

Applications of Mathematics

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Applications of Mathematics, Vol. 67 (2022), No. 1, 93–101

Persistent URL: <http://dml.cz/dmlcz/149361>

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UNIQUENESS OF WEAK SOLUTIONS
TO A KELLER-SEGEL-NAVIER-STOKES MODEL
WITH A LOGISTIC SOURCE

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Received March 7, 2020. Published online February 15, 2021.

Abstract. We prove a uniqueness result of weak solutions to the nD ($n \geq 3$) Cauchy problem of a Keller-Segel-Navier-Stokes system with a logistic term.

Keywords: Keller-Segel-Navier-Stokes system; uniqueness; weak solution

MSC 2020: 22E46, 53C35, 57S20, 35Q30

1. INTRODUCTION

We consider the following model of a Keller-Segel-Navier-Stokes with a logistic source [11], [17]:

$$(1.1) \quad \partial_t u + u \cdot \nabla u + \nabla \pi - \Delta u = n \nabla \varphi,$$

$$(1.2) \quad \operatorname{div} u = 0,$$

$$(1.3) \quad \partial_t n + u \cdot \nabla n - \Delta n + n^2 - an = -\nabla \cdot (n \nabla p) - \nabla \cdot (n \nabla q),$$

$$(1.4) \quad \partial_t p + u \cdot \nabla p - \Delta p = -np,$$

$$(1.5) \quad \partial_t q + u \cdot \nabla q - \Delta q + q = n \text{ in } \mathbb{R}^N \times (0, \infty),$$

$$(1.6) \quad (u, n, p, q)(\cdot, 0) = (u_0, n_0, p_0, q_0)(\cdot) \text{ in } \mathbb{R}^N \text{ (} N \geq 3\text{),}$$

where u is the velocity of the fluid, π is the pressure, n , p and q denote the density of amoebae, oxygen and chemical attractant, respectively. The smooth function $\varphi := \varphi(x)$ is a potential, a is a real constant.

This paper is supported by the key project of University natural science of Anhui province (No. KJ2017A453; No. KJ2017A454).

When $\varphi = 0$, system (1.1) and (1.2) reduces to the incompressible Navier-Stokes system. Ogawa and Taniuchi [14] obtained the uniqueness criterion

$$(1.7) \quad \nabla u \in L \log L(0, T; \dot{B}_{\infty, \infty}^0(\mathbb{R}^N))$$

with

$$(1.8) \quad L \log L(0, T; \dot{B}_{\infty, \infty}^0) := \left\{ f; \int_0^T \|f\|_{\dot{B}_{\infty, \infty}^0} \log(e + \|f\|_{\dot{B}_{\infty, \infty}^0}) dt < \infty \right\}.$$

We note that Kozono et al. [12] showed that u is smooth if

$$(1.9) \quad \nabla u \in L^1(0, T; \dot{B}_{\infty, \infty}^0).$$

Here $\dot{B}_{\infty, \infty}^0$ is the homogeneous Besov space.

On the other hand, when $u = 0$, system (1.3), (1.4) and (1.5) reduces to the Keller-Segel system [8], [9], [10], which has been dealt with in many studies [1], [2], [6], [7], [15], [16], [18].

Very recently, Fan-Zhao [5] (see also [3], [4]) established some regularity criteria when $q = 0$.

We will assume that

$$(1.10) \quad \nabla p \in L^{2/(1-r)}(0, T; \dot{X}_r), \nabla q \in L^{2/(1-s)}(0, T; \dot{X}_s) \quad \text{with } 0 < r, s < 1$$

and

$$(1.11) \quad \|f\|_{\dot{X}_r} := \sup \left\{ \frac{\|fg\|_{L^2}}{\|g\|_{\dot{H}^r}}, g \neq 0 \right\}.$$

The space \dot{X}_r of pointwise multipliers maps \dot{H}^r into L^2 . The pointwise multipliers between different spaces of differentiable functions have been studied [13]. They are a useful tool for stating minimal regularity requirements on the coefficients of partial differential operators for proving uniqueness or regularity of solutions.

The aim of this paper is to prove a uniqueness result:

Theorem 1.1. *Let $u_0 \in L^2$, $n_0 \in L^1 \cap H^{-1} \cap L^\infty$, $p_0 \in L^2 \cap L^\infty$, $q_0 \in L^2$, $\operatorname{div} u_0 = 0$, $n_0, p_0, q_0 \geq 0$ in \mathbb{R}^N . Suppose that $\varphi := \varphi(x)$ is a smooth function. If (1.7) and (1.10) hold, then problem (1.1)–(1.6) has at most one weak solution.*

Let η_j , $j = 0, \pm 1, \pm 2, \pm 3, \dots$, be the Littlewood-Paley dyadic decomposition of unity that satisfies $\hat{\eta} \in C_0^\infty(B_2 \setminus B_{1/2})$, $\hat{\eta}_j(\xi) = \hat{\eta}_j(2^{-j}\xi)$ and $\sum_{j=-\infty}^{\infty} \hat{\eta}_j(\xi) = 1$ except for $\xi = 0$. To fill the origin, we put a smooth cut of $\psi \in \mathcal{S}(\mathbb{R}^3)$ with $\hat{\psi}(\xi) \in C_0^\infty(B_1)$ such that

$$(1.12) \quad \hat{\psi} + \sum_{j=0}^{\infty} \hat{\eta}_j(\xi) = 1.$$

The homogeneous Besov space $\dot{B}_{p,q}^s := \{f \in \mathcal{S}' : \|f\|_{\dot{B}_{p,q}^s} < \infty\}$ is introduced by the norm

$$(1.13) \quad \|f\|_{\dot{B}_{p,q}^s} := \left(\sum_{j=-\infty}^{\infty} \|2^{js} \eta_j * f\|_{L^p}^q \right)^{1/q}$$

for $s \in \mathbb{R}$, $1 \leq p, q \leq \infty$.

Definition 1.1 (Weak solutions). We say that (u, n, p, q) is a *weak solution* to problem (1.1)–(1.6) if the following conditions are satisfied:

- (i) $0 \leq n, p, q, u, p, q \in L^\infty(0, T; L^2) \cap L^2(0, T; H^1)$, $n \in L^\infty(0, T; L^l) \cap L^2(0, T; H^1)$ with $2 \leq l < \infty$;
- (ii) $\int_0^T \int (-uw_t - u \otimes u : \nabla w + \nabla u : \nabla w) dx dt = \int_0^T \int n \nabla \varphi w dx dt$ holds for any $T > 0$, and any $w \in C_0^\infty(\mathbb{R}^N \times (0, T))$ with $\operatorname{div} w = 0$, and u satisfies the energy inequality

$$\frac{1}{2} \int |u|^2 dx + \int_0^t \int |\nabla u|^2 dx ds \leq \frac{1}{2} \int |u_0|^2 dx + \int_0^t \int n \nabla \varphi u dx ds, \text{ a.e. } 0 \leq t < T;$$

- (iii) $\int_0^T \int u \nabla v dx dt = 0$ holds for any $T > 0$ and any $v \in C_0^\infty(\mathbb{R}^N \times (0, T))$;
- (iv) $\int_0^T \int (-n \xi_t - un \nabla \xi + \nabla n \nabla \xi + n^2 \xi - an \xi) dx dt = \int_0^T \int (n \nabla p + n \nabla q) \nabla \xi dx dt$ holds for any $T > 0$ and any $\xi \in C_0^\infty(\mathbb{R}^N \times (0, T))$;
- (v) $\int_0^T \int (-p \eta_t - up \nabla \eta + \nabla p \nabla \eta + np \eta) dx dt = 0$ holds for any $T > 0$ and any $\eta \in C_0^\infty(\mathbb{R}^N \times (0, T))$;
- (vi) $\int_0^T \int (-q \xi_t - uq \nabla \xi + \nabla q \nabla \xi + q \xi) dx dt = \int_0^T \int n \xi dx dt$ holds for any $T > 0$ and any $\xi \in C_0^\infty(\mathbb{R}^N \times (0, T))$.

It is easy to prove the existence of weak solutions [17] and thus we omit the details here; we only need to prove the uniqueness.

2. PROOF OF THEOREM 1.1

This section is devoted to the proof of Theorem 1.1. First, from the equations of n, p, q and the maximum principle we see that

$$(2.1) \quad n, p, q \geq 0, \quad p \leq C.$$

For any $l \geq 2$, testing (1.3) by n^{l-1} , using (1.10) and denoting $w := n^{l/2}$, we obtain

$$\begin{aligned} & \frac{1}{l} \frac{d}{dt} \int w^2 dx + \frac{4(l-1)}{l^2} \int |\nabla w|^2 dx + \int n^{l+1} dx \\ &= C \int w(\nabla p + \nabla q) \nabla w dx + \int a n^l dx \\ &\leq C(\|w \nabla p\|_{L^2} + \|w \nabla q\|_{L^2}) \|\nabla w\|_{L^2} + C \|w\|_{L^2}^2 \\ &\leq C(\|\nabla p\|_{\dot{X}_r} \|w\|_{\dot{H}^r} + \|\nabla q\|_{\dot{X}_s} \|w\|_{\dot{H}^s}) \|\nabla w\|_{L^2} + C \|w\|_{L^2}^2 \\ &\leq C \|\nabla p\|_{\dot{X}_r} \|w\|_{L^2}^{1-r} \|\nabla w\|_{L^2}^{1+r} + C \|\nabla q\|_{\dot{X}_s} \|w\|_{L^2}^{1-s} \|\nabla w\|_{L^2}^{1+s} + C \|w\|_{L^2}^2 \\ &\leq \frac{l-1}{l^2} \|\nabla w\|_{L^2}^2 + C(\|\nabla p\|_{\dot{X}_r}^{2/(1-r)} + \|\nabla q\|_{\dot{X}_s}^{2/(1-s)}) \|w\|_{L^2}^2 + C \|w\|_{L^2}^2, \end{aligned}$$

which gives

$$(2.2) \quad \|n\|_{L^\infty(0,T;L^l)} + \|n\|_{L^2(0,T;H^1)} \leq C \quad \text{for any } l \geq 2.$$

Testing (1.1) by u , using (1.2) and (2.2), we observe that

$$\frac{1}{2} \frac{d}{dt} \int |u|^2 dx + \int |\nabla u|^2 dx = - \int n \nabla \varphi u dx \leq \|n\|_{L^2} \|\nabla \varphi\|_{L^\infty} \|u\|_{L^2} \leq C \|u\|_{L^2},$$

which implies

$$(2.3) \quad \|u\|_{L^\infty(0,T;L^2)} + \|u\|_{L^2(0,T;H^1)} \leq C.$$

Testing (1.4) by p and using (1.2) and (2.1), we deduce

$$\frac{1}{2} \frac{d}{dt} \int p^2 dx + \int |\nabla p|^2 dx + \int n p^2 dx = 0,$$

which implies

$$(2.4) \quad \|p\|_{L^\infty(0,T;L^2)} + \|p\|_{L^2(0,T;H^1)} \leq C.$$

Similarly, testing (1.5) by q and using (1.2), (2.1) and (2.2), we infer that

$$\frac{1}{2} \frac{d}{dt} \int q^2 dx + \int |\nabla q|^2 dx + \int q^2 dx = \int nq dx \leq \|n\|_{L^2} \|q\|_{L^2} \leq C \|q\|_{L^2},$$

which gives

$$(2.5) \quad \|q\|_{L^\infty(0,T;L^2)} + \|q\|_{L^2(0,T;H^1)} \leq C.$$

Now we are in a position to show the uniqueness of weak solutions. Let $(u_j, \pi_j, n_j, p_j, q_j)$ ($j = 1, 2$) be the two weak solutions. We denote

$$u := u_1 - u_2, \quad \pi := \pi_1 - \pi_2, \quad n := n_1 - n_2, \quad p := p_1 - p_2, \quad q := q_1 - q_2.$$

It is easy to see that

$$(2.6) \quad \partial_t u + u_1 \cdot \nabla u + \nabla \pi - \Delta u = n \nabla \varphi - u \cdot \nabla u_2,$$

$$(2.7) \quad \begin{aligned} \partial_t n - \Delta n = & -\nabla \cdot (u_2 n + u n_1) - \nabla \cdot (n \nabla p_1 + n_2 \nabla p) \\ & - \nabla \cdot (n \nabla q_1 + n_2 \nabla q) + a n - (n_1 + n_2) n, \end{aligned}$$

$$(2.8) \quad \partial_t p + u_1 \cdot \nabla p - \Delta p + n_1 p = -u \cdot \nabla p_2 - n p_2,$$

$$(2.9) \quad \partial_t q + u_1 \cdot \nabla q - \Delta q + q = n - u \cdot \nabla q_2.$$

Define ξ satisfying

$$(2.10) \quad -\Delta \xi = n.$$

Testing (2.6) by u and using (1.2), we deduce by a formal argument:

$$(2.11) \quad \frac{1}{2} \frac{d}{dt} \int |u|^2 dx + \int |\nabla u|^2 dx = \int n \nabla \varphi u dx - \int u \cdot \nabla u_2 \cdot u dx =: I_1 + I_2.$$

Indeed, this process can be justified by a very similar argument in [14] and thus we omit it.

We bound I_1 as follows:

$$(2.12) \quad I_1 \leq \|n\|_{L^2} \|\nabla \varphi\|_{L^\infty} \|u\|_{L^2} \leq C \|\Delta \xi\|_{L^2} \|u\|_{L^2} \leq \frac{1}{16} \|\Delta \xi\|_{L^2}^2 + C \|u\|_{L^2}^2.$$

We decompose u_2 into three parts in the phase variable:

$$(2.13) \quad u_2 = \sum_{j < -M} \eta_j * u_2 + \sum_{j = -M}^M \eta_j * u_2 + \sum_{j > M} \eta_j * u_2 =: u_2^l + u_2^m + u_2^h.$$

Thus,

$$(2.14) \quad I_2 = \int u \cdot \nabla u \cdot u_2^l \, dx - \int u \cdot \nabla u_2^m \cdot u \, dx + \int u \cdot \nabla u \cdot u_2^h \, dx =: I_2^l + I_2^m + I_2^h.$$

Recalling the Bernstein inequality

$$(2.15) \quad \|\eta_j * u\|_{L^q} \leq C 2^{jN(1/p-1/q)} \|\eta_j * u\|_{L^p}, \quad 1 \leq p \leq q \leq \infty,$$

the low-frequency part is estimated as

$$(2.16) \quad \begin{aligned} I_2^l &\leq \|u\|_{L^{2N/(N-2)}} \|\nabla u\|_{L^2} \|u_2^l\|_{L^N} \\ &\leq C \|\nabla u\|_{L^2}^2 \sum_{j < -M} 2^{jN(1/2-1/N)} \|\eta_j * u_2\|_{L^2} \\ &\leq C \|\nabla u\|_{L^2}^2 \left(\sum_{j < -M} 2^{(N-2)j} \right)^{1/2} \left(\sum_{j=-\infty}^{\infty} \|\eta_j * u_2\|_{L^2}^2 \right)^{1/2} \\ &\leq C \|\nabla u\|_{L^2}^2 2^{-(N-2)M/2} \|u_2\|_{L^2} \\ &\leq C \|\nabla u\|_{L^2}^2 2^{-(N-2)M/2}. \end{aligned}$$

The second term can be bounded as follows:

$$(2.17) \quad \begin{aligned} I_2^m &\leq \|u\|_{L^2}^2 \|\nabla u_2^m\|_{L^\infty} \leq C \|u\|_{L^2}^2 \sum_{j=-M}^M \|\eta_j * \nabla u_2\|_{L^\infty} \\ &\leq CM \|u\|_{L^2}^2 \|\nabla u_2\|_{\dot{B}_{\infty,\infty}^0}. \end{aligned}$$

On the other hand, the last term is simply bounded by the Hausdorff-Young inequality as

$$(2.18) \quad \begin{aligned} I_2^h &\leq \|\nabla u\|_{L^2} \|u\|_{L^2} \|u_2^h\|_{L^\infty} \\ &\leq \|\nabla u\|_{L^2} \|u\|_{L^2} \sum_{j > M} \|\{(-\Delta)^{-1/2}(\eta_{j-1} + \eta_j + \eta_{j+1})\} * \eta_j * (-\Delta)^{1/2} u_2\|_{L^\infty} \\ &\leq C \|\nabla u\|_{L^2} \|u\|_{L^2} \sum_{j > M} 2^{-j} \|\eta_j * (-\Delta)^{1/2} u_2\|_{L^\infty} \\ &\leq C 2^{-M} \|\nabla u\|_{L^2} \|u\|_{L^2} \|\nabla u_2\|_{\dot{B}_{\infty,\infty}^0} \\ &\leq C 2^{-M} \|u\|_{L^2}^2 \|\nabla u_2\|_{\dot{B}_{\infty,\infty}^0}^2 + C 2^{-M} \|\nabla u\|_{L^2}^2. \end{aligned}$$

Choosing M properly large so that $C 2^{-M/2} \leq \frac{1}{36}$ and $C 2^{-M} \|\nabla u_2\|_{\dot{B}_{\infty,\infty}^0} \leq 1$, we arrive at

$$(2.19) \quad I_2 \leq \frac{1}{16} \|\nabla u\|_{L^2}^2 + C \|u\|_{L^2}^2 \|\nabla u_2\|_{\dot{B}_{\infty,\infty}^0} (1 + \log(e + \|\nabla u_2\|_{\dot{B}_{\infty,\infty}^0})).$$

Testing (2.7) by ξ and using (2.10), we have

$$\begin{aligned}
(2.20) \quad & \frac{1}{2} \frac{d}{dt} \int |\nabla \xi|^2 dx + \int (\Delta \xi)^2 dx \\
& = - \int u_2 \nabla \xi \Delta \xi dx + \int u n \nabla \xi dx + \int (n \nabla p_1 + n_2 \nabla p) \nabla \xi dx \\
& \quad + \int (n \nabla q_1 + n_2 \nabla q) \nabla \xi dx + \int a n \xi dx + \int (n_1 + n_2) \xi \Delta \xi dx \\
& =: \sum_{j=3}^8 I_j.
\end{aligned}$$

Similarly to (2.19), we have

$$(2.21) \quad I_3 \leq \frac{1}{16} \|\Delta \xi\|_{L^2}^2 + C \|\nabla \xi\|_{L^2}^2 \|\nabla u_2\|_{\dot{B}_{\infty, \infty}^0} (1 + \log(e + \|\nabla u_2\|_{\dot{B}_{\infty, \infty}^0})).$$

We bound I_j ($j = 4, \dots, 8$) as follows

$$\begin{aligned}
(2.22) \quad & I_4 \leq \|u\|_{L^2} \|n_1\|_{L^{2N}} \|\nabla \xi\|_{L^{2N/(N-1)}} \leq C \|u\|_{L^2} \|\nabla \xi\|_{L^{2N/(N-1)}} \\
& \leq C \|u\|_{L^2} \|\nabla \xi\|_{L^2}^{1/2} \|\Delta \xi\|_{L^2}^{1/2} \leq \frac{1}{16} \|\Delta \xi\|_{L^2}^2 + C \|\nabla \xi\|_{L^2}^2 + C \|u\|_{L^2}^2.
\end{aligned}$$

Here we have used the Gagliardo-Nirenberg inequality

$$(2.23) \quad \|\nabla \xi\|_{L^{2N/(N-1)}} \leq C \|\nabla \xi\|_{L^2}^{1/2} \|\Delta \xi\|_{L^2}^{1/2}.$$

$$\begin{aligned}
(2.24) \quad & I_5 \leq \|n\|_{L^2} \|\nabla \xi \cdot \nabla p_1\|_{L^2} + \|n_2\|_{L^{2N}} \|\nabla p\|_{L^2} \|\nabla \xi\|_{L^{2N/(N-1)}} \\
& \leq C \|\Delta \xi\|_{L^2} \|\nabla p_1\|_{\dot{X}_r} \|\nabla \xi\|_{\dot{H}^r} + C \|\nabla p\|_{L^2} \|\nabla \xi\|_{L^2}^{1/2} \|\Delta \xi\|_{L^2}^{1/2} \\
& \leq \|\nabla p_1\|_{\dot{X}_r} \|\nabla \xi\|_{L^2}^{1-r} \|\Delta \xi\|_{L^2}^{1+r} + C \|\nabla p\|_{L^2} \|\nabla \xi\|_{L^2}^{1/2} \|\Delta \xi\|_{L^2}^{1/2} \\
& \leq \frac{1}{16} \|\Delta \xi\|_{L^2}^2 + C \|\nabla p_1\|_{\dot{X}_r}^{2/(1-r)} \|\nabla \xi\|_{L^2}^2 + C \|\nabla \xi\|_{L^2}^2 + \frac{1}{16} \|\nabla p\|_{L^2}^2.
\end{aligned}$$

Similarly to I_5 , we have

$$(2.25) \quad I_6 \leq \frac{1}{16} \|\Delta \xi\|_{L^2}^2 + C \|\nabla q_1\|_{\dot{X}_s}^{2/(1-s)} \|\nabla \xi\|_{L^2}^2 + C \|\nabla \xi\|_{L^2}^2 + \frac{1}{16} \|\nabla q\|_{L^2}^2,$$

$$(2.26) \quad I_7 = a \int |\nabla \xi|^2 dx,$$

$$\begin{aligned}
(2.27) \quad & I_8 \leq \|n_1 + n_2\|_{L^N} \|\Delta \xi\|_{L^2} \|\xi\|_{L^{2N/(N-2)}} \leq C \|\Delta \xi\|_{L^2} \|\nabla \xi\|_{L^2} \\
& \leq \frac{1}{16} \|\Delta \xi\|_{L^2}^2 + C \|\nabla \xi\|_{L^2}^2.
\end{aligned}$$

Inserting the above estimates into (2.20), we have

$$\begin{aligned}
(2.28) \quad & \frac{1}{2} \frac{d}{dt} \int |\nabla \xi|^2 dx + \frac{11}{16} \int |\Delta \xi|^2 dx \\
& \leq C \|\nabla \xi\|_{L^2}^2 \|\nabla u_2\|_{\dot{B}_{\infty,\infty}^0} (1 + \log(e + \|\nabla u_2\|_{\dot{B}_{\infty,\infty}^0})) \\
& \quad + C \|\nabla \xi\|_{L^2}^2 + C \|u\|_{L^2}^2 + C \|\nabla p_1\|_{\dot{X}_r}^{2/(1-r)} \|\nabla \xi\|_{L^2}^2 \\
& \quad + C \|\nabla q_1\|_{\dot{X}_s}^{2/(1-s)} \|\nabla \xi\|_{L^2}^2 + \frac{1}{16} \|\nabla p\|_{L^2}^2 + \frac{1}{16} \|\nabla q\|_{L^2}^2.
\end{aligned}$$

Testing (2.8) by p , using (1.2) and (2.1), we have

$$\begin{aligned}
(2.29) \quad & \frac{1}{2} \frac{d}{dt} \int p^2 dx + \int |\nabla p|^2 dx + \int n_1 p^2 dx \\
& = - \int u \cdot \nabla p_2 \cdot p dx - \int n p_2 p dx = \int u p_2 \nabla p dx - \int n p_2 p dx \\
& \leq \|u\|_{L^2} \|p_2\|_{L^\infty} \|\nabla p\|_{L^2} + \|p_2\|_{L^\infty} \|n\|_{L^2} \|p\|_{L^2} \\
& \leq \frac{1}{16} \|\nabla p\|_{L^2}^2 + \frac{1}{16} \|\Delta \xi\|_{L^2}^2 + C \|u\|_{L^2}^2 + C \|p\|_{L^2}^2.
\end{aligned}$$

Testing (2.9) by q and using (1.2), we have

$$\begin{aligned}
(2.30) \quad & \frac{1}{2} \frac{d}{dt} \int q^2 dx + \int |\nabla q|^2 dx + \int q^2 dx \\
& = \int n q dx - \int u \cdot \nabla q_2 q dx \\
& \leq \|n\|_{L^2} \|q\|_{L^2} + \|u \cdot \nabla q_2\|_{L^2} \|q\|_{L^2} \\
& \leq \|\Delta \xi\|_{L^2} \|q\|_{L^2} + C \|\nabla q_2\|_{\dot{X}_s} \|u\|_{\dot{H}^s} \|q\|_{L^2} \\
& \leq \|\Delta \xi\|_{L^2} \|q\|_{L^2} + C \|\nabla q_2\|_{\dot{X}_s} \|u\|_{L^2}^{1-s} \|\nabla u\|_{L^2}^s \|q\|_{L^2} \\
& \leq \frac{1}{16} \|\Delta \xi\|_{L^2}^2 + \frac{1}{16} \|\nabla u\|_{L^2}^2 + C \|q\|_{L^2}^2 + C \|\nabla q_2\|_{\dot{X}_s}^{2/(1-s)} \|u\|_{L^2}^2.
\end{aligned}$$

Inserting (2.12) and (2.19) into (2.11), then adding up to (2.28), (2.29) and (2.30) and using the Gronwall inequality, we conclude that

$$u = 0, \quad n = p = q = 0,$$

and thus

$$u_1 = u_2, \quad n_1 = n_2, \quad p_1 = p_2 \quad \text{and} \quad q_1 = q_2.$$

This completes the proof. \square

A c k n o w l e d g m e n t . The authors are indebted to the referee and editor Milan Pokorný for some nice suggestions.

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