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A VARIETY OF EULER'S SUM OF POWERS CONJECTURE

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Abstract. We consider a variety of Euler's sum of powers conjecture, i.e., whether the Diophantine system

$$\begin{cases} n = a_1 + a_2 + \dots + a_{s-1}, \\ a_1 a_2 \dots a_{s-1} (a_1 + a_2 + \dots + a_{s-1}) = b^s \end{cases}$$

has positive integer or rational solutions $n, b, a_i, i = 1, 2, \dots, s-1, s \geq 3$. Using the theory of elliptic curves, we prove that it has no positive integer solution for $s = 3$, but there are infinitely many positive integers n such that it has a positive integer solution for $s \geq 4$. As a corollary, for $s \geq 4$ and any positive integer n , the above Diophantine system has a positive rational solution. Meanwhile, we give conditions such that it has infinitely many positive rational solutions for $s \geq 4$ and a fixed positive integer n .

Keywords: Euler's sum of powers conjecture; elliptic curve; positive integer solution; positive rational solution

MSC 2020: 11D72, 11D41, 11G05

1. INTRODUCTION

In 1769, Euler (see [5], page 209) conjectured that the Diophantine equation

$$(1.1) \quad a_1^s + a_2^s + \dots + a_{s-1}^s = a_s^s$$

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has no positive integer solution (a_1, a_2, \dots, a_s) for $s \geq 3$. It is called *Euler's sum of powers conjecture*.

For $s = 3$, equation (1.1) corresponds to the case $n = 3$ of Fermat's Last Theorem: $x^n + y^n = z^n$, $n \geq 3$.

For $s = 4$, in 1988, Elkies in [4] disproved Euler's sum of powers conjecture by showing that $a_1^4 + a_2^4 + a_3^4 = a_4^4$ has infinitely many positive integer solutions. In particular, he gave the solution

$$2682440^4 + 15365639^4 + 18796760^4 = 20615673^4.$$

Shortly after, Frye (see [5], page 210) got the smallest counterexample

$$95800^4 + 217519^4 + 414560^4 = 422481^4.$$

For $s = 5$, in 1966, Lander and Parkin in [6] found the first counterexample

$$27^5 + 84^5 + 110^5 + 133^5 = 144^5.$$

In 2004, Frye (see [9]) obtained the only other known primitive solution for $s = 5$:

$$55^5 + 3183^5 + 28969^5 + 85282^5 = 85359^5.$$

For $s \geq 6$, there are no known solutions. More information about Euler's sum of powers conjecture can be found in [5], pages 209–218: *D1 Sums of like powers. Euler's conjecture*.

In 2013, Cai and Chen in [1] investigated the question to express a positive integer n as a sum of s positive integers whose product is a k th power, i.e.,

$$n = a_1 + a_2 + \dots + a_s$$

such that

$$a_1 a_2 \dots a_s = b^k$$

for positive integers n , a_i , b , k and $k \geq 2$. This can be considered a new variant of the Hilbert-Waring problem

$$n = a_1^k + a_2^k + \dots + a_s^k.$$

In 2015, Cai, Chen and Zhang in [2] studied the Diophantine system

$$A + B = C, \quad ABC = D^n,$$

where A, B, C, D, n are positive integers and $n \geq 3$, which is a new generalization of Fermat's Last Theorem.

Now we expand this idea to Euler's sum of powers conjecture and consider whether the Diophantine system

$$(1.2) \quad \begin{cases} n = a_1 + a_2 + \dots + a_{s-1}, \\ a_1 a_2 \dots a_{s-1} (a_1 + a_2 + \dots + a_{s-1}) = b^s \end{cases}$$

has positive integer or rational solutions $n, a_i, i = 1, 2, \dots, s-1, b$, and $s \geq 3$.

Obviously, the solutions of (1.1) are a subset of the solutions of (1.2). If $a_i, i = 1, 2, \dots, s-1$, and $a_1 + a_2 + \dots + a_{s-1}$ are relatively prime in pairs, (1.2) reduces to Euler's sum of powers conjecture. The motivation for studying (1.2) is that we try to find a counterexample to Euler's sum of powers conjecture for $s = 6$. Although we cannot obtain any counterexamples, we find some interesting results by the theory of elliptic curves.

For positive integer solutions of (1.2), we have:

Theorem 1.1. *For $s = 3$ and any positive integer n , (1.2) has no positive integer solution (a_1, a_2, b) .*

This is a special case studied in [1], Theorem 2, but we give a proof in a different way.

Theorem 1.2.

- (1) *For $s = 4$, there are infinitely many positive integers n such that (1.2) has a positive integer solution (a_1, a_2, a_3, b) .*
- (2) *For $s \geq 5$, there are infinitely many positive integers n such that (1.2) has a positive integer solution $(a_1, a_2, \dots, a_{s-1}, b)$, which depends on $s - 3$ positive rational parameters.*

For positive rational solutions, let us note that if (1.2) has a positive rational solution for some positive integer n , then it has a positive rational solution for each positive integer N , by the following transformation

$$\begin{cases} N = \frac{a_1 N}{n} + \frac{a_2 N}{n} + \dots + \frac{a_{s-1} N}{n}, \\ \frac{a_1 N}{n} \cdot \frac{a_2 N}{n} \cdot \dots \cdot \frac{a_{s-1} N}{n} \cdot \left(\frac{a_1 N}{n} + \frac{a_2 N}{n} + \dots + \frac{a_{s-1} N}{n} \right) = \left(\frac{bN}{n} \right)^s. \end{cases}$$

Therefore, from Theorem 1.2, we obtain:

Corollary 1.1. *For $s \geq 4$ and any positive integer n , (1.2) has a positive rational solution.*

Furthermore, we can ask the question: for which positive integer n does (1.2) have infinitely many positive rational solutions? We give a partial answer to it under some conditions in the following theorem.

Theorem 1.3. *For $s \geq 4$ and a fixed positive integer n , if (1.2) has a positive rational solution $(a'_1, a'_2, \dots, a'_{s-1}, b')$ and the elliptic curve*

$$\mathcal{E}_3: Y^2 = X^3 - 27u'^3v'(u'v'^3 - 24)X + 54u'^4(u'^2v'^6 - 36u'v'^3 + 216)$$

has positive rank, then (1.2) has infinitely many positive rational solutions.

2. PROOFS OF THE THEOREMS

P r o o f of Theorem 1.1. For $s = 3$, (1.2) reduces to

$$\begin{cases} \frac{n}{b} = b_1 + b_2, \\ b_1 b_2 (b_1 + b_2) = 1, \end{cases}$$

where

$$b_i = \frac{a_i}{b} \in \mathbb{Q}^+, \quad i = 1, 2.$$

Let us consider the cubic curve

$$b_1 b_2 (b_1 + b_2) = 1,$$

which equals

$$\left(\frac{b_1}{b_2}\right)^2 + \frac{b_1}{b_2} = \frac{1}{b_2^2}.$$

Let

$$u = \frac{b_1}{b_2}, \quad v = \frac{1}{b_2},$$

we have

$$u^2 + u = v^3.$$

Taking $y = 16u + 8$, $x = 4v$, we get the elliptic curve

$$y^2 = x^3 + 64.$$

Using the Magma package (see [7]), the rank of it is 0, and the only rational points are

$$(x, \pm y) = (8, 24), (0, 8), (-4, 0).$$

Tracing back, there is no positive integer solution of (1.2). □

Proof of Theorem 1.2. (1) For $s = 4$, (1.2) leads to

$$\begin{cases} \frac{n}{b} = b_1 + b_2 + b_3, \\ b_1 b_2 b_3 (b_1 + b_2 + b_3) = 1, \end{cases}$$

where

$$b_i = \frac{a_i}{b} \in \mathbb{Q}^+, \quad i = 1, 2, 3.$$

It's easy to see that $(a_1, a_2, a_3) = (1, 2, 24)$ satisfies (1.2), then

$$(b_1, b_2, b_3) = \left(\frac{1}{6}, \frac{1}{3}, 4\right).$$

Let us consider the positive rational solutions of the Diophantine system

$$\begin{cases} b_1 b_2 b_3 = \frac{2}{9}, \\ b_1 + b_2 + b_3 = \frac{9}{2}. \end{cases}$$

Eliminating b_3 , we get

$$18b_1^2 b_2 + 18b_1 b_2^2 - 81b_1 b_2 + 4 = 0,$$

then

$$18\frac{b_2}{b_1} + 18\left(\frac{b_2}{b_1}\right)^2 - 81\frac{b_2}{b_1}\frac{1}{b_1} + 4\left(\frac{1}{b_1}\right)^3 = 0.$$

Taking

$$u = \frac{b_2}{b_1}, \quad v = \frac{1}{b_1},$$

we have the curve

$$\mathcal{C}_1: 18u^2 + 18u - 81uv + 4v^3 = 0.$$

By the map $\varphi_1: \mathcal{C}_1 \mapsto \mathcal{E}_1$,

$$y = 384u - 864v + 192, \quad x = -32v + 243$$

and we obtain the elliptic curve

$$\mathcal{E}_1: y^2 = x^3 - 166779x + 26215254.$$

Using the Magma package (see [7]), the rank of \mathcal{E}_1 is 1 and then there are infinitely many rational points on it.

From the above transformations, we have

$$\begin{cases} b_1 = \frac{32}{243-x}, \\ b_2 = \frac{y-27x+6369}{12(243-x)}, \\ b_3 = \frac{-y-27x+6369}{12(243-x)}. \end{cases}$$

Then

$$\begin{cases} a_1 = \frac{32}{243-x}b, \\ a_2 = \frac{y-27x+6369}{12(243-x)}b, \\ a_3 = \frac{-y-27x+6369}{12(243-x)}b \end{cases}$$

is the solution of (1.2).

In view of $b_i > 0$, we get the condition

$$x < \frac{2123}{9}, \quad |y| < -27x + 6369.$$

It is easy to see that the point $P = (235, 8)$ satisfies this condition. By the theorem of Poincaré and Hurwitz (see [10], Chapter V, page 78, Satz 11) about the density of rational points, then there are infinitely many rational points on \mathcal{E}_1 satisfying

$$x < \frac{2123}{9}, \quad |y| < -27x + 6369.$$

Therefore, we can find infinitely many positive rational solutions b_i , $i = 1, 2, 3$, which leads to infinitely many positive integers a_i , $i = 1, 2, 3$, by multiplying the least common denominator of b_i . This proves that for $s = 4$ there exist infinitely many positive integers n such that (1.2) has a positive integer solution.

(2) For $s \geq 5$, (1.2) reduces to

$$\begin{cases} \frac{n}{b} = b_1 + b_2 + \dots + b_{s-1}, \\ b_1 b_2 \dots b_{s-1} (b_1 + b_2 + \dots + b_{s-1}) = 1, \end{cases}$$

where

$$b_i = \frac{a_i}{b} \in \mathbb{Q}^+, \quad i = 1, \dots, s-1.$$

Take

$$x = b_1, \quad y = b_2, \quad z = b_3, \quad u = b_4 \dots b_{s-1}, \quad v = b_4 + \dots + b_{s-1},$$

then

$$\begin{cases} \frac{n}{b} = x + y + z + v, \\ xyz u(x + y + z + v) = 1. \end{cases}$$

We need to study the rational points on the cubic curve

$$xyz u(x + y + z + v) = 1.$$

Let $z = ut^2y$, then we get

$$(2.1) \quad t^2u^2y^2x^2 + t^2u^2y^2(t^2uy + v + y)x - 1 = 0.$$

Consider the above equation as a quadratic equation in x . If it has rational solutions, then the discriminant

$$\Delta(y) = u^2t^2y^2(u^2t^2(ut^2 + 1)^2y^4 + 2vu^2t^2(ut^2 + 1)y^3 + u^2v^2t^2y^2 + 4)$$

should be a perfect square. It follows from the study of the rational parametric solutions in $\mathbb{Q}(t)$ on the quartic curve

$$\mathcal{C}_2: w^2 = u^2t^2(ut^2 + 1)^2y^4 + 2vu^2t^2(ut^2 + 1)y^3 + u^2v^2t^2y^2 + 4.$$

The discriminant of \mathcal{C}_2 is

$$\Delta(t) = 256u^6t^6(ut^2 + 1)^4(64u^2t^4 + u(uv^4 + 128)t^2 + 64)$$

and is nonzero as an element of $\mathbb{Q}(t)$ since $u, v \in \mathbb{Q}^+$. Then \mathcal{C}_2 is smooth.

By the method described in [8], page 77 (or [3], page 476, Proposition 7.2.1), we can transform \mathcal{C}_2 into the family of elliptic curves

$$\mathcal{E}_2: Y^2 = X(X^2 + u^2v^2t^2X - 16u^2(ut^2 + 1)^2t^2)$$

by the inverse birational map $\varphi_2: \mathcal{C}_2 \mapsto \mathcal{E}_2$ with

$$y = \frac{Y - uvtX}{ut(ut^2 + 1)X}, \quad w = \frac{Y^2 - u^2v^2t^2X^2 - 2X^3}{4ut(ut^2 + 1)X^2}$$

and

$$\begin{aligned} X &= 2ut(ut^2 + 1)(t^3u^2y^2 + tvy + tuy^2 - w), \\ Y &= 2u^2t^2(ut^2 + 1)(2t^2uy + v + 2y)(t^3u^2y^2 + tvy + tuy^2 - w). \end{aligned}$$

In view of $u, v \in \mathbb{Q}^+$, we can take

$$u = \frac{p}{q}, \quad v = \frac{c}{d},$$

where p, q, c, d are positive integers. Let

$$U = q^4 d^2 X, \quad V = q^6 d^3 Y,$$

then \mathcal{E}_2 reduces to

$$\mathcal{E}'_2: V^2 = U(U^2 + c^2 p^2 q^2 t^2 U - 16d^4 p^2 q^4 t^2 (pt^2 + q)^2).$$

Note that the point

$$P = (4pd^2 q^2 (pt^2 + q)t, 4cd^2 p^2 q^3 (pt^2 + q)t^2)$$

lies on \mathcal{E}'_2 . Using the group law on the elliptic curve, we obtain the point

$$2P = \left(\frac{16q^2 d^4 (pt^2 + q)^2}{c^2}, -\frac{64q^3 d^6 (pt^2 + q)^3}{c^3} \right).$$

Next, we determine the positive integer solutions of (1.2). From the birational map φ_2 and the point $-2P$, i.e., the reflected point of $2P$, we get

$$x = \frac{uv^3 t}{2(4ut^2 - uv^2 t + 4)}, \quad y = \frac{4ut^2 - uv^2 t + 4}{2uvt(ut^2 + 1)}, \quad z = \frac{(4ut^2 - uv^2 t + 4)t}{2v(ut^2 + 1)}.$$

To get $x > 0, y > 0, z > 0$ from $u > 0, v > 0$, we need

$$4ut^2 - uv^2 t + 4 > 0,$$

the discriminant of it is $\delta = u(uv^4 - 64)$. If $\delta < 0$, then for any $t \in \mathbb{Q}$, we have

$$4ut^2 - uv^2 t + 4 > 0.$$

If $\delta > 0$, then for any

$$t \in \left(0, \frac{uv^2 - \sqrt{u(uv^4 - 64)}}{8u} \right) \cup \left(\frac{uv^2 + \sqrt{u(uv^4 - 64)}}{8u}, \infty \right),$$

we have

$$4ut^2 - uv^2 t + 4 > 0.$$

Hence, by the density of rational numbers, for any $u > 0, v > 0$, there are infinitely many positive rational numbers t such that $x > 0, y > 0, z > 0$. Then for any given positive rational numbers b_4, \dots, b_{s-1} , there are infinitely many positive rational numbers t such that $b_i > 0, i = 1, 2, 3$. Multiplying the least common multiple of the denominators of $t, b_i, i = 1, \dots, s-1$, we can get $a_i \in \mathbb{Z}^+, i = 1, \dots, s-1$.

Therefore, for $s \geq 5$ there are infinitely many positive integers n such that (1.2) has a positive integer solution $(a_1, a_2, \dots, a_{s-1}, b)$, which depends on $s-3$ positive rational parameters $t, b_i, i = 4, \dots, s-1$. \square

Example 2.1.

(1) For $s = 4$, the points

$$(x, y) = (235, 8), \left(\frac{60266587}{257049}, \frac{3852230624}{130323843} \right)$$

on \mathcal{E}_1 lead to

$$(a_1, a_2, a_3) = (1, 2, 24), (781943058, 138991832, 18609625).$$

Then

$$\begin{cases} 27 = 1 + 2 + 24, \\ 1 \cdot 2 \cdot 24 \cdot (1 + 2 + 24) = 6^4 \end{cases}$$

and

$$\begin{cases} 939544515 = 781943058 + 138991832 + 18609625, \\ 781943058 \cdot 138991832 \cdot 18609625 \\ \quad \times (781943058 + 138991832 + 18609625) = 208787670^4. \end{cases}$$

(2) For $s = 5$, we have

$$x = \frac{uw^3t}{2(4ut^2 - uv^2t + 4)}, y = \frac{4ut^2 - uv^2t + 4}{2uvt(ut^2 + 1)}, z = \frac{(4ut^2 - uv^2t + 4)t}{2v(ut^2 + 1)}, u = v = b_4.$$

Take

$$b = 2uvt(ut^2 + 1)(4ut^2 - uv^2t + 4),$$

then

$$\begin{aligned} a_1 &= t^2 u^6 (ut^2 + 1), & a_2 &= (4ut^2 - tu^3 + 4)^2, \\ a_3 &= ut^2 (4ut^2 - tu^3 + 4)^2, & a_4 &= 2tu^3 (ut^2 + 1)(4ut^2 - tu^3 + 4). \end{aligned}$$

If $t = 1, u = 1$, we have

$$\begin{cases} 128 = 2 + 49 + 49 + 28, \\ 2 \cdot 49 \cdot 49 \cdot 28 \cdot (2 + 49 + 49 + 28) = 28^5. \end{cases}$$

When $t = 2, u = 1$, we obtain

$$\begin{cases} 2000 = 20 + 324 + 1296 + 360, \\ 20 \cdot 324 \cdot 1296 \cdot 360 \cdot (20 + 324 + 1296 + 360) = 360^5, \end{cases}$$

which reduces to

$$\begin{cases} 500 = 5 + 81 + 324 + 90, \\ 5 \cdot 81 \cdot 324 \cdot 90 \cdot (5 + 81 + 324 + 90) = 90^5. \end{cases}$$

Proof of Theorem 1.3. For $s \geq 4$, (1.2) is equivalent to

$$\begin{cases} \frac{n}{b} = b_1 + b_2 + \dots + b_{s-1}, \\ b_1 b_2 \dots b_{s-1} (b_1 + b_2 + \dots + b_{s-1}) = 1, \end{cases}$$

where

$$b_i = \frac{a_i}{b} \in \mathbb{Q}^+, \quad i = 1, \dots, s-1.$$

Put

$$x = b_1, \quad y = b_2, \quad u = b_3 \dots b_{s-1}, \quad v = b_3 + \dots + b_{s-1},$$

then

$$\begin{cases} \frac{n}{b} = x + y + v, \\ xyu(x + y + v) = 1. \end{cases}$$

For $s \geq 4$ and a fixed positive integer n , if (1.2) has a positive rational solution $(a'_1, a'_2, \dots, a'_{s-1}, b')$, then we have

$$u' = b'_3 \dots b'_{s-1}, \quad v' = b'_3 + \dots + b'_{s-1},$$

where

$$b'_i = \frac{a'_i}{b'} \in \mathbb{Q}^+, \quad i = 3, \dots, s-1.$$

Now we need to study the rational points on the cubic curve

$$\mathcal{C}_3: xyu'(x + y + v') = 1,$$

which can be transformed into the elliptic curve

$$\mathcal{E}_3: Y^2 = X^3 - 27u'^3v'(u'v'^3 - 24)X + 54u'^4(u'^2v'^6 - 36u'v'^3 + 216).$$

The map $\varphi_3: \mathcal{C}_3 \mapsto \mathcal{E}_3$ is given by

$$X = \frac{3u'(u'v'^2x + 12)}{x}, \quad Y = -\frac{108u'^2(x + 2y + v')}{x}$$

and its inverse map φ_3^{-1} is

$$x = \frac{36u'}{X - 3u'^2v'^2}, \quad y = -\frac{Y + 3u'(v'X - 3u'^2v'^3 + 36u')}{6u'(X - 3u'^2v'^2)}.$$

To get positive rational solutions x and y from the inverse map φ_3^{-1} , we have the condition

$$X > 3u'^2v'^2, \quad Y + 3u'(v'X - 3u'^2v'^3 + 36u') < 0.$$

From the first assumption of Theorem 1.3, \mathcal{C}_3 has a rational point

$$Q = (x, y) = \left(\frac{a'_1}{b'}, \frac{a'_2}{b'} \right),$$

so the rational point

$$R = (X, Y) = \varphi_3(Q) = \left(\frac{3u'(a'_1u'v'^2 + 12b')}{a'_1}, -\frac{108u'^2(b'v' + a'_1 + 2a'_2)}{a'_1} \right)$$

on \mathcal{E}_3 satisfies the above condition. If \mathcal{E}_3 has positive rank, by the theorem of Poincaré and Hurwitz (see [10], Chapter V, page 78, Satz 11) about the density of rational points, \mathcal{E}_3 has infinitely many rational points satisfying

$$X > 3u'^2v'^2, \quad Y + 3u'(v'X - 3u'^2v'^3 + 36u') < 0.$$

Therefore, there are infinitely many positive rational solutions (x, y) satisfying

$$\begin{cases} \frac{n}{b'} = x + y + v', \\ xyu'(x + y + v') = 1, \end{cases}$$

i.e.,

$$\begin{cases} n = xb' + yb' + a'_3 + \dots + a'_{s-1}, \\ (xb')(yb')(a'_3) \dots (a'_{s-1})(xb' + yb' + a'_3 + \dots + a'_{s-1}) = b'^s. \end{cases}$$

Then for $s \geq 4$ and a fixed positive integer n , (1.2) has infinitely many positive rational solutions $(a_1, a_2, a_3, \dots, a_{s-1}, b) = (xb', yb', a'_3, \dots, a'_{s-1}, b')$. \square

Example 2.2. For $s = 4$ and $n = 27$, (1.2) has a positive integer solution

$$\begin{cases} 27 = 1 + 2 + 24, \\ 1 \cdot 2 \cdot 24 \cdot (1 + 2 + 24) = 6^4. \end{cases}$$

Then

$$u' = v' = 4$$

and \mathcal{E}_3 reduces to

$$\mathcal{E}_3(4, 4): Y^2 = X^3 - 1603584X + 781553664.$$

Using the Magma package (see [7]), the rank of $\mathcal{E}_3(4, 4)$ is 1, then there are infinitely many rational points on it. From the positive rational solution

$$(x, y) = \left(\frac{1}{6}, \frac{1}{3} \right)$$

on $\mathcal{C}_3(4, 4)$: $4xy(x + y + 4) = 1$, we can get the point

$$(X, Y) = (1632, -50112)$$

on $\mathcal{E}_3(4, 4)$, which satisfies the condition

$$X > 768, \quad Y < -48X + 35136.$$

By the theorem of Poincaré and Hurwitz, there are infinitely many rational points satisfying this condition. Thus, for $s = 4$ and $n = 27$, (1.2) has infinitely many positive rational solutions $(a_1, a_2, a_3, b) = (6x, 6y, 24, 6)$. From the points

$$(X, Y) = (26752, -4370752), \left(\frac{1595044704}{616225}, -\frac{57181013356608}{483736625} \right), \\ \left(\frac{1119732063118503868848}{216521728021016209}, -\frac{36437253415550579162358351801216}{100751663784490024367909177} \right)$$

on $\mathcal{E}_3(4, 4)$, we obtain the positive rational solutions

$$(x, y) = \left(\frac{9}{1624}, \frac{841}{168} \right), \left(\frac{616225}{7790166}, \frac{2550409}{3829230} \right), \\ \left(\frac{216521728021016209}{662113455544190419}, \frac{4217686043540528329}{3000378379947935118} \right).$$

Remark 2.1. For $s = 4$ and $n = 1$, from the smallest counterexample of Euler's sum of powers conjecture found by Frye, see [5], page 210:

$$95800^4 + 217519^4 + 414560^4 = 422481^4,$$

we see that (1.2) has a positive rational solution

$$(a_1, a_2, a_3, b) = \left(\frac{95800^4}{422481^4}, \frac{217519^4}{422481^4}, \frac{414560^4}{422481^4}, \frac{95800 \cdot 217519 \cdot 414560}{422481^3} \right),$$

i.e.,

$$\begin{cases} 1 = \frac{95800^4}{422481^4} + \frac{217519^4}{422481^4} + \frac{414560^4}{422481^4}, \\ \frac{95800^4}{422481^4} \cdot \frac{217519^4}{422481^4} \cdot \frac{414560^4}{422481^4} \cdot 1 = \left(\frac{95800 \cdot 217519 \cdot 414560}{422481^3} \right)^4. \end{cases}$$

Then

$$u' = v' = \frac{414560^3}{95800 \cdot 217519 \cdot 422481}.$$

By the same method, we can show that for $s = 4$ and $n = 1$, (1.2) has infinitely many positive rational solutions

$$(a_1, a_2, a_3, b) = \left(\frac{95800 \cdot 217519 \cdot 414560}{422481^3} x, \frac{95800 \cdot 217519 \cdot 414560}{422481^3} y, \frac{95800^4}{422481^4}, \frac{95800 \cdot 217519 \cdot 414560}{422481^3} \right).$$

Because the solutions are huge, we just list an example:

$$(x, y) = \left(\frac{6388801718463172692616726574041101080486537011865759800000}{16386283355822543519946496843630803411504474368317131012609}, \frac{350016124522332823673623853541028266838547099813138623879281}{9409075428286934037926649233826637879987515730518054325184000} \right).$$

3. SOME RELATED QUESTIONS

From

$$n = a_1 + a_2 + \dots + a_{s-1}, \quad a_i \in \mathbb{Z}^+,$$

we can see that (1.2) has finitely many or no positive integer solutions for $s \geq 4$ and a fixed positive integer n . So we can ask the following question.

Question 3.1. For $s \geq 4$, which is the least positive integer n such that (1.2) has positive integer solutions (a_1, \dots, a_{s-1}, b) ?

When $s = 4, 5, 6$, it is easy to get the least positive integers $n = 18, 27, 8$, where

$$\begin{cases} 18 = 1 + 8 + 9, \\ 1 \cdot 8 \cdot 9 \cdot (1 + 8 + 9) = 6^4, \end{cases} \quad \begin{cases} 27 = 1 + 2 + 12 + 12, \\ 1 \cdot 2 \cdot 12 \cdot 12 \cdot (1 + 2 + 12 + 12) = 6^5, \end{cases}$$

and

$$\begin{cases} 8 = 1 + 1 + 2 + 2 + 2, \\ 1 \cdot 1 \cdot 2 \cdot 2 \cdot 2 \cdot (1 + 1 + 2 + 2 + 2) = 2^6. \end{cases}$$

For $s \geq 4$, let $N(n, s)$ denote the number of the positive integer solutions of (1.2), we raise the following question.

Question 3.2. For $s \geq 4$ and a fixed positive integer $n \geq 1$, can we give the formula for $N(n, s)$?

In Theorem 1.3, we give conditions such that (1.2) has infinitely many positive rational solutions for $s \geq 4$ and a fixed positive integer n . However, the conditions are not easy to check. Meanwhile, we have the following question.

Question 3.3. For positive rational numbers u and v , are there infinitely many positive rational solutions b_3, \dots, b_{s-1} of the Diophantine system

$$\begin{cases} u = b_3 \dots b_{s-1}, \\ v = b_3 + \dots + b_{s-1} \end{cases} \quad \text{for } s \geq 5?$$

When $s = 5$, we have

$$\begin{cases} u = b_3 b_4, \\ v = b_3 + b_4. \end{cases}$$

It is easy to get the condition, which is

$$u = \frac{v^2 - w^2}{4}, \quad v > w,$$

then

$$b_3 = \frac{v+w}{2}, \quad b_4 = \frac{v-w}{2}.$$





In 2014, Ulas in [11] proved that for each $k \geq 4$ and rational numbers A, B with $AB \neq 0$, the Diophantine system

$$\begin{cases} A = x_1 + x_2 + \dots + x_k, \\ B = x_1 x_2 \dots x_k \end{cases}$$

has infinitely many solutions depending on $k - 3$ free parameters. In his proof, $x_3 = -4Bt^2x_1$, so if $B > 0$, then x_1 and x_3 cannot be positive simultaneously. Therefore, we cannot get a positive answer to Question 3.3 by Ulas' result.

References

- [1] *T. Cai, D. Chen*: A new variant of the Hilbert-Waring problem. *Math. Comput.* 82 (2013), 2333–2341. [zbl](#) [MR](#) [doi](#)
- [2] *T. Cai, D. Chen, Y. Zhang*: A new generalization of Fermat's last theorem. *J. Number Theory* 149 (2015), 33–45. [zbl](#) [MR](#) [doi](#)
- [3] *H. Cohen*: Number Theory. Vol. I: Tools and Diophantine Equations. Graduate Texts in Mathematics 239. Springer, New York, 2007. [zbl](#) [MR](#) [doi](#)
- [4] *N. D. Elkies*: On $A^4 + B^4 + C^4 = D^4$. *Math. Comput.* 184 (1988), 825–835. [zbl](#) [MR](#) [doi](#)
- [5] *R. K. Guy*: Unsolved Problems in Number Theory. Problem Books in Mathematics. Springer, New York, 2004. [zbl](#) [MR](#) [doi](#)
- [6] *L. J. Lander, T. R. Parkin*: Counterexample to Euler's conjecture on sums of like powers. *Bull. Am. Math. Soc.* 72 (1966), 1079. [zbl](#) [MR](#) [doi](#)
- [7] Magma computational algebra system for algebra, number theory and geometry. Available at <http://magma.maths.usyd.edu.au/magma/>. [sw](#)
- [8] *L. J. Mordell*: Diophantine Equations. Pure and Applied Mathematics 30. Academic Press, London, 1969. [zbl](#) [MR](#)

- [9] *E. S. Rowland*: Elliptic curve and integral solutions to $A^4 + B^4 + C^4 = D^4$. Available at https://ericrowland.github.io/papers/Elliptic_curves_and_integral_solutions_to_A^4+B^4+C^4=D^4.pdf (2004), 7 pages.
- [10] *T. Skolem*: Diophantische Gleichungen. Ergebnisse der Mathematik und ihrer Grenzgebiete 5. Springer, Berlin, 1938. (In German.) 
- [11] *M. Ulas*: On some Diophantine systems involving symmetric polynomials. *Math. Comput.* 83 (2014), 1915–1930.   

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