Isaac Harris

The direct and inverse problem for sub-diffusion equations with a generalized impedance subregion

Applications of Mathematics, Vol. 67 (2022), No. 1, 1–20

Persistent URL: http://dml.cz/dmlcz/149354

67 (2022) APPLICATIONS OF MATHEMATICS No. 1, 1–20

THE DIRECT AND INVERSE PROBLEM FOR SUB-DIFFUSION EQUATIONS WITH A GENERALIZED IMPEDANCE SUBREGION

Isaac Harris, West Lafayette

Received April 15, 2020. Published online November 16, 2021.

Abstract. In this paper, we consider the direct and inverse problem for time-fractional diffusion in a domain with an impenetrable subregion. Here we assume that on the boundary of the subregion the solution satisfies a generalized impedance boundary condition. This boundary condition is given by a second order spatial differential operator imposed on the boundary. A generalized impedance boundary condition can be used to model corrosion and delimitation. The well-posedness for the direct problem is established where the Laplace transform is used to study the time dependent boundary value problem. The inverse impedance problem of determining the parameters from the Cauchy data is also studied provided the boundary of the subregion is known. The uniqueness of recovering the boundary parameters from the Neumann to Dirichlet mapping is proven.

Keywords: fractional diffusion; Laplace transform; inverse impedance problem

MSC 2020: 35R11, 35R30

1. Introduction

Here we are interested in studying the direct and inverse problem for a sub-diffusive partial differential equation in a domain with an impenetrable subregion. To close the system we require that the solution has a given flux on the outer boundary and satisfies a homogeneous generalized impedance boundary condition on the interior boundary. We assume that the model is given by the fractional diffusion equation, where the spatial partial differential operator is given by a symmetric elliptic operator. The temporal derivative is given by the Caputo fractional derivative denoted ∂_t^α for a fixed $\alpha \in (0,1)$. There has been a lot of interest in the study of sub-diffusive process in recent years, see, e.g., [14] and the references therein. It has even been shown in [16] that sub-diffusive processes can be used as a regularization strategy for classical severely ill-posed backward diffusive processes. In

DOI: 10.21136/AM.2021.0107-20

general, we have seen in the literature that the generalized impedance boundary condition models complex features such as coating and corrosion (see, e.g., [3]). Therefore, we see that the boundary condition considered here can be used to model different physical situations. Even though we only consider the inverse coefficient problem of recovering the impedance parameters, another interesting inverse problem corresponds to recovering that shape of the impedance region. This can be done using the so-called factorization method (see [17] for details). The factorization method only provides a way of recovering the impedance region and here we focus on recovering the coefficents assuming the region is known. In [3] a generalized impedance condition is derived to asymptotically describe delimitation for the acoustic scattering problem. In [11] the factorization method is employed to solve the inverse shape problem of recovering an inclusion with a generalized impedance condition from electrostatic data and unique recovery of the impedance coefficients is proven. Recently, in [9] the factorization method was studied for a heat equation to reconstruct interior cavities. The interior cavity is given by a thermal insulating region which gives a zero flux on the interior boundary. See the manuscript [17] for an in-depth study of the factorization method applied to inverse scattering problems. Even though it is not considered here, the question of employing the factorization method to recover the interior boundary is an interesting open problem for either the heat equation or the sud-diffusive equation. See [4], [7] for other examples of the inverse problem for recovering the impedance coefficients from electrostatic data.

Just as in these manuscripts, we are interested in the inverse impedance problem of unique recovery of the impedance coefficients. Here we will assume that we have the Cauchy data coming from the fractional diffusion equation. For other recent manuscripts that have considered inverse problems for an impedance condition and fractional derivatives see [12], [13], [15], [23]. Just as in [20] we will use the Laplace transform to study the well-posedness of a diffusion equation. In order to prove solvability in the time-domain, we will formally take the Laplace transform of the time-fractional diffusion equation in question then appealing to Laplace inversion formula from Chapter 3 of [22]. The Laplace and Fourier transforms are very useful tools for studying time-domain problems. In many manuscripts such as [2], [20] the Laplace and Fourier transform are used to prove the solvability of hyperbolic and parabolic equations. This is done by reducing the time-domain to an auxiliary problem in the frequency-domain, where one proves well-posedness for the auxiliary problem. In order to establish well-posedness in the time-domain, one must establish explicit bounds on the frequency variable and appeal to the inverse transform. Once in the frequency-domain, one can employ techniques used for elliptic equations. In order to prove the well-posedness of the sub-diffusion equation studied here, we will study the transformed equation and carefully derive the stability estimates with respect to the frequency variable just as in the aforementioned works.

The rest of the paper is organized as follows. In Section 2 we rigorously define the direct and inverse problems under consideration. To do so, we will define the boundary value problem that will be studied as well as the appropriate assumption on the coefficients. Then in Section 3, we prove well-posedness of the direct problem by studying the corresponding problem in the frequency domain given by the Laplace transform of the time dependent problem. Section 4 is dedicated to studying the inverse impedance problem of recovering the generalized impedance boundary parameters from the knowledge of the Neumann-to-Dirichlet mapping. Lastly, in the final section, we conclude by summarizing the result from the previous sections and discuss future problems under consideration.

2. Problem statement

In this section, we will formulate the direct and inverse problem to be analyzed in Sections 3 and 4. The problems will be rigorously defined so that we may employ variational methods for solving these problems. We begin by considering the direct problem associated with the sub-diffusion equation with an impenetrable interior inclusion with a generalized impedance boundary condition. Here, we let $D \subset \mathbb{R}^2$ be a simply connected open set with C^2-boundary Γ_1 with unit outward normal ν. Now let $D_0 \subset D$ be a (possible multiple) connected open set with C^2-boundary Γ_0, where we assume that $\text{dist}(\Gamma_1, \overline{D}_0) \geqslant d > 0$. This gives that the annular region $D_1 = D \setminus \overline{D}_0$ is a connected set with boundary $\partial D_1 = \Gamma_1 \cup \Gamma_0$. See Figure 1 for example.

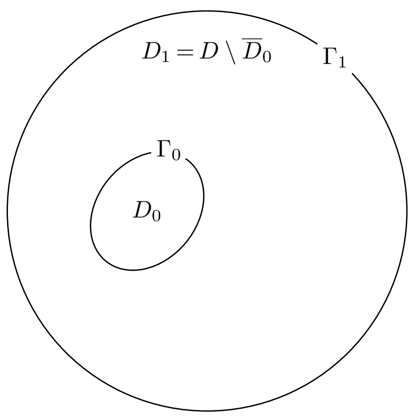

Figure 1. Example of a circular domain D with and elliptical subregion D_0.

In order to study this problem, we will consider the space of tempered distributions which vanish for $t \leqslant 0$ (i.e., causal). Now we define $u(x,t)$ as the causal tempered distribution solution to the sub-diffusion equation with a generalized impedance boundary condition that takes values in $H^1(D_1)$ for any $t > 0$. The boundary value problem under consideration is given by

$$\partial_t^\alpha u = \nabla \cdot A(x)\nabla u - c(x)u \quad \text{in } D_1 \times \mathbb{R}_+ \text{ with } u(x,t) = 0 \quad \forall\, t \leqslant 0, \tag{2.1}$$

$$\partial_{\nu_A} u(\cdot,t)|_{\Gamma_1} = f(x)g(t) \quad \text{and} \quad \mathscr{B}[u(\cdot,t)]|_{\Gamma_0} = 0 \quad \forall\, t > 0. \tag{2.2}$$

We will assume that the parameter $\alpha \in (0,1)$ is fixed. The fractional time derivative is assumed to be the Caputo derivative defined by

$$\partial_t^\alpha u = \frac{1}{\Gamma(1-\alpha)} \int_0^t \frac{\partial_\tau u(\cdot,\tau)}{(t-\tau)^\alpha}\, \mathrm{d}\tau,$$

where $\Gamma(1-\alpha)$ is the Gamma function evaluated at $1-\alpha$. Here the boundary operator in (2.2) is defined as

$$\mathscr{B}[u] = \partial_{\nu_A} u - \frac{\mathrm{d}}{\mathrm{d}\sigma}\eta(x)\frac{\mathrm{d}}{\mathrm{d}\sigma}u + \gamma(x)u, \tag{2.3}$$

where $\mathrm{d}/\mathrm{d}\sigma$ be the tangential derivative and σ is the arc-length parameter on Γ_0. Here we take ν to be the unit outward normal to the domain D_1 and $\nu \cdot A\nabla = \partial_{\nu_A}$ be the corresponding conormal derivative. Also, the generalized impedance boundary condition on the boundary Γ_0 is understood in the weak sense such that

$$0 = \int_{\Gamma_0} \left(\overline{\varphi}\partial_{\nu_A} u(\cdot,t) + \eta \frac{\mathrm{d}u(\cdot,t)}{\mathrm{d}\sigma}\frac{\mathrm{d}\overline{\varphi}}{\mathrm{d}\sigma} + \gamma u(\cdot,t)\overline{\varphi} \right) \mathrm{d}\sigma \quad \forall\, \varphi \in H^1(\Gamma_0) \quad \text{and} \quad t > 0.$$

To study problem (2.1)–(2.2) we assume that the spatial partial differential operator is symmetric and elliptic. To this end, we let the matrix-valued coefficient $A(x) \in C^{0,1}(D_1, \mathbb{R}^{2\times 2})$ be symmetric positive definite such that

$$\overline{\xi} \cdot A(x)\xi \geqslant A_{\min}|\xi|^2 \quad \text{for a.e. } x \in D_1.$$

The scalar coefficient $c(x) \in L^\infty(D_1)$ is such that

$$c(x) \geqslant 0 \quad \text{for a.e. } x \in D_1.$$

Notice that the assumptions on the coefficients give that the differential operator defined by the right-hand side of (2.1) is a symmetric elliptic partial differential operator. The regularity assumptions on the domain and coefficients ensure that we can apply unique continuation for the elliptic operator, which is important for considering the inverse problem. The flux on the boundary is given by the separated function $f(x)g(t)$, where $f \in H^{-1/2}(\Gamma_1)$ and g is piecewise continuous for all $t \geqslant 0$ of exponential order such that $g(t) = 0$ for all $t \leqslant 0$. Now, assume that the impedance parameters $\eta \in L^\infty(\Gamma_0)$ and $\gamma \in L^\infty(\Gamma_0)$. For analytical considerations throughout the paper, we will assume that the coefficients satisfy

$$\eta \geqslant \eta_{\min} > 0 \quad \text{and} \quad \gamma \geqslant \gamma_{\min} > 0 \quad \text{for a.e. } x \in \Gamma_0.$$

Note that in three spatial dimensions the operator $\frac{\mathrm{d}}{\mathrm{d}\sigma}\eta\frac{\mathrm{d}}{\mathrm{d}\sigma}$ is replaced by the Laplace-Beltrami operator $\operatorname{div}_{\Gamma_0}(\eta \operatorname{grad}_{\Gamma_0})$ and the analysis in Section 3 holds. The analysis in Section 4 does not hold in three spatial dimensions. There is little known for the recovery on the impedance parameters in three dimensions. Also, the analysis presented in the following sections can be simply augmented for the classical diffusion process, where the fractional derivative is replaced with the classical first order temporal derivative.

For completeness, we will state the result that will be used in Section 3 to prove well-posedness. This gives a characterization of which analytic functions with values in Banach space is the Laplace transform of a causal tempered distribution. To this end, let $\mathbb{C}_+ = \{s \in \mathbb{C},\ \operatorname{Re}(s) > 0\}$ and X be a Banach space. Assume that the mapping $\Phi\colon\ \mathbb{C}_+ \mapsto X$ is an analytic function such that

$$\|\Phi(s)\|_X \leqslant \mathcal{C}(\operatorname{Re}(s))|s|^\mu \quad \text{with } \mu < -1, \tag{2.4}$$

where $\mathcal{C}\colon\ (0,\infty) \mapsto (0,\infty)$ is non-increasing with $\mathcal{C}(\sigma) = \mathcal{O}(\sigma^{-l})$ as $\sigma \to 0$ for some $l \in \mathbb{N}$. Then there exists a unique X-valued causal tempered distribution $\varphi(t)$ whose Laplace transform is $\Phi(s)$, see [22] Chapter 3 for details.

Now when we have formulated the direct problem, we define the inverse problem under consideration. Here we are interested in the inverse impedance problem of determining the boundary operator $\mathscr{B}$ (i.e., the impedance parameters) from the knowledge of the solution u on the outer boundary of Γ_1. To this end, assume that the temporal function g is fixed and that we have *Neumann-to-Dirichlet* (NtD) mapping denoted by Λ that maps

$$f \mapsto u(\cdot,t)|_{\Gamma_1} \quad \forall\, t > 0.$$

It is clear that Λ depends on the boundary parameters and we wish to study the injectivity of the mapping $(\eta,\gamma) \mapsto \Lambda$. Since the temporal function g is assumed to be fixed, we have that the NtD operator can be viewed as linear mapping given by

$$\Lambda f = u(\cdot,t)|_{\Gamma_1}$$

for any $f \in H^{-1/2}(\Gamma_1)$. Using the NtD mapping to estimate the coefficients in a differential equation goes back to at least the Calderon problem (1980) in electrical impedance tomography [6]. It is clear that the NtD mapping will depend non-linearly on the coefficients but one can ask questions about uniqueness and stability. This is the basis of electrical impedance tomography, where there has been a lot of research in using the NtD mapping (see, e.g., [18]). In our analysis, we will assume the knowledge of the NtD mapping. A similar inverse impedance problems have been considered in [11] for the electrical impedance tomography problem.

3. Analysis of the direct problem

To analyze the direct problem we will use the Laplace transform. Therefore, let X be a Banach space, where we let $\mathrm{TD}[X]$ denote the X-valued causal tempered distribution with values in the Banach space X (see [22] for details). In order for the solution $u(\cdot, t)$ of (2.1)–(2.2) to be a causal tempered distribution, we will assume that g is a causal (real-valued) piecewise continuous function for all $t \geqslant 0$ of exponential order. This gives that the boundary data $f(x)g(t)$ is a causal tempered distribution with values in $H^{-1/2}(\Gamma_1)$. We define the Laplace transform for a causal tempered distribution $w \in \mathrm{TD}[X]$ as

$$\mathscr{L}\{w(t)\} = \int_0^\infty w(t)\mathrm{e}^{-st}\,\mathrm{d}t \quad \text{denoted } W(s) = \mathscr{L}\{w(t)\}$$

for any $s \in \mathbb{C}_+ = \{s \in \mathbb{C}, \text{ where } \mathrm{Re}(s) > 0\}$. By our assumptions on $g(t)$ we have that the Laplace transform of the boundary data exists and is given by $f(x)G(s)$, where $\mathscr{L}\{g(t)\} = G(s)$. We will further assume that there is a constant independent of $s \in \mathbb{C}_+$, where the Laplace transform for g satisfies

$$|G(s)| \leqslant \frac{C}{|s|^p} \quad \text{for some } p > 1 \; \forall\, s \in \mathbb{C}_+. \tag{3.1}$$

Now we consider the function space for the solution to the direct problem. Due to the generalized impedance condition (2.3) we consider the solution as a causal tempered distribution that has values in $H^1(D_1, \Gamma_0)$. Therefore, we wish to show the existence and uniqueness of the solution $u \in \mathrm{TD}[H^1(D_1, \Gamma_0)]$ that is the solution to (2.1)–(2.2) for given boundary data $f(x)g(t) \in \mathrm{TD}[H^{-1/2}(\Gamma_1)]$. We now define the space for which we attempt to find the solution as

$$H^1(D_1, \Gamma_0) = \{\varphi \in H^1(D_1) \text{ such that } \varphi|_{\Gamma_0} \in H^1(\Gamma_0)\}$$

with that associated norm/inner-product

$$\|\varphi\|^2_{H^1(D_1,\Gamma_0)} = \|\varphi\|^2_{H^1(D_1)} + \|\varphi\|^2_{H^1(\Gamma_0)}.$$

It is clear that $H^1(D_1, \Gamma_0)$ is a Hilbert Space with the graph norm defined above. Here the Sobolev spaces on the boundary are defined by the dual pairing between $H^p(\Gamma_j)$ and $H^{-p}(\Gamma_j)$ (for $p \geqslant 0$) with $L^2(\Gamma_j)$ as the pivot space, where Γ_j for $j = 0, 1$ are the closed curves defined in the previous section. The definition of the aforementioned Sobolev spaces can be found in [8], [21].

In order to prove the well-posedness of (2.1)–(2.2) with respect to any given spatial boundary data $f \in H^{-1/2}(\Gamma_1)$ and fixed causal temporal data g satisfying (3.1), we use the Laplace transform. We formally take the Laplace transform of equation (2.1)–(2.2) and by appealing to the fact that the solution u is causal, we obtain

$$-\nabla \cdot A(x)\nabla U + (c(x) + s^\alpha)U = 0 \text{ in } D_1 \quad \forall\, s \in \mathbb{C}_+ \tag{3.2}$$

$$\partial_{\nu_A} U(\cdot; s)|_{\Gamma_1} = f(x)G(s) \quad \text{and} \quad \mathscr{B}[U(\cdot; s)]|_{\Gamma_0} = 0 \quad \forall\, s \in \mathbb{C}_+. \tag{3.3}$$

Equations (3.2)–(3.3) are obtained by interchanging the spatial differential operators with the Laplace transform, e.g.,

$$\mathscr{L}\{\nabla \cdot A(x)\nabla u - c(x)u\} = \nabla \cdot A(x)\nabla \mathscr{L}\{u\} - c(x)\mathscr{L}\{u\},$$

where we use the fact that the coefficients and spatial differential operators are independent of t. This is done similarly to the boundary conditions to obtain (3.3). Here, $U(\cdot; s)$ denotes that Laplace transform of $u(\cdot, t)$. We have used the fact that

$$\mathscr{L}\{\partial_t^\alpha u(\cdot, t)\} = s^\alpha U(\cdot; s)$$

by appealing to the definition of the fractional time derivative and the convolution theorem for Laplace transforms. Indeed, notice that the Caputo derivative $\partial_t^\alpha u(\cdot, t)$ is the convolution integral of $\partial_t u(\cdot, t)$ and $t^{-\alpha}/\Gamma(1 - \alpha)$. Therefore, we formally see that convolution theorem would give that

$$\mathscr{L}\{\partial_t^\alpha u\} = \mathscr{L}\{\partial_t u\}\mathscr{L}\{t^{-\alpha}/\Gamma(1 - \alpha)\} = [sU(\cdot; s) - u(\cdot, 0)]\frac{1}{s^{1-\alpha}},$$

using the fact that $u(\cdot, 0) = 0$ (see, e.g., equation (2.253) of [19]). We can consider (3.2)–(3.3) as the frequency-domain boundary value problem associated with (2.1)–(2.2). Using the Laplace (or Fourier) transform to study time-domain problems is commonly done for hyperbolic problems (see, e.g., [2], [5], [10]). To prove the well-posedness of (2.1)–(2.2) for $u \in \mathrm{TD}[H^1(D_1, \Gamma_0)]$ will be done in two steps. First, we prove that (3.2)–(3.3) has a unique solution in $H^1(D_1,\Gamma_0)$. Then we can apply the Laplace inversion theorem (see [22] Chapter 3) to conclude that (2.1)–(2.2) is well-posed. This means we need to prove that $U(\cdot;s)$ satisfies the estimate (2.4). To this end, we will employ a variational technique for proving the well-posedness of (3.2)–(3.3), where we must establish the stability estimate, where the dependence on the frequency variable $s \in \mathbb{C}_+$ is explicit to apply the Laplace inversion theorem.

We have that for any given $V \in H^1(D_1,\Gamma_0)$ the equivalent variational formulation of (3.2)–(3.3) is obtained by appealing to Green's 1st Theorem and is given by

$$a_s(U,V) + b(U,V) = l_s(V). \tag{3.4}$$

Here the sesquilinear forms $a_s(\cdot,\cdot)$ and $b(\cdot,\cdot)\colon\ H^1(D_1,\Gamma_0)^2 \mapsto \mathbb{C}$ are defined by

$$a_s(U,V) = \int_{D_1} (A(x)\nabla U \cdot \nabla\overline{V} + (c(x)+s^\alpha)U\overline{V})\,\mathrm{d}x, \tag{3.5}$$

$$b(U,V) = \int_{\Gamma_0} \Big(\eta\frac{\mathrm{d}U}{\mathrm{d}\sigma}\frac{\mathrm{d}\overline{V}}{\mathrm{d}\sigma} + \gamma U\overline{V}\Big)\,\mathrm{d}\sigma \tag{3.6}$$

and the conjugate linear functional $l_s(\cdot)\colon\ H^1(D_1,\Gamma_0) \mapsto \mathbb{C}$ is defined as

$$l_s(V) = G(s)\int_{\Gamma_1} f\overline{V}\,\mathrm{d}\sigma. \tag{3.7}$$

It is clear that the sesquilinear forms are continuous for any given $s \in \mathbb{C}_+$ by appealing to the boundedness of the coefficients and the Cauchy-Schwartz inequality. In order to prove the well-posedness we will use the Lax-Milgram Lemma (see [21] Theorem 6.5), where the coercivity constant will depend on s. Then, in order to prove that the solution $U(\cdot;s) \in H^1(D_1,\Gamma_0)$ to (3.4) (and therefore (3.2)–(3.3)) is the Laplace transform of a tempered distribution $u \in \mathrm{TD}[H^1(D_1,\Gamma_0)]$ that solves (2.1)–(2.2), we prove that the reciprocal of the coercivity constant satisfies the assumption of the Laplace inversion formula given by equation (3.2) in [22].

Theorem 3.1. *The sesquilinear form $a_s(\cdot,\cdot)$ defined by* (3.5) *satisfies the estimate*

$$|a_s(U,U)| \geqslant C\cos(\alpha\pi/2)\min(1,\mathrm{Re}(s)^\alpha)\|U\|^2_{H^1(D_1)},$$

where $C > 0$ is a constant depending only on the coefficient matrix.

P r o o f. To prove the claim, notice that

$$|a_s(U,U)| = |\mathrm{e}^{-\mathrm{i}\alpha\,\mathrm{Arg}(s)}a_s(U,U)| \geqslant |\mathrm{Re}(\mathrm{e}^{-\mathrm{i}\alpha\,\mathrm{Arg}(s)}a_s(U,U))|.$$

Here $\mathrm{Arg}(s)$ denoted the argument of the complex number (i.e., the angular variable when represented in polar coordinates) such that $s = |s|e^{i\,\mathrm{Arg}(s)}$. Recall that for any $s \in \mathbb{C}_+$ which gives that $|\alpha\,\mathrm{Arg}(s)| \leqslant \alpha\pi/2$ and therefore,

$$1 \geqslant \cos(\alpha\,\mathrm{Arg}(s)) \geqslant \cos(\alpha\pi/2) > 0 \quad \forall\, \alpha \in (0,1).$$

This is due to the fact that $\cos(z)$ is minimized at $\pm\alpha\pi/2$ for any $z \in [-\alpha\pi/2, \alpha\pi/2]$. Therefore, we see that for all $U \in H^1(D_1, \Gamma_0)$

$$\begin{aligned} e^{-i\alpha\,\mathrm{Arg}(s)} a_s(U,U) = e^{-i\alpha\,\mathrm{Arg}(s)} \int_{D_1} (A(x)\nabla U \cdot \nabla \overline{U} + c(x)|U|^2)\,\mathrm{d}x \\ + s^{\alpha} e^{-i\alpha\,\mathrm{Arg}(s)} \int_{D_1} |U|^2\,\mathrm{d}x \end{aligned}$$

for all $U \in H^1(D_1,\Gamma_0)$ and notice that $s^\alpha = |s|^\alpha e^{i\alpha\,\mathrm{Arg}(s)}$, which gives that $s^\alpha e^{-i\alpha\,\mathrm{Arg}(s)} = |s|^\alpha$. Also, notice that the volume integrals in $a_s(U,U)$ are real-valued based on the assumptions of the coefficients. Now, using the fact that $c(x) \geqslant 0$, we can then estimate

$$\begin{aligned} \mathrm{Re}(e^{-i\alpha\,\mathrm{Arg}(s)} a_s(U,U)) &\geqslant \cos(\alpha\,\mathrm{Arg}(s)) \int_{D_1} A(x)\nabla U \cdot \nabla\overline{U}\,\mathrm{d}x + |s|^\alpha \int_{D_1} |U|^2\,\mathrm{d}x \\ &\geqslant \cos(\alpha\pi/2)[A_{\min}\|\nabla U\|^2_{L^2(D_1)} + \mathrm{Re}(s)^\alpha \|U\|^2_{L^2(D_1)}] \\ &\geqslant \cos(\alpha\pi/2)\min(1, A_{\min})\min(1, \mathrm{Re}(s)^\alpha)\|U\|^2_{H^1(D_1)}. \end{aligned}$$

This proves the claim. □

This gives us an explicit s-dependent coercivity estimate in $H^1(D_1)$ for $a(\cdot,\cdot)$. We now prove a coercivity estimate in $H^1(\Gamma_0)$ for the sesquilinear form $b(\cdot,\cdot)$, which would imply that the sum of the sesquilinear forms is coercive in $H^1(D_1,\Gamma_0)$.

Theorem 3.2. *The sesquilinear form $b(\cdot,\cdot)$ defined by (3.6) satisfies the estimate*

$$|b(U,U)| \geqslant C\cos(\alpha\pi/2)\|U\|^2_{H^1(\Gamma_0)},$$

where $C > 0$ is a constant depending only on the impedance parameters.

Proof. Similarly to proving the lower bound, we consider

$$|b(U,U)| \geqslant |\mathrm{Re}(e^{-i\alpha\,\mathrm{Arg}(s)} b(U,U))|,$$

where again $\mathrm{Arg}(s)$ is the argument of the complex number s. We still have that

$$1 \geqslant \cos(\alpha\,\mathrm{Arg}(s)) \geqslant \cos(\alpha\pi/2) > 0 \quad \forall\, \alpha \in (0,1),$$

where again we use the fact that $\cos(z)$ is minimized at $\pm\alpha\pi/2$ for any $z \in [-\alpha\pi/2, \alpha\pi/2]$. Therefore, by (3.6) we have that

$$\mathrm{e}^{-\mathrm{i}\alpha\,\mathrm{Arg}(s)} b(U,U) = \mathrm{e}^{-\mathrm{i}\alpha\,\mathrm{Arg}(s)}\left[\int_{\Gamma_0} \eta\left|\frac{\mathrm{d}U}{\mathrm{d}\sigma}\right|^2 + \gamma|U|^2\,\mathrm{d}\sigma\right]$$

for all $U \in H^1(D_1, \Gamma_0)$. Notice that the line integrals in $b(U,U)$ are real-valued based on the assumptions of the coefficients. Now, using the lower bounds on the impedance parameters

$$\eta \geqslant \eta_{\min} > 0 \quad \text{and} \quad \gamma \geqslant \gamma_{\min} > 0 \quad \text{for a.e. } x \in \Gamma_0$$

we have that

$$\begin{aligned}
\mathrm{Re}(\mathrm{e}^{-\mathrm{i}\alpha\,\mathrm{Arg}(s)} b(U,U)) &\geqslant \cos(\alpha\,\mathrm{Arg}(s))\left[\int_{\Gamma_0} \eta\left|\frac{\mathrm{d}U}{\mathrm{d}\sigma}\right|^2 + \gamma|U|^2\,\mathrm{d}\sigma\right] \\
&\geqslant \cos(\alpha\pi/2)\left[\eta_{\min}\left\|\frac{\mathrm{d}U}{\mathrm{d}\sigma}\right\|^2_{L^2(\Gamma_0)} + \gamma_{\min}\|U\|^2_{L^2(\Gamma_0)}\right] \\
&\geqslant \cos(\alpha\pi/2)\min(\gamma_{\min}, \eta_{\min})\|U\|^2_{H^1(\Gamma_0)}.
\end{aligned}$$

Recall that the norm in $H^1(\Gamma_0)$ is given by

$$\|U\|^2_{H^1(\Gamma_0)} = \int_{\Gamma_0}\left(\left|\frac{\mathrm{d}U}{\mathrm{d}\sigma}\right|^2 + |U|^2\right)\mathrm{d}\sigma.$$

This proves the claim. □

Notice that the Lax-Milgram Lemma implies that the sesquilinear form given by $a_s(\cdot,\cdot) + b(\cdot,\cdot)$ defined by (3.5)–(3.6) can be represented by an invertible operator $\mathbb{T}(s)$ that maps $H^1(D_1, \Gamma_0)$ into itself so that

$$a_s(U,V) + b(U,V) = (\mathbb{T}(s)U, V)_{H^1(D_1,\Gamma_0)} \quad \forall\, U, V \in H^1(D_1, \Gamma_0).$$

Since the sesquilinear form $a_s(\cdot,\cdot)$ is analytic for $s \in \mathbb{C}_+$, we have that $\mathbb{T}(s)$ depends analytically on $s \in \mathbb{C}_+$. Indeed, to prove the analyticity of $a_s(\cdot,\cdot)$ we see that by the Riesz representation theorem there exist bounded linear mappings from $H^1(D_1, \Gamma_0)$ into itself such that

$$\int_{D_1} (A(x)\nabla U \cdot \nabla\overline{V} + c(x)U\overline{V})\,\mathrm{d}x = (\mathbb{A}_1 U, V)_{H^1(D_1,\Gamma_0)}$$

and

$$\int_{D_1} U\overline{V}\,\mathrm{d}x = (\mathbb{A}_2 U, V)_{H^1(D_1,\Gamma_0)} \quad \forall\, U, V \in H^1(D_1, \Gamma_0).$$

Now, appealling to (3.5) we see that

$$a_s(U,V) = (\mathbb{A}_1 U, V)_{H^1(D_1,\Gamma_0)} + s^\alpha (\mathbb{A}_2 U, V)_{H^1(D_1,\Gamma_0)}$$

and since the mapping $s \mapsto \mathbb{A}_1 + s^\alpha \mathbb{A}_2$ is analytic from $\mathbb{C}_+$ to the space of bounded linear operators, it proves the analyticity of $a_s(\cdot,\cdot)$. Now provided that $\mathbb{T}(s)$ is invertible its inverse would also depend analytically on $s \in \mathbb{C}_+$.

We will now derive a norm estimate for the inverse of $\mathbb{T}(s)$ for any $s \in \mathbb{C}_+$, where the dependence on the frequency variable is made explicit. To this end, the lower bounds given in the above results imply that

$$|(\mathbb{T}(s)U, U)_{H^1(D_1,\Gamma_0)}| \geqslant C \cos(\alpha\pi/2) \min(1, \mathrm{Re}(s)^\alpha) \|U\|^2_{H^1(D_1,\Gamma_0)},$$

where the constant C is independent of $s \in \mathbb{C}_+$. Notice that we have used that

$$\begin{aligned}\mathrm{Re}(\mathrm{e}^{-\mathrm{i}\alpha \operatorname{Arg}(s)} b(U,U)) &\geqslant C\cos(\alpha\pi/2)\|U\|^2_{H^1(\Gamma_0)} \\ &\geqslant C\cos(\alpha\pi/2)\min(1,\mathrm{Re}(s)^\alpha)\|U\|^2_{H^1(\Gamma_0)}.\end{aligned}$$

From the coercivity estimate we have that

$$\|\mathbb{T}^{-1}(s)\|_{\mathcal{B}(H^1(D_1,\Gamma_0))} \leqslant \frac{C\sec(\alpha\pi/2)}{\min(1,\mathrm{Re}(s)^\alpha)}$$

(see [21] Theorem 6.5) in the operator norm, where $\mathcal{B}(H^1(D_1,\Gamma_0))$ is the space of bounded linear transformations from $H^1(D_1,\Gamma_0)$ into itself. Therefore, in the inversion theorem we can conclude that $\mathcal{C}(\mathrm{Re}(s))$ from (2.4) is given by

$$\mathcal{C}(\mathrm{Re}(s)) = \frac{C\sec(\alpha\pi/2)}{\min(1,\mathrm{Re}(s)^\alpha)}.$$

Now we derive a norm estimate for the conjugate linear functional $l_s(\cdot)$.

Theorem 3.3. *The conjugate linear functional $l_s(\cdot)$ defined by (3.7) satisfies the estimate*

$$|l_s(V)| \leqslant C|G(s)|\|f\|_{H^{-1/2}(\Gamma_1)}\|V\|_{H^1(D_1,\Gamma_0)},$$

where $C > 0$ is a constant depending only on the domain.

Proof. This is a consequence of the duality between $H^{\pm 1/2}$ with L^2 as the pivot space and the trace theorem (see, e.g., [8]) which gives that

$$|l_s(V)| \leqslant |G(s)|\|f\|_{H^{-1/2}(\Gamma_1)}\|V\|_{H^{1/2}(\Gamma_0)} \leqslant C|G(s)|\|f\|_{H^{-1/2}(\Gamma_1)}\|V\|_{H^1(D_1,\Gamma_0)}.$$

Here C is the constant from the trace theorem, proving the claim. □

By appealing to the Riesz representation theorem we can conclude that the variational problem (3.4) is equivalent to

$$\mathbb{T}(s)U = L_s \quad \text{where} \quad l_s(V) = (L_s, V)_{H^1(D_1,\Gamma_0)} \quad \forall\, V \in H^1(D_1, \Gamma_0).$$

We have that

$$\|L_s\|_{H^1(D_1,\Gamma_0)} \leqslant C|G(s)|\|f\|_{H^{-1/2}(\Gamma_1)}.$$

Provided that $G(s)$ depends analytically on $s \in \mathbb{C}_+$, we can conclude that L_s depends analytically on $s \in \mathbb{C}_+$. The analyticity assumption on $G(s)$ is needed so that we may apply the Laplace inversion formula. This implies that $U = U(\cdot, s) \in H^1(D_1, \Gamma_0)$ is given by $U(\cdot, s) = \mathbb{T}(s)^{-1}L_s$ and is therefore analytic with respect to $s \in \mathbb{C}_+$. By appealing to the estimate of the norm for the inverse of $\mathbb{T}(s)$ we have that the solution U to the variational problem (3.4) satisfies the norm estimate

$$\|U(\cdot, s)\|_{H^1(D_1,\Gamma_0)} \leqslant \frac{C\sec(\alpha\pi/2)}{\min(1, \mathrm{Re}(s)^\alpha)}|G(s)|\|f\|_{H^{-1/2}(\Gamma_1)}, \tag{3.8}$$

where the constant $C > 0$ is independent of $s \in \mathbb{C}_+$. From the above analysis we have that there is a unique solution to (3.2)–(3.3) satisfying the stability estimate (3.8). We recall that (3.2)–(3.3) was obtained by taking the Laplace transform of the time dependent equations (2.1)–(2.2). In order to prove the well-posedness of (2.1)–(2.2), we still need to show that $U(\cdot, s)$ is the Laplace transform of a causal tempered distribution $u(\cdot, t)$ that takes values in $H^1(D_1, \Gamma_0)$. To do so, we will appeal to the Laplace inversion theorem which can be applied, since we have assumed that the Laplace transform for $g(t)$ satisfies (3.1). Applying equation 3.3 from [22] gives the following result.

Theorem 3.4. *Assume that $f \in H^{-1/2}(\Gamma_1)$ and the Laplace transform of $g(t)$ given by $G(s)$ depends analytically on $s \in \mathbb{C}_+$ satisfying* (3.1). *Then we have that there is a unique solution $u \in \mathrm{TD}[H^1(D_1, \Gamma_0)]$ to* (2.1)–(2.2). *Moreover, we have the estimate*

$$\|u(\cdot, t)\|_{H^1(D_1,\Gamma_0)} \leqslant Ct^{\alpha+|1-p|}\|f\|_{H^{-1/2}(\Gamma_1)} \quad \forall\, f \in H^{-1/2}(\Gamma_1)$$

when $t \geqslant 1$, where the constant $C > 0$ is independent of t.

Notice that Theorem 3.4 gives that there is a solution $u(\cdot, t)$ to (2.1)–(2.2) that has at most polynomial growth in t. The proof of Theorem 3.4 is a direct consequence of the previous analysis in this section along with the strong inversion formula for the Laplace transform. The polynomial growth will play a role in an estimate in the proceeding section to study the inverse problem.

4. Analysis of the inverse problem

In this section, we consider the inverse impedance problem of recovering the impedance parameters η and γ from the Cauchy data. These types of inverse problems have applications where one needs to infer about the interior structure of a medium from boundary measurements. Indeed, this comes up in medical imaging and non-destructive testing when investigating the interior structure must be done without being able physically access the interior. These problems are frequently found in engineering applications of non-destructive testing. The mathematical questions are uniqueness, existence, and continuity with respect to the given measurements as well as developing numerical inversion algorithms. These questions have been studied for the elliptic problem coming from electrical impedance tomography in [4], [7], [11], where uniqueness results are given as well as numerical methods for recovering the impedance parameters. Note that the generalized impedance condition given in (2.3) depends on the material parameters η and γ linearly. Therefore, one hopes to derive a direct algorithm for recovering the coefficients. This is useful, since it would not require initial estimate on the material parameters. In [11] this is done in the case of electrical impedance tomography as well as developed a factorization method for recovering the interior boundary. Whereas in [4] a system of non-linear boundary integral equations is used to recover the impedance parameters and interior boundary Γ_0. Here we will only focus on the question of uniquely determining the impedance parameters on the interior boundary from measurement on the exterior boundary.

To begin, we assume that the temporal part of the flux $g(t)$ is a causal tempered distribution that is again *fixed* such that its Laplace transform $G(s)$ is well-defined and depends analytically on $s \in \mathbb{C}_+$ satisfying (3.1). Therefore, by Theorem 3.4 we have that there is a unique solution u to (2.1)–(2.2) that is a causal tempered distribution that takes values in $H^1(D_1, \Gamma_0)$ for all $t > 0$. Then we consider the *Neumann-to-Dirichlet* (NtD) mapping denoted by Λ that maps

$$H^{-1/2}(\Gamma_1) \to \mathrm{TD}[H^{1/2}(\Gamma_1)]$$

such that

$$f \mapsto u(\cdot, t)|_{\Gamma_1} \quad \forall\, t > 0.$$

We will assume that the NtD is known, i.e., one can measure the solution on the boundary Γ_1 given the flux f. By appealing to Theorem 3.4 and the trace theorem we have that the NtD operator is a well defined linear operator. Indeed, we have that by Theorem 3.4, the mapping $f \mapsto u(\cdot, t)|_{D_1}$ exists as a bounded linear map from $H^{-1/2}(\Gamma_1)$ into $\mathrm{TD}[H^1(D_1, \Gamma_0)]$. Then by the trace theorem the mapping $u(\cdot,t)|_{D_1} \mapsto u(\cdot,t)|_{\Gamma_1}$ is a well defined bounded linear mapping from $\mathrm{TD}[H^1(D_1,\Gamma_0)]$ into $\mathrm{TD}[H^{1/2}(\Gamma_1)]$ and we see that the NtD is the composition of the aforementioned bounded linear mappings. The main idea in this section is to extend the theory developed in [11] for the electrical impedance tomography problem for our inverse problem by appealing to the Laplace transform. This employs variational techniques to prove the uniqueness of the coefficients from the knowledge of the NtD mapping. Since variational techniques are used, less regularity is needed in the analysis than in [4] but one requires the knowledge of the full NtD mapping. We will assume that the NtD mapping is known for any $f \in H^{-1/2}(\Gamma_1)$ and for all $t > 0$ denoted

$$\Lambda = \Lambda(\eta,\gamma) \text{ with } \Lambda f = u(\cdot,t)|_{\Gamma_1} \quad \forall\, t > 0.$$

Since the NtD mapping is known for all $t > 0$, we can consider the Laplace transform of the NtD mapping

$$\mathscr{L}\{\Lambda f\} = \int_0^\infty u(\cdot,t)|_{\Gamma_1} \mathrm{e}^{-st}\,\mathrm{d}t$$

that maps

$$f \mapsto U(\cdot,s)|_{\Gamma_1} \quad \text{for any } s \in \mathbb{C}_+,$$

where U is the solution to (3.2)–(3.3). Since U solves an elliptic problem, it is easier to study the uniqueness in the frequency-domain which would imply uniqueness in the time-domain by the inversion formula. Before we can prove the main uniqueness result we first prove an auxiliary density result.

Theorem 4.1. *Define the set*

$$\mathcal{U} = \{U|_{\Gamma_0} \colon\ U \in H^1(D_1,\Gamma_0) \textit{ solving } (3.2)\text{–}(3.3)\ \forall\, f \in H^{-1/2}(\Gamma_1)\} \subset H^1(\Gamma_0).$$

Then $\mathcal{U}$ is a dense subspace of $L^2(\Gamma_0)$ for any $s \in \mathbb{R}_+$ such that $G(s) \neq 0$.

Proof. It is clear that the mapping $f \mapsto U(\cdot,s)|_{\Gamma_0}$ is linear since it is the composition of the solution operator for (3.2)–(3.3) and the Trace operator. This implies that $\mathcal{U}$ defines a linear subspace of $L^2(\Gamma_0)$. Now to prove the claim we will show that the set $\mathcal{U}^\perp = \{0\}$. To this end, we let $\varphi \in \mathcal{U}^\perp$ and let $V \in H^1_0(D_1,\Gamma_1)$ be the solution to the dual problem

$$-\nabla \cdot A(x)\nabla V + (c(x) + \overline{s}^\alpha)V = 0 \quad \text{in } D_1 \quad \forall\, s \in \mathbb{C}_+$$
$$\partial_{\nu_A} V|_{\Gamma_1} = 0 \quad \text{and} \quad \mathscr{B}[V]|_{\Gamma_0} = \varphi \quad \forall\, s \in \mathbb{C}_+.$$

It is clear that there is a unique solution $V \in H_0^1(D_1, \Gamma_0)$ to the dual problem above by appealing to similar arguments as in Section 3. Now let $s \in \mathbb{R}_+$ such that $G(s) \neq 0$. Therefore, we obtain that

$$
\begin{aligned}
0 &= \int_{\Gamma_0} U\varphi \, \mathrm{d}s = \int_{\Gamma_0} U\mathscr{B}[V] \, \mathrm{d}s = \int_{\Gamma_0} (U\partial_{\nu_A} V - V\partial_{\nu_A} U) \, \mathrm{d}s \\
&= -\int_{\Gamma_1} (U\partial_{\nu_A} V - V\partial_{\nu_A} U) \, \mathrm{d}s = G(s) \int_{\Gamma_1} fV \, \mathrm{d}s \quad \forall f \in H^{-1/2}(\Gamma_1),
\end{aligned}
$$

where we have used Green's 2nd Theorem. Due to the duality of $H^{\pm 1/2}$ the Hahn-Banach Theorem implies that $V = 0$ on Γ_1. Since V has zero Cauchy data on Γ_1, we can conclude that $V = 0$ in D_1 by unique continuation due to the assumptions on the domain and coefficients. The generalized impedance boundary condition implies that $\varphi = 0$, proving the claim. $\square$

In order to prove the uniqueness result, we will require that the impedance parameters $(\eta, \gamma) \in C(\Gamma_0) \times L^\infty(\Gamma_0)$. Even though less regularity is needed to prove the well-posedness of the problem, we will see that the increased regularity is needed for the proof of the uniqueness result presented in this section. This is not uncommon that the well-posedness can be established for weaker assumptions on the coefficients. The extra regularity for η is expected since it turns up in the second order differential operator on the boundary. This is standard in the analysis of PDEs just as in standard elliptic regularity results [8].

Theorem 4.2. *Let Λ be the Neumann-to-Dirichlet operator for* (2.1)–(2.2) *such that*

$$f \mapsto u(\cdot, t)|_{\Gamma_1} \quad \forall\, t > 0.$$

Then the mapping $(\eta, \gamma) \mapsto \Lambda(\eta, \gamma)$ is injective provided that $(\eta, \gamma) \in C(\Gamma_0) \times L^\infty(\Gamma_0)$.

P r o o f. In order to prove the claim, we proceed by way of contradiction. So assume that there are two sets of impedance parameters denoted $(\eta_j, \gamma_j) \in C(\Gamma_0) \times L^\infty(\Gamma_0)$ that produce the same NtD data for all $t > 0$. Then we have that the corresponding NtD mappings

$$\Lambda_j = \Lambda(\eta_j, \gamma_j) \quad \text{for } j = 1, 2$$

coincide for all $f \in H^{-1/2}(\Gamma_1)$. Now define the corresponding solutions to (2.1)–(2.2) by $u^{(j)}$ and its Laplace transform by $U^{(j)}$ which is the solution to (3.2)–(3.3). Since the Cauchy data for $u^{(j)}$ on Γ_1 coincide for all $t > 0$, we have that $U^{(1)} = U^{(2)}$ in D_1

for all $s \in \mathbb{C}_+$ and for any $f \in H^{-1/2}(\Gamma_1)$. We will assume that $s \in \mathbb{R}_+$ so that the Laplace transforms of the solution are real-valued. Now denote $U = U^{(1)} = U^{(2)}$, which satisfies the generalized impedance conditions

$$\partial_\nu U - \frac{\mathrm{d}}{\mathrm{d}\sigma}\eta_1 \frac{\mathrm{d}}{\mathrm{d}\sigma}U + \gamma_1 U = \partial_\nu U - \frac{\mathrm{d}}{\mathrm{d}\sigma}\eta_2 \frac{\mathrm{d}}{\mathrm{d}\sigma}U + \gamma_2 U = 0 \quad \text{on } \Gamma_0.$$

By subtracting the equations we obtain

$$0 = -\frac{\mathrm{d}}{\mathrm{d}\sigma}(\eta_1 - \eta_2)\frac{\mathrm{d}}{\mathrm{d}\sigma}U + (\gamma_1 - \gamma_2)U \quad \text{on } \Gamma_0$$

and integrating over Γ_0 gives that

$$0 = \int_{\Gamma_0} -\frac{\mathrm{d}}{\mathrm{d}\sigma}(\eta_1 - \eta_2)\frac{\mathrm{d}}{\mathrm{d}\sigma}U + (\gamma_1 - \gamma_2)U \,\mathrm{d}\sigma = \int_{\Gamma_0} (\gamma_1 - \gamma_2)U \,\mathrm{d}\sigma,$$

where the equality comes from integration by parts with the arc length variable σ. Since the above equality holds for all $f \in H^{-1/2}(\Gamma_1)$, appealing to Theorem 4.1 we can conclude that $\gamma_1 = \gamma_2$ a.e. on Γ_0.

Now assume that $f \in L^2(\Gamma_1) \subset H^{-1/2}(\Gamma_1)$ is real-valued, then by the similar analysis as in Section 2 of [7] we can conclude that $U \in H^{3/2}(D_1)$, which implies that $\partial_\nu U \in L^2(\Gamma_0)$. Then the generalized impedance boundary condition implies that

$$\eta_1 \frac{\mathrm{d}U}{\mathrm{d}\sigma} \in H^1(\Gamma_0) \quad \forall f \in L^2(\Gamma_1),$$

which implies that $U \in C^1(\Gamma_0)$, since $H^1(\Gamma_0) \subset C(\Gamma_0)$ and $\eta_1 \in C(\Gamma_0)$ with η_1 strictly positive. Since $\gamma_1 = \gamma_2$, subtracting the generalized impedance conditions gives

$$\frac{\mathrm{d}}{\mathrm{d}\sigma}(\eta_1 - \eta_2)\frac{\mathrm{d}}{\mathrm{d}\sigma}U = 0 \quad \forall f \in L^2(\Gamma_1).$$

Whence

$$(\eta_1 - \eta_2)\frac{\mathrm{d}U}{\mathrm{d}\sigma} = C \quad \forall f \in L^2(\Gamma_1),$$

where C is some constant. Now define $x(\sigma)\colon [0,l] \mapsto \mathbb{R}^2$ as an l-periodic C^2 representation of the closed curve Γ_0, where l is the length of the curve. Then we identify the space $H^1(\Gamma_0)$ with the auxiliary space $H^1_{\mathrm{per}}[0,l]$ of l-periodic functions. It is clear that due to the periodic condition, $U(x(0)) = U(x(l))$ for all real-valued $f \in L^2(\Gamma_1)$. Rolle's Theorem gives that the tangential derivative for U is zero for at least one point on the curve which gives that

$$(\eta_1 - \eta_2)\frac{\mathrm{d}U}{\mathrm{d}\sigma} = 0 \quad \text{for all real-valued } f \in L^2(\Gamma_1).$$

Now to prove that $\eta_1 = \eta_2$, we proceed by contradiction and assume that there is some $x^* \in \Gamma_0$, where $(\eta_1 - \eta_2)(x^*) > 0$. Due to the continuity there exists $\delta > 0$ such that $(\eta_1 - \eta_2) > 0$ for all $x \in \Gamma_0^\delta = \Gamma_0 \cap B(x^*, \delta)$. We can conclude that

$$\frac{\mathrm{d}U}{\mathrm{d}\sigma} = 0 \quad \text{on } \Gamma_0^\delta \text{ for all real-valued } f \in L^2(\Gamma_1). \tag{4.1}$$

Now for any f_1 and f_2 linearly independent real-valued $L^2(\Gamma_1)$ functions, we have that the corresponding U_{f_1} and U_{f_2} are linearly independent (see Theorem 2.2 in [7]). Therefore, we can conclude that the Wronskian given by

$$(U_{f_1}, U_{f_2}) \mapsto U_{f_1} \frac{\mathrm{d}}{\mathrm{d}\sigma} U_{f_2} - U_{f_2} \frac{\mathrm{d}}{\mathrm{d}\sigma} U_{f_1}$$

cannot be identically zero on any open subset of Γ_0. By (4.1) we have that Wronskian is identically zero on Γ_0^δ, which contradicts the linear independence of f_1 and f_2 proving the claim. □

Notice that from the proof of Theorem 4.2 we have that Cauchy data for f and $U(\cdot, s)$ on Γ_1 uniquely determine the impedance parameters. Assuming that the NtD, as well as Γ_0, is known, this implies that we can use a data completion algorithm to recover $U_f(\cdot, s)$ and $\partial_{\nu_A} U_f(\cdot, s)$ on the inner boundary Γ_0. Recently, in [1] a stable data completion algorithm was derived using boundary integral equations for the Helmholtz equation. Provided that $A = I$ and $c = 0$, the numerical method for recovering the interior Cauchy data in [1] can be employed for a given $s \in \mathbb{R}_+$. Once $U_f(\cdot, s)$ and $\partial_\nu U_f(\cdot, s)$ are known on Γ_0, we can employ the reconstruction algorithm in Section 4 of [11] to recover the impedance parameters. This method constructs a linear system of equations to recover the impedance parameters. This gives a direct method for recovering the parameters where one does not need a priori estimates for η and γ. To do this, we need the compute the Laplace transform of the data. This would require infinite temporal measurements on Γ_1 which is not physically feasible. Therefore, we will show that one can take partial temporal measurements on the outer boundary Γ_1 to approximate the Laplace transform of the NtD mapping.

To this end, we now define the partial temporal NtD measurements on the outer boundary Γ_1. This is that mapping such that the spatial flux component $f \in H^{-1/2}(\Gamma_1)$ is mapped to

$$\widetilde{u}_f(\cdot, t)|_{\Gamma_1} = \begin{cases} u_f(\cdot, t)|_{\Gamma_1}, & t \leqslant T \\ 0, & t > T \end{cases} \quad \text{for some } T \geqslant 1.$$

It is clear that $\widetilde{u}_f(\cdot, t)|_{\Gamma_1} \in \mathrm{TD}[H^{1/2}(\Gamma_1)]$ and it denotes the measured partial temporal data on the finite time-interval $(0, T)$. This can be seen as an approximation of the measured data, where we extend that data for all unknown temporal values by zero. Note that we can write

$$\widetilde{u}_f(\cdot,t)|_{\Gamma_1} = \chi_{[0,T]}(t)u_f(\cdot,t)|_{\Gamma_1} \quad \forall\, t>0, \tag{4.2}$$

where χ is the indicator function. Now, we will estimate the error in the Laplace transforms in the NtD measurements with respect to the finite time of measurements taken on $(0,T)$, where $T \geqslant 1$.

Theorem 4.3. *Let $\widetilde{U}_f(\cdot,s)|_{\Gamma_1} \in H^{1/2}(\Gamma_1)$ denote the Laplace transform of the partial temporal NtD measurements given by* (4.2) *for any $f \in H^{-1/2}(\Gamma_1)$. Then we have that there is a $m \in \mathbb{N}$ such that*

$$\|U_f(\cdot,s) - \widetilde{U}_f(\cdot,s)\|_{H^{1/2}(\Gamma_1)} \leqslant CT^m \mathrm{e}^{-\mathrm{Re}(s)T}\|f\|_{H^{-1/2}(\Gamma_1)} \quad \text{for any } T \geqslant 1$$

with the constant $C>0$ being independent of T and $f \in H^{-1/2}(\Gamma_1)$.

P r o o f. We begin by noticing that

$$[u_f(\cdot,t) - \widetilde{u}_f(\cdot,t)]|_{\Gamma_1} = [1-\chi_{[0,T]}(t)]u_f(\cdot,t)|_{\Gamma_1}$$

for any $f \in H^{-1/2}(\Gamma_1)$. By taking the Laplace transform on both sides we have that

$$U_f(\cdot,s) - \widetilde{U}_f(\cdot,s) = \int_T^\infty u(\cdot,t)|_{\Gamma_1}\mathrm{e}^{-st}\,\mathrm{d}t.$$

From the above equality we are able to estimate the $H^{1/2}(\Gamma_1)$ norm. Therefore, by the trace theorem we have

$$\begin{aligned}\|U_f(\cdot,s) - \widetilde{U}_f(\cdot,s)\|_{H^{1/2}(\Gamma_1)} &\leqslant \int_T^\infty \|u(\cdot,t)\|_{H^{1/2}(\Gamma_1)}\mathrm{e}^{-\mathrm{Re}(s)t}\,\mathrm{d}t\\ &\leqslant C\int_T^\infty \|u(\cdot,t)\|_{H^1(D_1,\Gamma_0)}\mathrm{e}^{-\mathrm{Re}(s)t}\,\mathrm{d}t.\end{aligned}$$

Now by the norm estimate in Theorem 3.4 we have that

$$\|U_f(\cdot,s) - \widetilde{U}_f(\cdot,s)\|_{H^{1/2}(\Gamma_1)} \leqslant C\|f\|_{H^{-1/2}(\Gamma_1)}\int_T^\infty t^{\alpha+|1-p|}\mathrm{e}^{-\mathrm{Re}(s)t}\,\mathrm{d}t$$

since we have assumed that $T \geqslant 1$. We now let $m = \lceil \alpha + |1-p| \rceil$ and whence

$$\int_T^\infty t^m \mathrm{e}^{-\mathrm{Re}(s)t}\,\mathrm{d}t = \mathrm{e}^{-\mathrm{Re}(s)T}\sum_{k=0}^m \binom{m}{k}\frac{(m-k)!}{\mathrm{Re}(s)^{m-k+1}}T^k,$$

which is obtained by the binomial theorem and using standard calculus to evaluate the improper integral. Therefore, we can conclude that

$$\|U_f(\cdot,s)-\widetilde{U}_f(\cdot,s)\|_{H^{1/2}(\Gamma_1)} \leqslant C\|f\|_{H^{-1/2}(\Gamma_1)}\mathrm{e}^{-\mathrm{Re}(s)T}\sum_{k=0}^{m}\binom{m}{k}\frac{(m-k)!}{\mathrm{Re}(s)^{m-k+1}}T^k$$

and again, the fact that $T \geqslant 1$ proves the claim. $\square$

Notice that by Theorem 4.3 we have that the Laplace transform of the partial temporal finite time NtD measurements converge in the operator norm to the Laplace transform of the NtD measurements for (3.2)–(3.3) as $T \to \infty$. This gives that for the case when $A = I$ and $c = 0$ one can use the stabilized data completion algorithm in [1] to recover the Cauchy data on the inner boundary for a fixed $T \gg 1$ and whence reconstruct the impedance parameters according to [11].

5. Summary and conclusions

Here we have studied the direct and inverse impedance problems for a sub-diffusion equations with a generalized impedance boundary condition. The analysis for the direct problem holds in both $\mathbb{R}^2$ and $\mathbb{R}^3$. The analysis uses the Laplace transform to study the problem in the frequency-domain and to assure that one can use the inversion formula to infer the solvability in the time-domain. There is still a need to test numerical methods for solving the direct problem. The uniqueness results for the inverse impedance problem strongly depend on analysis unique to the $\mathbb{R}^2$ case. We have also discussed a possible method for recovering the impedance parameters for the NtD measurements on the outer boundary. The inversion algorithm uses a method for the case when the elliptic operator is given by the Laplacian. A numerical study for the proposed inversion algorithm also needs to be established.

References

[1] *Y. Boukari, H. Haddar*: A convergent data completion algorithm using surface integral equations. Inverse Probl. *31* (2015), Article ID 035011, 21 pages. zbl MR doi

[2] *T. S. Brown, S. Du, H. Eruslu, F.-J. Sayas*: Analysis of models for viscoelastic wave propagation. Appl. Math. Nonlinear Sci. *3* (2018), 55–96. MR doi

[3] *F. Cakoni, I. de Teresa, H. Haddar, P. Monk*: Nondestructive testing of the delaminated interface between two materials. SIAM J. Appl. Math. *76* (2016), 2306–2332. zbl MR doi

[4] *F. Cakoni, R. Kress*: Integral equation methods for the inverse obstacle problem with generalized impedance boundary condition. Inverse Probl. *29* (2013), Article ID 015005, 19 pages. zbl MR doi

[5] *F. Cakoni P. Monk, V. Selgas*: Analysis of the linear sampling method for imaging penetrable obstacles in the time domain. Anal. PDE *14* (2021), 667–688. zbl MR doi

[6] *A.-P. Calderón*: On an inverse boundary value problem. Seminar on Numerical Analysis and Its Applications to Continuum Physics. Soc. Brasil. Mat., Rio de Janeiro, 1980, pp. 65–73. MR
[7] *S. Chaabane, B. Charfi, H. Haddar*: Reconstruction of discontinuous parameters in a second order impedance boundary operator. Inverse Probl. *32* (2016), Article ID 105004, 22 pages. zbl MR doi
[8] *L. C. Evans*: Partial Differential Equations. Graduate Studies in Mathematics 19. American Mathematical Society, Providence, 2010. zbl MR doi
[9] *J. Guo, G. Nakamura, H. Wang*: The factorization method for recovering cavities in a heat conductor. Available at `https://arxiv.org/abs/1912.11590` (2019), 15 pages.
[10] *H. Haddar, A. Lechleiter, S. Marmorat*: An improved time domain linear sampling method for Robin and Neumann obstacles. Appl. Anal. *93* (2014), 369–390. zbl MR doi
[11] *I. Harris*: Detecting an inclusion with a generalized impedance condition from electrostatic data via sampling. Math. Methods Appl. Sci. *42* (2019), 6741–6756. zbl MR doi
[12] *I. Harris*: A direct method for reconstructing inclusions and boundary conditions from electrostatic data. To appear in Appl. Anal. (2021), 19 pages. doi
[13] *M. I. Ismailov, M. Çiçek*: Inverse source problem for a time-fractional diffusion equation with nonlocal boundary conditions. Appl. Math. Modelling *40* (2016), 4891–4899. zbl MR doi
[14] *B. Jin, W. Rundell*: A tutorial on inverse problems for anomalous diffusion processes. Inverse Probl. *31* (2015), Article ID 035003, 40 pages. zbl MR doi
[15] *B. Kaltenbacher, W. Rundell*: On an inverse potential problem for a fractional reaction-diffusion equation. Inverse Probl. *35* (2019), Article ID 065004, 31 pages. zbl MR doi
[16] *B. Kaltenbacher, W. Rundell*: Regularization of a backwards parabolic equation by fractional operators. Inverse Probl. Imaging *13* (2019), 401–430. zbl MR doi
[17] *A. Kirsch, N. Grinberg*: The Factorization Method for Inverse Problems. Oxford Lecture Series in Mathematics and Its Applications 36. Oxford University Press, Oxford, 2008. zbl MR doi
[18] *J. F. Mueller, S. Siltanen*: Linear and Nonlinear Inverse Problems with Practical Applications. Computational Science & Engineering 10. SIAM, Philadelphia, 2012. zbl MR doi
[19] *I. Podlubny*: Fractional Differential Equations: An Introduction to Fractional Derivatives, Fractional Differential Equations, to Methods of Their Solution and Some of Their Applications. Mathematics in Science and Engineering 198. Academic Press, San Diego, 1999. zbl MR doi
[20] *T. Qiu, A. Rieder, F.-J. Sayas, S. Zhang*: Time-domain boundary integral equation modeling of heat transmission problems. Numer. Math. *143* (2019), 223–259. zbl MR doi
[21] *S. Salsa*: Partial Differential Equations in Action: From Modelling to Theory. Universitext. Springer, Berlin, 2008. zbl MR doi
[22] *F.-J. Sayas*: Retarded Potentials and Time Domain Boundary Integral Equations: A Road Map. Springer Series in Computational Mathematics 50. Springer, Cham, 2016. zbl MR doi
[23] *T. N. Thach, T. N. Huy, P. T. M. Tam, M. N. Minh, N. H. Can*: Identification of an inverse source problem for time-fractional diffusion equation with random noise. Math. Methods Appl. Sci. *42* (2019), 204–218. zbl MR doi

Author's address: *Isaac Harris*, Department of Mathematics, Purdue University, 150 N. University Street, West Lafayette, IN 47907, U.S.A., e-mail: `harri814@purdue.edu`.