}$$

In particular, $p(x_i) = s(x_i)$, $i = 1, 2$. Set $\overline{x}_1 := \max\{u, x_1\}$ and $\overline{x}_2 := \min\{v, x_2\}$. Then

$$u \leq \overline{x}_1 < z < \overline{x}_2 \leq v.$$

If $u \leq x_1$, then $\overline{x}_1 = x_1 \in [u, v)$ and

$$q(\overline{x}_1) = q(x_1) \leq s(x_1) = p(x_1) = p(\overline{x}_1).$$

Otherwise $x_1 < u = \overline{x}_1$. Then we can find $l \in (0, 1)$ such that $u = lx_1 + (1 - l)x_2$. Thus (using the semiconvexity of f)

$$\begin{aligned} q(\overline{x}_1) &= q(u) \leq f(u) = f(lx_1 + (1 - l)x_2) \\ &\leq lf(x_1) + (1 - l)f(x_2) + l(1 - l)(x_2 - x_1)\omega(x_2 - x_1) \\ &\leq ls(x_1) + (1 - l)s(x_2) + l(1 - l)(x_2 - x_1)\omega(x_2 - x_1) \\ &= p(lx_1 + (1 - l)x_2) = p(u) = p(\overline{x}_1). \end{aligned}$$

Hence we have shown that $q(\overline{x}_1) \leq p(\overline{x}_1)$, and we can analogously show that $q(\overline{x}_2) \leq p(\overline{x}_2)$. Thus (using the convexity of q and the concavity of p)

$$q(z) \leq \frac{\overline{x}_2 - z}{\overline{x}_2 - \overline{x}_1} q(\overline{x}_1) + \frac{z - \overline{x}_1}{\overline{x}_2 - \overline{x}_1} q(\overline{x}_2) \leq \frac{\overline{x}_2 - z}{\overline{x}_2 - \overline{x}_1} p(\overline{x}_1) + \frac{z - \overline{x}_1}{\overline{x}_2 - \overline{x}_1} p(\overline{x}_2) \leq p(z)$$

and so

$$s(z) = q(z) \leq p(z) = \lambda s(x_1) + (1 - \lambda)s(x_2) + \lambda(1 - \lambda)(x_2 - x_1)\omega(x_2 - x_1).$$

□

Theorem 4.3. *Let $I \subset \mathbb{R}$ be an interval, $f_1, f_2: I \rightarrow \mathbb{R}$ and $\omega_1, \omega_2 \in \mathcal{M}$. Suppose that f_1 is semiconvex with modulus ω_1 , f_2 is semiconcave with modulus ω_2 and $f_1 \leq f_2$. Denote by \mathcal{S} the set of all $s: I \rightarrow \mathbb{R}$ which are semiconvex with modulus ω_1 and satisfy $s \leq f_2$. Then the function f defined by*

$$f(x) = \sup\{s(x) : s \in \mathcal{S}\}, \quad x \in I,$$

is semiconvex with modulus ω_1 , semiconcave with modulus ω_2 and satisfies $f_1 \leq f \leq f_2$.

PROOF: It is clear that $f_1 \leq f \leq f_2$. By [3, Proposition 2.1.5], f is semiconvex with modulus ω_1 . Let $u, v \in I$, $u < v$. We want to show that

$$f(\lambda u + (1 - \lambda)v) \geq \lambda f(u) + (1 - \lambda)f(v) - \lambda(1 - \lambda)(v - u)\omega_2(v - u), \quad \lambda \in [0, 1].$$

Define a function q by

$$q(x) := \frac{v - x}{v - u} f(u) + \frac{x - u}{v - u} f(v) - \frac{v - x}{v - u} \frac{x - u}{v - u} (v - u)\omega_2(v - u), \quad x \in [u, v],$$

and a function s by

$$s(x) := \begin{cases} \max\{q(x), f(x)\}, & x \in [u, v], \\ f(x), & x \in I \setminus [u, v]. \end{cases}$$

Then q is convex and

$$\begin{aligned} q(\lambda u + (1 - \lambda)v) &= \lambda f(u) + (1 - \lambda)f(v) - \lambda(1 - \lambda)(v - u)\omega_2(v - u), \quad \lambda \in [0, 1]. \end{aligned}$$

Since f_2 is semiconcave with modulus ω_2 , we have

$$\begin{aligned} q(\lambda u + (1 - \lambda)v) &= \lambda f(u) + (1 - \lambda)f(v) - \lambda(1 - \lambda)(v - u)\omega_2(v - u) \\ &\leq \lambda f_2(u) + (1 - \lambda)f_2(v) - \lambda(1 - \lambda)(v - u)\omega_2(v - u) \\ &\leq f_2(\lambda u + (1 - \lambda)v), \quad \lambda \in [0, 1]. \end{aligned}$$

This implies that $s \leq f_2$. And since $q(u) = f(u)$ and $q(v) = f(v)$, s is semiconvex with modulus ω_1 by Lemma 4.2. Consequently, $s \in \mathcal{S}$ and thus $s \leq f$. Hence

$$\begin{aligned} f(\lambda u + (1 - \lambda)v) &\geq s(\lambda u + (1 - \lambda)v) \geq q(\lambda u + (1 - \lambda)v) \\ &= \lambda f(u) + (1 - \lambda)f(v) - \lambda(1 - \lambda)(v - u)\omega_2(v - u), \quad \lambda \in [0, 1]. \end{aligned}$$

□

Corollary 4.4. *Let $I \subset \mathbb{R}$ be an interval, $\omega \in \mathcal{M}$, $f_1 \in SC^\omega(I)$ and $f_2 \in -SC^\omega(I)$. Suppose that $f_1 \leq f_2$. Then there exists a continuous $f: I \rightarrow \mathbb{R}$ such that $f_1 \leq f \leq f_2$ and $f \upharpoonright_{\text{int } I} \in C^{1,\omega}(\text{int } I)$.*

PROOF: By Theorem 4.3 there exists $f \in SC^\omega(I) \cap (-SC^\omega(I))$ such that $f_1 \leq f \leq f_2$. Then f is continuous by Proposition 4.1 (i) and $f \upharpoonright_{\text{int } I} \in C^{1,\omega}(\text{int } I)$ by Proposition 4.1 (ii). □

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REFERENCES

- [1] Benyamini Y., Lindenstrauss J., *Geometric Nonlinear Functional Analysis*, Vol. 1, American Mathematical Society Colloquium Publications, 48, American Mathematical Society, Providence, 2000.
- [2] Bernard P., *Lasry–Lions regularization and a lemma of Ilmanen*, Rend. Sem. Mat. Univ. Padova **124** (2010), 221–229.

- [3] Cannarsa P., Sinestrari C., *Semiconcave Functions, Hamilton–Jacobi Equations, and Optimal Control*, Progress in Nonlinear Differential Equations and Their Applications, 58, Birkhäuser, Boston, 2004.
- [4] Deville R., Godefroy G., Zizler V., *Smoothness and Renormings in Banach Spaces*, Pitman Monographs and Surveys in Pure and Applied Mathematics, 64, Longman Scientific & Technical, Harlow, John Wiley & Sons, New York, 1993.
- [5] Duda J., Zajíček L., *Semiconvex functions: representations as suprema of smooth functions and extensions*, J. Convex Anal. **16** (2009), no. 1, 239–260.
- [6] Duda J., Zajíček L., *Smallness of singular sets of semiconvex functions in separable Banach spaces*, J. Convex Anal. **20** (2013), no. 2, 573–598.
- [7] Hájek P., Johannis M., *Smooth Analysis in Banach Spaces*, De Gruyter Series in Nonlinear Analysis and Applications, 19, De Gruyter, Berlin, 2014.
- [8] Ilmanen T., *The level-set flow on a manifold*, Differential Geometry: Partial Differential Equations on Manifolds, Los Angeles, 1990, Proc. Sympos. Pure Math. **54** (1993), Part 1, 193–204.
- [9] Kryštof V., *Generalized versions of Ilmanen lemma: insertion of $C^{1,\omega}$ or $C_{\text{loc}}^{1,\omega}$ functions*, Comment. Math. Univ. Carolin. **59** (2018), no. 2, 223–231.

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