

Mehdi Hassani

Remarks on Ramanujan's inequality concerning the prime counting function

*Communications in Mathematics*, Vol. 29 (2021), No. 3, 473–482

Persistent URL: <http://dml.cz/dmlcz/149330>

## Terms of use:

© University of Ostrava, 2021

Institute of Mathematics of the Czech Academy of Sciences provides access to digitized documents strictly for personal use. Each copy of any part of this document must contain these *Terms of use*.



This document has been digitized, optimized for electronic delivery and stamped with digital signature within the project *DML-CZ: The Czech Digital Mathematics Library* <http://dml.cz>

# Remarks on Ramanujan’s inequality concerning the prime counting function

Mehdi Hassani

**Abstract.** In this paper we investigate Ramanujan’s inequality concerning the prime counting function, asserting that  $\pi(x)^2 < \frac{ex}{\log x} \pi\left(\frac{x}{e}\right)$  for  $x$  sufficiently large. First, we study its sharpness by giving full asymptotic expansions of its left and right hand sides expressions. Then, we discuss the structure of Ramanujan’s inequality, by replacing the factor  $\frac{x}{\log x}$  on its right hand side by the factor  $\frac{x}{\log x-h}$  for a given  $h$ , and by replacing the numerical factor  $e$  by a given positive  $\alpha$ . Finally, we introduce and study inequalities analogous to Ramanujan’s inequality.

## 1 Introduction

Among several conjectures and results concerning the distribution of prime numbers, on page 310 of his second notebook [11], Ramanujan asserts that the inequality

$$\pi(x)^2 < \frac{ex}{\log x} \pi\left(\frac{x}{e}\right), \tag{1}$$

holds for  $x$  sufficiently large. Here, as usual,  $\pi(x)$  denotes the number of primes not exceeding  $x$ . To confirm (1) for  $x$  sufficiently large, we note that the prime number theorem with error term gives the expansion

$$\pi(x) = x \sum_{k=0}^n \frac{k!}{\log^{k+1} x} + O\left(\frac{x}{\log^{n+2} x}\right), \tag{2}$$

for any integer  $n \geq 0$ . Using (2) with  $n = 4$  implies

$$\pi(x)^2 - \frac{ex}{\log x} \pi\left(\frac{x}{e}\right) = -\frac{x^2}{\log^6 x} + O\left(\frac{x^2}{\log^7 x}\right),$$

---

2020 MSC: 11A41

Key words: Prime numbers, Ramanujan’s inequality, Riemann hypothesis

Affiliation:

Department of Mathematics, University of Zanjan, University Blvd., 45371-38791,  
 Zanjan, Iran  
 E-mail: mehdi.hassani@znu.ac.ir

and this proves (1) for  $x$  sufficiently large. We refer the interested reader to pages 111–137 of [2] and pages 22–47 of [5] for more details on Ramanujan’s ideas concerning the analytic theory of primes.

The most important studies regarding Ramanujan’s inequality (1) ask about the positive integer  $x_{\mathcal{R}}$  for which (1) holds if  $x \geq x_{\mathcal{R}}$  and fails for  $x < x_{\mathcal{R}}$ . In 2012 the author [7] approximated  $x_{\mathcal{R}}$  under the assumption of the existence of some very good bounds for the function  $\pi(x)$ . In 2015 Dudek and Platt [3], based on the sharp bounds due to Trudgian, which appeared some months after their work in [12], obtained such very good bounds for  $\pi(x)$  implying that  $x_{\mathcal{R}} \leq e^{9658}$ . Dudek and Platt note, on page 292, that using a result by Mossinghoff and Trudgian [8] they can prove that  $x_{\mathcal{R}} \leq e^{9394}$ . In 2018 Axler [1] proved that  $x_{\mathcal{R}} \leq e^{9032}$ . He also proved that (1) holds unconditionally for every  $x$  satisfying  $38,358,837,683 \leq x \leq 10^{19}$ .

Under the assumption that the Riemann hypothesis is true, the author [7] proved that  $x_{\mathcal{R}} \leq 138,766,146,692,471,228$ . Dudek and Platt [3] refined this conditional result by showing that  $x_{\mathcal{R}} \leq 1.15 \times 10^{16}$ . Moreover, they proved that assuming the Riemann hypothesis, the largest integer counterexample to Ramanujan’s inequality (1) is at  $x = 38,358,837,682$ .

In 2013 the author [6] studied the following generalization of (1) for a given positive integer  $n$ ,

$$\pi(x)^{2^n} < \frac{e^n}{\prod_{k=1}^n \left(1 - \frac{k-1}{\log x}\right)^{2^{n-k}}} \left(\frac{x}{\log x}\right)^{2^{n-1}} \pi\left(\frac{x}{e^n}\right).$$

For  $n = 1$  the above generalization coincides with (1).

In this paper we are motivated by some questions, including “How sharp is Ramanujan’s inequality (1)?” and “How subtle is the form of Ramanujan’s inequality (1)?”. We will consider these questions in Sections 2 and 3, respectively. In Section 4 we consider some inequalities analogous to Ramanujan’s inequality. We end the paper with a computational observation concerning the constant  $x_{\mathcal{R}}$ .

## 2 Sharpness of Ramanujan’s inequality

To work on the sharpness of Ramanujan’s inequality we give full asymptotic expansions of its left and right hand sides expressions, providing some very nice corollaries.

**Theorem 1.** *Let  $\ell_k = \sum_{j=0}^k j!(k-j)!$  and  $r_k = \sum_{j=0}^k j! \binom{k}{j}$ . Then, for a given integer  $n \geq 0$  we have*

$$\pi(x)^2 = x^2 \sum_{k=0}^n \frac{\ell_k}{\log^{k+2} x} + O\left(\frac{x^2}{\log^{n+3} x}\right), \tag{3}$$

and

$$\frac{ex}{\log x} \pi\left(\frac{x}{e}\right) = x^2 \sum_{k=0}^n \frac{r_k}{\log^{k+2} x} + O\left(\frac{x^2}{\log^{n+3} x}\right). \tag{4}$$

As an immediate corollary, we obtain full asymptotic expansions of

$$\pi(x)^2 - \frac{ex}{\log x} \pi\left(\frac{x}{e}\right) \text{ as } x \rightarrow \infty.$$

**Corollary 1.** *For a given integer  $n \geq 4$  we have*

$$\pi(x)^2 - \frac{ex}{\log x} \pi\left(\frac{x}{e}\right) = x^2 \sum_{k=4}^n \frac{d_k}{\log^{k+2} x} + O\left(\frac{x^2}{\log^{n+3} x}\right), \tag{5}$$

where  $d_k = \ell_k - r_k = \sum_{j=0}^k j!((k-j)! - \binom{k}{j})$ .

Note that  $d_0 = d_1 = d_2 = d_3 = 0$ , and some more initial values of  $d_k$  are  $d_4 = -1, d_5 = -14, d_6 = -145, d_7 = -1412, d_8 = -13985$  and so on. Since  $d_k < 0$  for any  $k \geq 4$  (see Remark 1 below) we obtain the following refinement of Ramanujan's inequality (1).

**Corollary 2.** *Let  $m \geq 4$  be a fixed integer. Then, for  $x$  sufficiently large*

$$\pi(x)^2 < \frac{ex}{\log x} \pi\left(\frac{x}{e}\right) + x^2 \sum_{k=4}^m \frac{d_k}{\log^{k+2} x}.$$

**Remark 1.** The sequences  $(\ell_n)_{n \geq 0}$  and  $(r_n)_{n \geq 0}$  are known in the OEIS as A003149 and A000522, respectively. The recurrence  $\ell_n = n! + \frac{n+1}{2} \ell_{n-1}$  holds for each  $n \geq 1$ . Thus, by induction we obtain

$$2n! \leq \ell_n \leq \frac{8}{3}n! \tag{6}$$

for each  $n \geq 1$ . Moreover, the above recurrence implies that  $\ell_n \sim 2n!$  as  $n \rightarrow \infty$ . On the other hand, we observe that if  $e_n = \sum_{k=0}^n \frac{1}{k!}$  then

$$0 < e - e_n = \sum_{k=1}^{\infty} \frac{1}{(n+k)!} = \frac{1}{n!} \sum_{k=1}^{\infty} \prod_{j=1}^k \frac{1}{n+j} < \frac{1}{n!} \sum_{k=1}^{\infty} \frac{1}{(n+1)^k} = \frac{1}{nn!}.$$

Thus, for each  $n \geq 1$  we obtain  $r_n = n!e_n = \lfloor en! \rfloor$ , where  $\lfloor x \rfloor$  denotes the largest integer not exceeding  $x$ . Meanwhile, for  $n \geq 4$  we have  $n! \geq 24 > 1/(e - \frac{8}{3})$ . Thus,  $\frac{8}{3}n! < en! - 1$ , and considering the right hand side of (6) we obtain  $\ell_n < r_n$  for each  $n \geq 4$ .

### 3 Structure of Ramanujan's inequality

To study the structure of Ramanujan's inequality (1), we make observations concluding that he was quite clever when he was creating his inequality on the prime counting function. Our method is mainly to use the expansion (2) with  $n = 1$  and  $n = 4$ , and to simplify the expressions following methods introduced in the proof of Theorem 1.

The first observation concerns the factor  $\frac{x}{\log x}$  on the right hand side of (1). This factor basically comes from the true size of  $\pi(x)$ , which is  $\frac{x}{\log x}$ . Although we know that  $\frac{x}{\log x - 1}$  is a much closer approximation for  $\pi(x)$ , the following result asserts that the factor  $\frac{x}{\log x}$  on the right hand side of (1) is the best choice among several approximations of the form  $\frac{x}{\log x - h}$ .

**Theorem 2.** *Let  $h$  be a given real number. If  $h \geq 0$  then for  $x$  sufficiently large*

$$\pi(x)^2 < \frac{ex}{\log x - h} \pi\left(\frac{x}{e}\right).$$

*If  $h < 0$  then the above inequality reverses.*

**Remark 2.** More precisely, the expansion (2) with  $n = 4$  gives

$$\pi(x)^2 = x^2 \left( \frac{1}{\log^2 x} + \frac{2}{\log^3 x} + \frac{5}{\log^4 x} + \frac{16}{\log^5 x} + \frac{64}{\log^6 x} + O\left(\frac{1}{\log^7 x}\right) \right), \quad (7)$$

and

$$\frac{ex}{\log x - h} \pi\left(\frac{x}{e}\right) = x^2 \left( \frac{1}{\log^2 x} + \frac{2 + A_1(h)}{\log^3 x} + \frac{5 + B_1(h)}{\log^4 x} + \frac{16 + C_1(h)}{\log^5 x} + \frac{65 + D_1(h)}{\log^6 x} + O\left(\frac{1}{\log^7 x}\right) \right),$$

where

$$\begin{aligned} A_1(h) &= h, \\ B_1(h) &= h(h + 2), \\ C_1(h) &= h(h^2 + 2h + 5), \\ D_1(h) &= h(h^3 + 2h^2 + 5h + 16). \end{aligned}$$

A comparison of the coefficients shows that  $h = 0$  is the most critical choice, as in Ramanujan’s inequality (1).

The second observation on the structure of Ramanujan’s inequality (1) concerns the factor  $e$ . The following result is a formulation of Theorem 1.3 of [7]; however, in the present paper we give a much simpler proof.

**Theorem 3.** *If  $\alpha \geq e$  then for  $x$  sufficiently large*

$$\pi(x)^2 < \frac{\alpha x}{\log x} \pi\left(\frac{x}{\alpha}\right).$$

*If  $0 < \alpha < e$  then the above inequality reverses.*

**Remark 3.** Assume that  $\alpha > 0$ , and let  $u = \log \alpha$ . Then the expansion (2) with  $n = 4$  gives

$$\frac{\alpha x}{\log x} \pi\left(\frac{x}{\alpha}\right) = x^2 \left( \frac{1}{\log^2 x} + \frac{A_2(u)}{\log^3 x} + \frac{B_2(u)}{\log^4 x} + \frac{C_2(u)}{\log^5 x} + \frac{D_2(u)}{\log^6 x} + O\left(\frac{1}{\log^7 x}\right) \right),$$

where

$$\begin{aligned} A_2(u) &= u + 1, \\ B_2(u) &= u^2 + 2u + 2, \\ C_2(u) &= u^3 + 3u^2 + 6u + 6, \\ D_2(u) &= u^4 + 4u^3 + 12u^2 + 24u + 24. \end{aligned}$$

Note that for  $\alpha = e$  one has  $A_2(1) = 2$ ,  $B_2(1) = 5$ ,  $C_2(1) = 16$  and  $D_2(1) = 65$ . Comparing these values and the coefficients in (7) shows that  $\alpha = e$  is the most critical choice, as in Ramanujan's inequality (1).

As another observation on the structure of Ramanujan's inequality, in (1) we replace  $x$  by  $e x$ . Thus Ramanujan's inequality takes the following equivalent form:

$$\pi(e x)^2 < \frac{e^2 x}{1 + \log x} \pi(x). \tag{8}$$

The expansion (2) with  $n = 4$  gives

$$\pi(e x)^2 = e^2 x^2 \left( \frac{1}{\log^2 x} + \frac{2}{\log^4 x} + \frac{4}{\log^5 x} + \frac{19}{\log^6 x} + O\left(\frac{1}{\log^7 x}\right) \right),$$

and

$$\frac{e^2 x}{1 + \log x} \pi(x) = e^2 x^2 \left( \frac{1}{\log^2 x} + \frac{2}{\log^4 x} + \frac{4}{\log^5 x} + \frac{20}{\log^6 x} + O\left(\frac{1}{\log^7 x}\right) \right).$$

Thus we obtain (8) for  $x$  sufficiently large. Here, the point is missing the factor  $\frac{1}{\log^3 x}$  in both expansions; to see the reason, for a given  $h$  we have

$$\frac{e^2 x}{h + \log x} \pi(x) = e^2 x^2 \left( \frac{1}{\log^2 x} + \frac{A_3(h)}{\log^3 x} + \frac{B_3(h)}{\log^4 x} + \frac{C_3(h)}{\log^5 x} + \frac{D_3(h)}{\log^6 x} + O\left(\frac{1}{\log^7 x}\right) \right),$$

where

$$\begin{aligned} A_3(h) &= -h + 1, \\ B_3(h) &= h^2 - h + 2, \\ C_3(h) &= -h^3 + h^2 - 2h + 6, \\ D_3(h) &= h^4 - h^3 + 2h^2 - 6h + 24. \end{aligned}$$

Comparing the coefficients shows that  $h = 1$  is the most critical choice, as in (8). Moreover, we get the following result.

**Theorem 4.** *Let  $h$  be a given real number. If  $h \leq 1$  then for  $x$  sufficiently large*

$$\pi(e x)^2 < \frac{e^2 x}{h + \log x} \pi(x).$$

*If  $h > 1$  then the above inequality reverses.*

### 4 Inequalities analogous to Ramanujan's inequality

Not only is beautiful and curious, Ramanujan's inequality compels us to improve our knowledge on the distribution of prime numbers. In search of analogues of Ramanujan's inequality, first we try to consider  $\pi(x)^3$  instead of  $\pi(x)^2$ , and we compare it with expressions analogous to the right hand side of (1). Our observations end in the following double sided inequality.

**Theorem 5.** For  $x$  sufficiently large

$$\left(\frac{\sqrt{e}x}{\log x}\right)^2 \pi\left(\frac{x}{e}\right) < \pi(x)^3 < \frac{e^2x}{\log x} \pi\left(\frac{x}{e}\right)^2. \tag{9}$$

Finally, we propose the question of studying inequalities of Ramanujan type concerning other functions of prime numbers.

1. *Ramanujan’s inequality concerning the Chebyshev functions.* Since  $\psi(x) \sim x$  and  $\theta(x) \sim x$  as  $x \rightarrow \infty$ , we may ask about the validity of the following inequalities for  $x$  sufficiently large:

$$\psi(x)^2 < ex\psi\left(\frac{x}{e}\right), \quad \text{and} \quad \theta(x)^2 < ex\theta\left(\frac{x}{e}\right). \tag{10}$$

2. *Ramanujan’s inequality concerning the  $n$ -th prime.* Since  $p_n \sim n \log n$  as  $n \rightarrow \infty$ , an analogue of Ramanujan’s inequality concerning the  $n$ -th prime reads as

$$p_n^2 < en \log n p_{\lfloor \frac{n}{e} \rfloor},$$

or equivalently, by replacing  $n$  by  $en$ , as

$$p_{\lfloor en \rfloor}^2 < e^2 n (1 + \log n) p_n.$$

3. *Ramanujan’s inequality concerning  $\pi(x; q, a)$ .* For given positive integer  $q$  and integer  $a$  with  $\gcd(a, q) = 1$  let  $\pi(x; q, a)$  denote the number of primes  $p$  not exceeding  $x$  with  $p \equiv a \pmod{q}$ . The prime number theorem for arithmetic progressions asserts that  $\pi(x; q, a) \sim \frac{1}{\varphi(q)} \frac{x}{\log x}$ . Hence, analogous to Ramanujan’s inequality concerning the number of primes in arithmetic progressions, we may ask about the validity of the following inequality for  $x$  sufficiently large:

$$\pi(x; q, a)^2 < \varphi(q) \frac{ex}{\log x} \pi\left(\frac{x}{e}; q, a\right).$$

## 5 Proofs

*Proof of Theorem 1.* For a given integer  $n \geq 0$ , multiplying twice (2) gives (3). Also, (2) gives

$$\pi\left(\frac{x}{e}\right) = \frac{x}{e} \sum_{k=0}^n \frac{k!}{(\log x - 1)^{k+1}} + O\left(\frac{x}{\log^{n+2} x}\right).$$

Let  $c_0 = 1$  and  $c_m = \frac{1}{m!} \prod_{i=1}^m (k+i)$  for  $m \geq 1$ . By using the binomial expansion we get  $(1-t)^{-(k+1)} = \sum_{m=0}^n c_m t^m + O(t^{n+1})$  as  $t \rightarrow 0$ . Hence

$$\begin{aligned} \sum_{k=0}^n \frac{k!}{(\log x - 1)^{k+1}} &= \sum_{k=0}^n \frac{k!}{\log^{k+1} x} \left(1 - \frac{1}{\log x}\right)^{-(k+1)} \\ &= \sum_{k=0}^n \frac{k!}{\log^{k+1} x} \sum_{m=0}^n \frac{c_m}{\log^m x} + O\left(\frac{1}{\log^{n+2} x}\right). \end{aligned}$$

Collecting the diagonal terms of the above double sum gives

$$\sum_{k=0}^n \frac{k!}{\log^{k+1} x} \sum_{m=0}^n \frac{c_m}{\log^m x} = \sum_{k=0}^n \frac{r_k}{\log^{k+1} x} + O\left(\frac{1}{\log^{n+2} x}\right),$$

where  $r_k = \sum_{j=0}^k j! c_{k-j}$ . Note that  $c_m = \binom{k+m}{m}$  for  $m \geq 0$ . Thus we obtain (4). □

*Proof of Theorem 2.* For a given  $h$ , by using (2) with  $n = 1$  we obtain

$$\pi(x)^2 = x^2 \left( \frac{1}{\log^2 x} + \frac{2}{\log^3 x} + O\left(\frac{1}{\log^4 x}\right) \right),$$

and

$$\frac{ex}{\log x - h} \pi\left(\frac{x}{e}\right) = x^2 \left( \frac{1}{\log^2 x} + \frac{2+h}{\log^3 x} + O\left(\frac{1}{\log^4 x}\right) \right).$$

Comparing the coefficients of the term  $\frac{1}{\log^3 x}$  gives the result for  $h \neq 0$ . The case  $h = 0$  is Ramanujan's inequality (1). □

*Proof of Theorem 3.* For a given  $\alpha > 0$ , by using (2) with  $n = 1$  we obtain

$$\pi(x)^2 = x^2 \left( \frac{1}{\log^2 x} + \frac{2}{\log^3 x} + O\left(\frac{1}{\log^4 x}\right) \right),$$

and

$$\frac{\alpha x}{\log x} \pi\left(\frac{x}{\alpha}\right) = x^2 \left( \frac{1}{\log^2 x} + \frac{1 + \log \alpha}{\log^3 x} + O\left(\frac{1}{\log^4 x}\right) \right).$$

Comparing the coefficients of the term  $\frac{1}{\log^3 x}$  gives the result for  $\alpha \neq e$ . The case  $\alpha = e$  is Ramanujan's inequality (1). □

*Proof of Theorem 4.* For  $h \neq 1$  we consider the expansion (2) with  $n = 1$ , to get

$$\pi(ex)^2 = e^2 x^2 \left( \frac{1}{\log^2 x} + O\left(\frac{1}{\log^4 x}\right) \right),$$

and

$$\frac{e^2 x}{h + \log x} \pi(x) = e^2 x^2 \left( \frac{1}{\log^2 x} + \frac{-h+1}{\log^3 x} + O\left(\frac{1}{\log^4 x}\right) \right).$$

The case  $h = 1$  has been considered in (8). □

*Proof of Theorem 5.* We use the expansion (2) with  $n = 1$  to obtain

$$\pi(x)^3 = x^3 \left( \frac{1}{\log^3 x} + \frac{3}{\log^4 x} + O\left(\frac{1}{\log^5 x}\right) \right),$$

and similar expansions for the left and right hand sides of (9) with factors 2 and 4 instead of 3 for the term  $\frac{1}{\log^4 x}$ , respectively, hence concluding the proof. □



### 6 A Computational Remark

We conclude the paper with another observation, of computational nature, on Ramanujan’s inequality. If we let

$$\mathcal{R}(x) := \frac{\pi(x)}{x},$$

then (1) is equivalent to

$$\mathcal{R}(x)^2 < \frac{1}{\log x} \mathcal{R}\left(\frac{x}{e}\right). \tag{11}$$

The main point of this equivalent form is the missing term  $x$  in the expansion of both sides. Hence, in numerical investigations, by taking  $z = \log x$  we arrive to equivalent inequalities only in terms of  $z$ , in logarithmic scale and much smaller to work with. To make this point clear, we rework  $x_{\mathcal{R}}$  by using sharp bounds of Trudgian [12] for  $\pi(x)$ , who proved unconditionally

$$|\pi(x) - \text{li}(x)| \leq x \mathcal{T}(x) \quad (\text{for } x \geq 229), \tag{12}$$

with

$$\mathcal{T}(x) = \frac{0.2795}{(\log x)^{\frac{3}{4}}} \exp\left(-\sqrt{\frac{\log x}{6.455}}\right).$$

Also, it is known [9] that for a given  $n \geq 0$ ,

$$|\text{li}(x) - x S_n(n)| \leq 3\sqrt{n}(n+1)! \frac{x}{\log^{n+2} x} \quad (\text{for } x > 1), \tag{13}$$

where

$$S_n(x) = \sum_{k=0}^n \frac{k!}{\log^{k+1} x}.$$

By combining (12) and (13) we get  $L_n(x) \leq \mathcal{R}(x) \leq U_n(x)$  for  $x \geq 229$  with

$$L_n(x) = S_n(x) - \frac{3\sqrt{n}(n+1)!}{\log^{n+2} x} - \mathcal{T}(x),$$

and

$$U_n(x) = S_n(x) + \frac{3\sqrt{n}(n+1)!}{\log^{n+2} x} + \mathcal{T}(x).$$

Thus, (11) (and consequently Ramanujan’s inequality (1)) holds if  $U_n(x)^2 \log x < L_n\left(\frac{x}{e}\right)$  for some integer  $n \geq 0$ , and letting  $z = \log x$ , this is equivalent to  $f_n(z) > 0$ , with

$$f_n(z) = \sum_{k=0}^n \frac{k!}{(z-1)^{k+1}} - \frac{3\sqrt{n}(n+1)!}{(z-1)^{n+2}} - \frac{0.2795}{(z-1)^{\frac{3}{4}}} \exp\left(-\sqrt{\frac{z-1}{6.455}}\right) - z \left( \sum_{k=0}^n \frac{k!}{z^{k+1}} + \frac{3\sqrt{n}(n+1)!}{z^{n+2}} + \frac{0.2795}{z^{\frac{3}{4}}} \exp\left(-\sqrt{\frac{z}{6.455}}\right) \right)^2.$$

Running some careful computations in Maple, taking  $n = 5$ , we observe that  $f_n(z) > 0$  for  $z \geq 9048.857$ . Hence we obtain  $x_{\mathcal{R}} \leq e^{9048.857}$ , unconditionally. Regarding to this computational observation, we mention that Platt and Trudgian [10] proved that  $x_{\mathcal{R}} \leq e^{4041}$ . Also, they proved that Ramanujan's inequality (1) holds unconditionally for every  $x$  satisfying  $38,358,837,683 \leq x \leq e^{56}$ .

A recent conditional bound concerning  $\pi(x)$  is due to Dusart [4], asserting that if the Riemann hypothesis is true then

$$|\pi(x) - \text{li}(x)| \leq x \mathcal{D}(x) \quad (\text{for } x \geq 5639), \tag{14}$$

with

$$\mathcal{D}(x) = \frac{\log x - \log \log x}{8\pi \sqrt{x}}.$$

We combine (13) and (14) to get  $\bar{L}_n(x) \leq \mathcal{R}(x) \leq \bar{U}_n(x)$  for  $x \geq 5639$  under the assumption that the Riemann hypothesis is true, with

$$\bar{L}_n(x) = S_n(x) - \frac{3\sqrt{n}(n+1)!}{\log^{n+2} x} - \mathcal{D}(x),$$

and

$$\bar{U}_n(x) = S_n(x) + \frac{3\sqrt{n}(n+1)!}{\log^{n+2} x} + \mathcal{D}(x).$$

Thus, (11) and consequently (1) hold assuming the Riemann hypothesis is true, if  $\bar{U}_n(x)^2 \log x < \bar{L}_n(\frac{x}{e})$  for some integer  $n \geq 0$ , and letting  $z = \log x$ , this is equivalent to  $g_n(z) > 0$ , with

$$g_n(z) = \sum_{k=0}^n \frac{k!}{(z-1)^{k+1}} - \frac{3\sqrt{n}(n+1)!}{(z-1)^{n+2}} - \frac{z-1-\log(z-1)}{8\pi e^{\frac{z-1}{2}}} - z \left( \sum_{k=0}^n \frac{k!}{z^{k+1}} + \frac{3\sqrt{n}(n+1)!}{z^{n+2}} + \frac{z-\log z}{8\pi e^{\frac{z}{2}}} \right)^2.$$

Computations on Maple with  $n = 12$  show that  $g_n(z) > 0$  for  $z \geq 39.18$ . Hence we obtain  $x_{\mathcal{R}} \leq e^{39.18} \cong 1.037 \times 10^{17}$  under the assumption that the Riemann hypothesis is true. Note that Dudek and Platt [3] obtained  $x_{\mathcal{R}} \leq 1.15 \times 10^{16}$  under the same condition.

### Acknowledgement

I express my gratitude to the anonymous referee(s) for their careful reading of the manuscript and for giving many valuable suggestions and corrections, which improved the presentation of the paper.

### References

[1] C. Axler: Estimates for  $\pi(x)$  for large values of  $x$  and Ramanujan's prime counting inequality. *Integers* 18, Paper No. A61 (2018).  
 [2] B.C. Berndt: *Ramanujan's Notebooks. Part IV*. Springer-Verlag, New York (1994).

- [3] A.W. Dudek, D.J. Platt: On solving a curious inequality of Ramanujan. *Exp. Math.* 24 (3) (2015) 289–294.
- [4] P. Dusart: Estimates of the  $k$ th prime under the Riemann hypothesis. *Ramanujan J.* 47 (1) (2018) 141–154.
- [5] G.H. Hardy: *Ramanujan. Twelve lectures on subjects suggested by his life and work.* Cambridge University Press, Cambridge, England; Macmillan Company, New York (1940).
- [6] M. Hassani: Generalizations of an inequality of Ramanujan concerning prime counting function. *Appl. Math. E-Notes* 13 (2013) 148–154.
- [7] M. Hassani: On an inequality of Ramanujan concerning the prime counting function. *Ramanujan J.* 28 (3) (2012) 435–442.
- [8] M.J. Mossinghoff, T.S. Trudgian: Nonnegative trigonometric polynomials and a zero-free region for the Riemann zeta-function. *J. Number Theory* 157 (2015) 329–349.
- [9] F.W.J. Olver, D.W. Lozier, R.F. Boisvert, Ch.W. Clark: *NIST Handbook of Mathematical Functions.* U.S. Department of Commerce, National Institute of Standards and Technology, Washington, DC; Cambridge University Press, Cambridge (2010).
- [10] D. Platt, T. Trudgian: The error term in the prime number theorem. *arXiv: 1809.03134* (2018). Preprint.
- [11] S. Ramanujan: *Notebooks. Vols. 1, 2.* Tata Institute of Fundamental Research, Bombay (1957).
- [12] T. Trudgian: Updating the error term in the prime number theorem. *Ramanujan J.* 39 (2) (2016) 225–234.

*Received:* 8 September, 2019

*Accepted for publication:* 1 October, 2019

*Communicated by:* Karl Dilcher