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FINITE AND INFINITE ORDER OF GROWTH OF SOLUTIONS TO LINEAR DIFFERENTIAL EQUATIONS NEAR A SINGULAR POINT

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Abstract. In this paper, we investigate the growth of solutions of a certain class of linear differential equation where the coefficients are analytic functions in the closed complex plane except at a finite singular point. For that, we will use the value distribution theory of meromorphic functions developed by Rolf Nevanlinna with adapted definitions.

Keywords: linear differential equation; growth of solution; finite singular point

MSC 2020: 34M10, 30D35

1. Introduction and statement of results

Throughout this paper, we assume that the reader is familiar with the fundamental results and the standard notations of the Nevanlinna value distribution theory of a meromorphic function on the complex plane $\mathbb{C}$ and in the unit disc $D=\{z\in\mathbb{C}\colon\ |z|<1\}$ (see [7], [12], [17]). The importance of this theory has inspired many authors to find modifications and generalizations to different domains. Extensions of Nevanlinna theory to annuli have been made by [1], [8], [10], [11], [14]. In [4], Hamouda studied the growth of solutions of linear differential equations with analytic coefficients in the unit disc based on the behavior of the coefficients on a neighborhood of a point on the boundary of the unit disc. Recently in [2], [6], Fettouch and Hamouda investigated the growth of solutions of certain linear differential equations near a finite singular point. In this paper, we continue this investigation near a finite singular point to study other types of linear differential equations. First, we recall

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the appropriate definitions. Set $\overline{\mathbb{C}} = \mathbb{C} \cup \{\infty\}$ and suppose that $f(z)$ is meromorphic in $\overline{\mathbb{C}} \setminus \{z_0\}$ where $z_0 \in \mathbb{C}$. Define the counting function near z_0 by

$$(1.1)\qquad N_{z_0}(r, f) = -\int_{\infty}^{r} \frac{n(t, f) - n(\infty, f)}{t}\,\mathrm{d}t - n(\infty, f)\log r,$$

where $n(t, f)$ counts the number of poles of $f(z)$ in the region

$$\{z \in \mathbb{C}\colon\ t \leqslant |z - z_0|\} \cup \{\infty\}$$

each pole according to its multiplicity; and the proximity function by

$$(1.2)\qquad m_{z_0}(r, f) = \frac{1}{2\pi}\int_0^{2\pi} \ln^+ |f(z_0 - r\mathrm{e}^{\mathrm{i}\varphi})|\,\mathrm{d}\varphi.$$

The characteristic function of f is defined in the usual manner by

$$(1.3)\qquad T_{z_0}(r, f) = m_{z_0}(r, f) + N_{z_0}(r, f).$$

In addition, the order of the meromorphic function $f(z)$ near z_0 is defined by

$$(1.4)\qquad \sigma_T(f, z_0) = \limsup_{r\to 0} \frac{\log^+ T_{z_0}(r, f)}{-\log r}.$$

For an analytic function $f(z)$ in $\overline{\mathbb{C}} \setminus \{z_0\}$, we have also the definition

$$(1.5)\qquad \sigma_M(f, z_0) = \limsup_{r\to 0} \frac{\log^+ \log^+ M_{z_0}(r, f)}{-\log r},$$

where $M_{z_0}(r, f) = \max\{|f(z)|\colon\ |z - z_0| = r\}$.

By the usual manner of the definition of the iterated order of a meromorphic function in the complex plane (see [9]), we define the n-iterated order near z_0 as follows:

$$(1.6)\qquad \sigma_{n,T}(f, z_0) = \limsup_{r\to 0} \frac{\log_n^+ T_{z_0}(r, f)}{-\log r},$$

and for an analytic function $f(z)$ in $\overline{\mathbb{C}} \setminus \{z_0\}$, we have also the definition

$$(1.7)\qquad \sigma_{n,M}(f, z_0) = \limsup_{r\to 0} \frac{\log_{n+1}^+ M_{z_0}(r, f)}{-\log r},$$

where $\log_{n+1}^+(x) = \ln^+ \log_n^+(x)$ ($n \geqslant 1$ is an integer) and $\ln^+(x) = \max(\ln x, 0)$.

Remark 1.1. It is shown in [2] that if f is a non constant meromorphic function in $\overline{\mathbb{C}} - \{z_0\}$ and $g(w) = f(z_0 - 1/w)$, then $g(w)$ is meromorphic in $\mathbb{C}$ and we have

$$T(R, g) = T_{z_0}\Big(\frac{1}{R}, f\Big);$$

and so $\sigma(f, z_0) = \sigma(g)$. Also, if $f(z)$ is analytic in $\overline{\mathbb{C}} \setminus \{z_0\}$, then, $g(w)$ is entire and thus $\sigma_T(f, z_0) = \sigma_M(f, z_0)$ and in general $\sigma_{n,T}(f, z_0) = \sigma_{n,M}(f, z_0) n \geqslant 1$. So, we can use the notation $\sigma_n(f, z_0)$ without any ambiguity.

We recall the following definitions.

Definition 1.1. The linear measure of a set $E \subset (0, \infty)$ is defined as $\int_0^\infty \chi_E(t)\,\mathrm{d}t$ and the logarithmic measure of E is defined by $\int_0^\infty \chi_E(t)t^{-1}\,\mathrm{d}t$ where $\chi_E(t)$ is the characteristic function of the set E.

In 2016, Fettouch and Hamouda proved the following result.

Theorem A ([2]). *Let $A_0(z) \not\equiv 0$, $A_1(z), \ldots, A_{k-1}(z)$ be analytic functions in $\overline{\mathbb{C}} \setminus \{z_0\}$ satisfying $\max\{\sigma(A_j, z_0)\colon\ j \neq 0\} < \sigma(A_0, z_0)$. Then, every solution $f(z) \not\equiv 0$ of the differential equation*

$$f^{(k)} + A_{k-1}(z)f^{(k-1)} + \ldots + A_1(z)f' + A_0(z)f = 0$$

satisfies $\sigma(f, z_0) = \infty$ with $\sigma_2(f, z_0) = \sigma(A_0, z_0)$.

In the following two results, we will base our study on the domination of A_0 on only a curve tending to z_0. In this case, it may hapen that

$$\sigma(A_0, z_0) \leqslant \max\{\sigma(A_j, z_0)\colon\ j \neq 0\}.$$

Theorem 1.1. *Let $A_0(z) \not\equiv 0$, $A_1(z), \ldots, A_{k-1}(z)$ be analytic functions in $\overline{\mathbb{C}} \setminus \{z_0\}$. If there exists a subset γ of a curve tending to z_0 such that the set $\gamma_0 = \{|z_0 - z|\colon\ z \in \gamma\} \cap (0, 1)$ is of infinite logarithmic measure, such that for $z \in \gamma$, $r = |z_0 - z| \in \gamma_0$ and for any fixed $\mu > 0$, we have*

$$\lim_{r\to 0} \frac{1}{|A_0(z)|r^{\mu}} \Bigg(\sum_{j=1}^{k-1} |A_j(z)| + 1\Bigg) = 0, \tag{1.8}$$

then every solution $f(z) \not\equiv 0$ of the differential equation

$$f^{(k)} + A_{k-1}(z)f^{(k-1)} + \ldots + A_1(z)f + A_0(z)f = 0, \tag{1.9}$$

that is analytic in $\overline{\mathbb{C}} \setminus \{z_0\}$ is of infinite order.

Corollary 1.1. *Let $P_j(z)$, $j=1,2,\ldots,k-1$ be polynomials and $P_0(z)$ be a transcendental entire function; let $A_j(z)=P_j(1/(z_0-z))$; then every solution $f(z)\not\equiv 0$ of* (1.9)*, that is analytic in $\overline{\mathbb{C}}\setminus\{z_0\}$, is of infinite order.*

Example 1.1. The differential equation

$$f''' + \frac{1}{z^3}f'' + \frac{1}{z^2}f' + \sum_{n=1}^{\infty}\frac{1}{n^{n^2}z^n}f = 0, \tag{1.10}$$

fulfills the assumptions of Theorem 1.1 as z tends to $z_0=0$ on the ray $\arg\theta=0$. So, every solution $f(z)\not\equiv 0$ of (1.10) is of infinite order. We signal here that $\sigma(A_0,0)=\sigma(A_1,0)=\sigma(A_2,0)=0$.

Theorem 1.2. *Let $A_0(z)\not\equiv 0$, $A_1(z),\ldots,A_{k-1}(z)$ be analytic functions in $\overline{\mathbb{C}}\setminus\{z_0\}$. If there exists a subset γ of a curve tending to z_0 such that the set $\gamma_0=\{|z_0-z|\colon z\in\gamma\}\cap(0,1)$ is of infinite logarithmic measure, such that for $z\in\gamma$ and $r=|z_0-z|\in\gamma_0$, we have*

$$\lim_{r\to 0}\frac{1}{|A_0(z)|}\left(\sum_{j=1}^{k-1}|A_j(z)|+1\right)\exp_n\frac{\lambda}{r^{\mu}}=0 \tag{1.11}$$

where $n\geqslant 1$ is an integer, $\lambda>0$, $\mu>0$ are real constants, then every solution $f(z)\not\equiv 0$ of (1.9)*, that is analytic in $\overline{\mathbb{C}}\setminus\{z_0\}$, satisfies $\sigma_n(f,z_0)=\infty$ and furthermore $\sigma_{n+1}(f,z_0)\geqslant\mu$.*

Example 1.2. The differential equation

$$f''' + f''\exp\frac{1}{z} + f'\exp_2\frac{1}{z^3} + f\exp_2\frac{1}{z^2} = 0, \tag{1.12}$$

fulfills the assumptions of Theorem 1.2 as z tends to $z_0=0$ on the ray $\arg\theta=\frac{1}{5}\pi$. So, every solution $f(z)\not\equiv 0$ of (1.12) is of infinite order with $\sigma_3(f,0)\geqslant 2$.

Now, we will investigate the case when A_s, $s\neq 0$ dominates the other coefficients in a sector. Let $I(\varepsilon)=(\theta_1+\varepsilon,\theta_2-\varepsilon)\subset[0,2\pi)$ and $S(\varepsilon)$ denote the sector $\{z\colon \arg(z_0-z)\in I(\varepsilon)\}$, $\varepsilon\geqslant 0$.

Theorem 1.3. *Let $A_0(z),\ldots,A_{k-1}(z)$ be analytic functions in $\overline{\mathbb{C}}\setminus\{z_0\}$ satisfying that there exist real constants $0\leqslant\theta_1<\theta_2\leqslant 2\pi$ such that for any $\theta\in(\theta_1,\theta_2)$ there exists a set $\Gamma_\theta=\{r=|z-z_0|\colon \arg(z-z_0)=\theta\}\subset(0,1)$ of infinite logarithmic measure, and for every fixed $\mu>0$, we have*

$$\lim_{z\to z_0}\frac{1}{|A_s(z)|r^{\mu}}\left(\sum_{j=0,\,j\neq s}^{k-1}|A_j(z)|+1\right)=0,\quad s\neq 0 \tag{1.13}$$

where $\arg(z_0 - z) = \theta \in I(0)$ *and* $|z_0 - z| = r \in \Gamma_\theta$. *Given* $\varepsilon > 0$ *small enough, if* $f \not\equiv 0$ *is a solution of* (1.9) *that is analytic in* $\overline{\mathbb{C}} \setminus \{z_0\}$ *and of finite order* $\sigma(f, z_0) < \infty$, *then the following statements hold.*

(i) *There exist* $j \in \{0, \ldots, s-1\}$ *and a complex constant* $b_j \neq 0$ *such that* $f^{(j)}(z) \to b_j$ *as* $z \to z_0$ *in the sector* $S(\varepsilon)$. *More precisely, for every fixed* $\mu > 0$ *we have*

$$\lim_{z \to z_0} \frac{|f^{(j)}(z) - b_j|}{r^\mu} = 0 \tag{1.14}$$

with $z \in S(\varepsilon)$ *and* $|z_0 - z| = r \in \Gamma_\theta$.

(ii) *For each integer* $m \geqslant j+1$, $f^{(m)}(z) \to 0$ *as* $z \to z_0$ *in* $S(\varepsilon)$. *More precisely, for every fixed* $\mu > 0$ *we have*

$$\lim_{z \to z_0} \frac{|f^{(m)}(z)|}{r^\mu} = 0 \tag{1.15}$$

with $z \in S(\varepsilon)$ *and* $|z_0 - z| = r \in \Gamma_\theta$.

Example 1.3. The function $f(z) = \mathrm{e}^{1/z} - 1$ satisfies the differential equation

$$f''' + \mathrm{e}^{-1/z} f'' + \Big(\frac{2}{z} - \frac{5}{z^2} - \frac{6}{z^3} - \frac{1}{z^4}\Big) f' + \Big(\frac{2}{z^3} + \frac{1}{z^4}\Big) f = 0. \tag{1.16}$$

The differential equation (1.16) fulfills the assumptions of Theorem 1.3 in any sector $(\theta_1, \theta_2) \subset (\frac{1}{2}\pi, \frac{3}{2}\pi)$ with $z_0 = 0$. In this example, $A_2(z) = \mathrm{e}^{-1/z}$ is the dominating coefficient, while we have $j = 0$ and $b_j = -1$.

Theorem 1.4. *Let* $A_0(z), \ldots, A_{k-1}(z)$ *be analytic functions in* $\overline{\mathbb{C}} \setminus \{z_0\}$ *satisfying that there exist real constants* $0 \leqslant \theta_1 < \theta_2 \leqslant 2\pi$ *such that for any* $\theta \in (\theta_1, \theta_2)$ *there exists a set* $\Gamma_\theta = \{r = |z - z_0| \colon \arg(z - z_0) = \theta\} \subset (0,1)$ *of infinite logarithmic measure such that we have*

$$\lim_{z \to z_0} \frac{1}{|A_s(z)|} \Big(\sum_{j=0,\, j \neq s}^{k-1} |A_j(z)| + 1 \Big) \exp \frac{\lambda}{r^\alpha} = 0, \quad s \neq 0 \tag{1.17}$$

where $\arg(z_0 - z) = \theta \in I(0)$ *and* $|z_0 - z| = r \in \Gamma_\theta$, $\lambda > 0$, $\alpha > 0$ *are real constant. Given* $\varepsilon > 0$ *small enough, if* $f \not\equiv 0$ *is a solution of* (1.9), *analytic in* $\overline{\mathbb{C}} \setminus \{z_0\}$ *and of finite order* $\sigma(f, z_0) < \infty$, *then the following statements hold.*

(i) *There exists* $j \in \{0, \ldots, s-1\}$ *and a complex constant* $b_j \neq 0$ *such that* $f^{(j)}(z) \to b_j$ *as* $z \to z_0$ *in the sector* $S(\varepsilon)$. *More precisely, for* $\lambda > \lambda' > 0$ *we have*

$$|f^{(j)}(z) - b_j| < \exp\Big(-\frac{\lambda'}{r^\alpha}\Big)$$

for all $z \in S(\varepsilon)$ *with* $|z_0 - z| = r \in \Gamma_\theta$.

(ii) *For each integer* $m \geqslant j+1$, $f^{(m)}(z) \to 0$ *as* $z \to z_0$ *in* $S(\varepsilon)$. *More precisely, for* $\lambda' > 0$ *we have*

$$|f^{(m)}(z)| < \exp\Big(-\frac{\lambda'}{r^{\alpha}}\Big)$$

for all $z \in S(\varepsilon)$ *with* $|z_0 - z| = r \in \Gamma_\theta$.

Corollary 1.2. *Let* $A_0(z), \dots, A_{k-1}(z)$ *be analytic functions in* $\overline{\mathbb{C}} \setminus \{z_0\}$ *satisfying that there exist real constants* $0 \leqslant \theta_1 < \theta_2 \leqslant 2\pi$ *such that for any* $\theta \in (\theta_1, \theta_2)$ *there exists a set* $\Gamma_\theta = \{r = |z - z_0|\colon \arg(z - z_0) = \theta\} \subset (0,1)$ *of infinite logarithmic measure, we have*

$$|A_s(z)| \geqslant \exp\frac{\alpha}{r^{\mu}}, \quad s \neq 0,$$
$$|A_j(z)| \leqslant \exp\frac{\beta}{r^{\mu}}$$

where $\arg(z_0 - z) = \theta \in (\theta_1, \theta_2)$ *and* $|z_0 - z| = r \in \Gamma_\theta$, $\alpha > \beta \geqslant 0$, $\mu > 0$ *are real constant. Given* $\varepsilon > 0$ *small enough, if* $f \not\equiv 0$ *is a solution of* (1.9) *that is analytic in* $\overline{\mathbb{C}} \setminus \{z_0\}$ *and of finite order* $\sigma(f, z_0) < \infty$, *then the following statements hold.*

(i) *There exists* $j \in \{0, \dots, s-1\}$ *and a complex constant* $b_j \neq 0$ *such that* $f^{(j)}(z) \to b_j$ *as* $z \to z_0$ *in the sector* $S(\varepsilon)$. *More precisely, for* $\alpha - \beta > \lambda' > 0$ *we have*

$$|f^{(j)}(z) - b_j| < \exp\Big(-\frac{\lambda'}{r^{\mu}}\Big) \tag{1.18}$$

for all $z \in S(\varepsilon)$ *with* $|z_0 - z| = r \in \Gamma_\theta$.

(ii) *For each integer* $m \geqslant j+1$, $f^{(m)}(z) \to 0$ *as* $z \to z_0$ *in* $S(\varepsilon)$. *More precisely, for* $\alpha - \beta > \lambda' > 0$ *we have*

$$|f^{(m)}(z)| < \exp\Big(-\frac{\lambda'}{r^{\mu}}\Big) \tag{1.19}$$

for all $z \in S(\varepsilon)$ *with* $|z_0 - z| = r \in \Gamma_\theta$.

Indeed, by taking $\alpha - \beta > \lambda > 0$, the condition (1.17) holds; and then the assertions (1.18)–(1.19) hold by taking $\lambda > \lambda' > 0$. We can see similar results of these theorems in the complex plane and in the unit disc in [3], [5], [13].

2. Preliminary lemmas

To prove these results we need the following lemmas.

Lemma 2.1 ([2]). *Let f be a non constant meromorphic function in $\overline{\mathbb{C}} \setminus \{z_0\}$; let $\alpha > 0$, $\varepsilon > 0$ be given real constants and $j \in \mathbb{N}$; then*

(i) *there exists a set $E_1 \subset (0,1)$ that has finite logarithmic measure and a constant $A > 0$ that depends on α and j such that for all $r = |z - z_0|$ satisfying $r \in (0,1) \setminus E_1$, we have*

$$\Big|\frac{f^{(j)}(z)}{f(z)}\Big| \leqslant A\Big(\frac{1}{r^2} T_{z_0}(\alpha r, f) \log T_{z_0}(\alpha r, f)\Big)^j; \tag{2.1}$$

(ii) *there exists a set $E_2 \subset [0, 2\pi)$ that has a linear measure zero and a constant $A > 0$ that depends on α and j such that for all $\theta \in [0, 2\pi) \setminus E_2$ there exists a constant $r_0 = r_0(\theta) > 0$ such that (2.1) holds for all z satisfying* $\arg(z - z_0) \in [0, 2\pi) \setminus E_2$ *and $r = |z - z_0| < r_0$.*

Lemma 2.2 ([2]). *Let f be a non constant meromorphic function in $\overline{\mathbb{C}} \setminus \{z_0\}$ of finite order $\sigma(f, z_0) < \infty$; let $\varepsilon > 0$ be a given constant. Then,*

(i) *there exists a set $E_1 \subset (0,1)$ that has finite logarithmic measure such that for all $r = |z - z_0| \in (0,1) \setminus E_1$, we have*

$$\Big|\frac{f^{(k)}(z)}{f(z)}\Big| \leqslant \frac{1}{r^{k(\sigma+2+\varepsilon)}}, \quad k \in \mathbb{N}; \tag{2.2}$$

(ii) *there exists a set $E_2 \subset [0, 2\pi)$ that has a linear measure zero such that for all $\theta \in [0, 2\pi) \setminus E_2$ there exists a constant $r_0 = r_0(\theta) > 0$ such that for all z satisfying* $\arg(z - z_0) \in [0, 2\pi) \setminus E_2$ *and $r = |z - z_0| < r_0$, the inequality (2.2) holds.*

Lemma 2.3. *Let f be a non constant meromorphic function in $\overline{\mathbb{C}} \setminus \{z_0\}$ of finite order $\sigma_n(f, z_0) = \sigma_n < \infty$ $(n \geqslant 1)$ and let $\varepsilon > 0$ be a given constant. Then, there exists a set $E_1 \subset (0,1)$ that has finite logarithmic measure such that for all $r = |z - z_0| \in (0,1) \setminus E_1$, we have*

(i) *if $n = 1$, (2.2) holds,*

(ii) *and if $n \geqslant 2$*

$$\Big|\frac{f^{(k)}(z)}{f(z)}\Big| \leqslant \Big(\exp_{n-1} \frac{1}{r^{\sigma_n+\varepsilon}}\Big)^k, \quad k \in \mathbb{N}. \tag{2.3}$$

P r o o f. By the definition

$$\sigma_n(f,z_0)=\limsup_{r\to 0}\frac{\log_n T_{z_0}(r,f)}{-\log r}=\sigma_n,$$

for given $\varepsilon'>0$ there exists r_0 such that for $0<r<r_0$, we have

$$\frac{\log_n T_{z_0}(r,f)}{-\log r}<\sigma_n+\varepsilon';$$

which implies

$$T_{z_0}(r,f)<\exp_{n-1}\frac{1}{r^{\sigma_n+\varepsilon'}}. \tag{2.4}$$

Combining (2.4) with Lemma 2.1, for $\alpha>0$, there exists a set $E_1\subset(0,1)$ that has finite logarithmic measure and a constant $A>0$ that depends only on α such that for all $r=|z-z_0|$ satisfying $r\notin(0,1)\setminus E_1$, we have

$$\Bigl|\frac{f^{(k)}(z)}{f(z)}\Bigr|\leqslant A\Bigl(\frac{1}{r^2}\exp_{n-1}\Bigl(\frac{\alpha}{r}\Bigr)^{\sigma_n+\varepsilon'}\exp_{n-2}\Bigl(\frac{\alpha}{r}\Bigr)^{\sigma_n+\varepsilon'}\Bigr)^k.$$

Then, for $\varepsilon>\varepsilon'>0$ and r near enough to 0, we have

$$\Bigl|\frac{f^{(k)}(z)}{f(z)}\Bigr|\leqslant\Bigl(\exp_{n-1}\frac{1}{r^{\sigma_n+\varepsilon}}\Bigr)^k.$$

□

Lemma 2.4. *Let $f(z)$ be a non constant meromorphic function in $\overline{\mathbb{C}}\setminus\{z_0\}$. Then*

$$\sigma(f^{(j)},z_0)=\sigma(f,z_0),\quad j\in\mathbb{N}.$$

P r o o f. It is sufficient to prove that $\sigma(f',z_0)=\sigma(f,z_0)$. By Remark 1.1, $g(w)=f(z_0-1/w)$ is meromorphic in $\mathbb{C}$ and $\sigma(g)=\sigma(f,z_0)$. It is well known that for a meromorphic function in $\mathbb{C}$ we have $\sigma(g')=\sigma(g)$, (see [16], [15]). We have $f'(z)=g'(w)/w^2$. Set $h(w)=g'(w)/w^2$. Obviously, we have $\sigma(h)=\sigma(g')$. On the other hand, by Remark 1.1, we have $\sigma(h)=\sigma(f',z_0)$. So, we conclude that $\sigma(f',z_0)=\sigma(f,z_0)$. □

Lemma 2.5. *Let f be a non constant meromorphic function in $\overline{\mathbb{C}}\setminus\{z_0\}$ and suppose that $|f^{(k)}(z)|$ is unbounded on some ray $\arg(z_0-z)=\theta$. Then there exists an infinite sequence of points $z_m=z_0-r_m\mathrm{e}^{\mathrm{i}\theta}$, $m=1,2,\dots$, where $r_m\to 0$, such that $f^{(k)}(z_m)\to\infty$ and*

$$\Bigl|\frac{f^{(j)}(z_m)}{f^{(k)}(z_m)}\Bigr|\leqslant M,$$

where $M>0$ and $j\in(0,1,\dots,k-1)$.

P r o o f. Let $M(r,\theta,f^{(k)}) = \max|f^{(k)}(z)|$ where $z \in [z_0 - r_1 e^{i\theta}, z_0 - r e^{i\theta}]$. Clearly, we may construct a sequence of points $z_m = z_0 - r_m e^{i\theta}$, $m \geqslant 1$, $r_m \to 0$, such that $M(r,\theta,f^{(k)}) = |f^{(k)}(z_m)| \to \infty$. For each m, by $(k-j)$-fold iteration integration along the line segment $[z_1, z_m]$ we have

$$\begin{aligned} f^{(j)}(z_m) = f^{(j)}(z_1) &+ f^{(j+1)}(z_1)(z_m - z_1) \\ &+ \ldots + \frac{1}{(k-j-1)} f^{(k-1)}(z_1)(z_m - z_1)^{k-j-1} \\ &+ \int_{z_1}^{z_m} \ldots \int_{z_1}^{y} f^{(k)}(x)\,dx\,dy \ldots dt; \end{aligned}$$

and by an elementary triangle inequality estimate we obtain

$$\begin{aligned} |f^{(j)}(z_m)| \leqslant |f^{(j)}(z_1)| &+ |f^{(j+1)}(z_1)||(z_m - z_1)| \\ &+ \ldots + \frac{1}{(k-j-1)}|f^{(k-1)}(z_1)||(z_m - z_1)|^{k-j-1} \\ &+ \frac{1}{(k-j)}|f^{(k)}(z_m)||(z_m - z_1)|^{k-j}. \end{aligned} \tag{2.5}$$

From (2.5) and taking account that when $m \to \infty$, $f^{(k)}(z_m) \to \infty$, $z_m \to z_0$, we obtain

$$\Big|\frac{f^{(j)}(z_m)}{f^{(k)}(z_m)}\Big| \leqslant M, \quad M > 0.$$

□

Lemma 2.6. *Let f be an analytic function in $\overline{\mathbb{C}} \setminus \{z_0\}$. Let $a \geqslant \frac{1}{2}$ and*

$$G = \Big\{z\colon\ |\arg(z_0 - z)| < \frac{\pi}{2a}\Big\}.$$

Suppose that $\limsup_{z\to\varsigma} |f(z)| \leqslant M$ for all $\varsigma \in \partial G$, where M is a fixed constant. Suppose further that there exist constants K, $b < a$ such that

$$|f(z)| \leqslant K \exp\frac{1}{r^b} \quad \text{as } r \to 0,$$

where $r = |z_0 - z|$ and $z \in G$. Then, $|f(z)| \leqslant M$ for all $z \in G$.

P r o o f. The change of variable $w = 1/(z_0 - z)$ maps G onto $H = \{w\colon\ |\arg(w)| < \pi/(2a)\}$ and the function $g(w) = f(z)$ is an entire function on $w \in \mathbb{C}$ and we have $|\arg(z_0 - z)| = \pi/(2a) \Leftrightarrow |\arg(w)| = \pi/(2a)$ and $\limsup_{w\to\xi}|g(w)| = \limsup_{z\to\varsigma}|f(z)| \leqslant M$ for all $\xi \in \partial H$. Further, we have

$$|g(w)| = |f(z)| \leqslant K \exp\frac{1}{r^b} = K \exp R^b \quad \text{as } R \to \infty,$$

where $R = |w| = 1/r$. Then, by Phragmen-Lindelöf theorem we get $|g(w)| \leqslant M$ for all $w \in H$. Therefore, $|f(z)| \leqslant M$ for all $z \in G$. □

Lemma 2.7. *If f is analytic function in $\overline{\mathbb{C}} \setminus \{z_0\}$ such that for any $\mu > 0$, we have*

$$|f(z_0 - r\mathrm{e}^{\mathrm{i}\theta})| \leqslant r^{\mu} \quad \text{as } r \to 0$$

then $\int_0^r |f(z_0 - t\mathrm{e}^{\mathrm{i}\theta})| \, \mathrm{d}t$ converges and for every $\alpha > 0$, we have

$$\int_0^r |f(z_0 - t\mathrm{e}^{\mathrm{i}\theta})| \, \mathrm{d}t \leqslant r^{\alpha} \quad \text{as } r \to 0.$$

Proof. It is easy to show that $\int_0^r |f(z_0 - t\mathrm{e}^{\mathrm{i}\theta})| \, \mathrm{d}t$ converges; and we have

$$\int_0^r |f(z_0 - t\mathrm{e}^{\mathrm{i}\theta})| \, \mathrm{d}t \leqslant \int_0^r t^{\mu} \, \mathrm{d}t = \frac{r^{\mu+1}}{\mu+1}.$$

Let $\alpha > 0$. By taking $\mu + 1 > \alpha$, we have

$$\int_0^r |f(z_0 - t\mathrm{e}^{\mathrm{i}\theta})| \, \mathrm{d}t \leqslant \frac{r^{\mu+1}}{\mu+1} \leqslant r^{\alpha} \quad \text{as } r \to 0.$$

□

Lemma 2.8. *Let f be an analytic function in $\overline{\mathbb{C}} \setminus \{z_0\}$. The two following assertions are equivalent:*

(i) *for any $\mu > 0$, $|f(z_0 - r\mathrm{e}^{\mathrm{i}\theta})| \leqslant r^{\mu}$ as $r \to 0$,*
(ii) *for any $\alpha > 0$, $\lim\limits_{r \to 0} |f(z_0 - r\mathrm{e}^{\mathrm{i}\theta})|/r^{\alpha} = 0$.*

Proof. (ii) $\Rightarrow$ (i). Suppose that for any $\alpha > 0$, $\lim\limits_{r \to 0} |f(z_0 - r\mathrm{e}^{\mathrm{i}\theta})|/r^{\alpha} = 0$. For any $\alpha > 0$ and $\varepsilon > 0$, there exists $\delta > 0$ such that for $0 < r < \delta$ we have $|f(z_0 - r\mathrm{e}^{\mathrm{i}\theta})| \leqslant \varepsilon r^{\alpha}$. By taking $\varepsilon = 1$ we get the assertion (i).

(i) $\Rightarrow$ (ii). Suppose that for any $\mu > 0$, $|f(z_0 - r\mathrm{e}^{\mathrm{i}\theta})| \leqslant r^{\mu}$ as $r \to 0$. Let $\alpha > 0$. We have

$$\frac{|f(z_0 - r\mathrm{e}^{\mathrm{i}\theta})|}{r^{\alpha}} \leqslant \frac{r^{\mu}}{r^{\alpha}}.$$

By taking $\mu > \alpha$, we obtain

$$\lim_{r \to 0} \frac{|f(z_0 - r\mathrm{e}^{\mathrm{i}\theta})|}{r^{\alpha}} = 0.$$

□

Lemma 2.9. *If f is analytic function in $\overline{\mathbb{C}} \setminus \{z_0\}$ such that*

$$|f(z_0 - t\mathrm{e}^{\mathrm{i}\theta})| \leqslant \exp\Bigl(-\frac{\lambda}{t^\alpha}\Bigr),$$

where $\alpha > 0$, $\lambda > 0$, then $\int_0^r |f(z_0 - t\mathrm{e}^{\mathrm{i}\theta})|\,\mathrm{d}t$ converges and we have

$$\int_0^r |f(z_0 - t\mathrm{e}^{\mathrm{i}\theta})|\,\mathrm{d}t \leqslant \exp\Bigl(-\frac{\lambda}{r^\alpha}\Bigr) \quad \text{as } r \to 0.$$

Proof. It is easy to show that $\int_0^r |f(z_0 - t\mathrm{e}^{\mathrm{i}\theta})|\,\mathrm{d}t$ converges; and we have

$$\begin{aligned}\int_0^r |f(z_0 - t\mathrm{e}^{\mathrm{i}\theta})|\,\mathrm{d}t &\leqslant \int_0^r \exp\Bigl(-\frac{\lambda}{r^\alpha}\Bigr)\,\mathrm{d}t \leqslant \exp\Bigl(-\frac{\lambda}{r^\alpha}\Bigr)\int_0^r \mathrm{d}t \\ &\leqslant r\exp\Bigl(-\frac{\lambda}{r^\alpha}\Bigr) \leqslant \exp\Bigl(-\frac{\lambda}{r^\alpha}\Bigr) \quad \text{as } r \to 0.\end{aligned}$$

□

3. Proof of theorems

Proof of Theorem 1.1. Suppose that $f \not\equiv 0$ is a solution of (1.9) of finite order $\sigma(f, z_0) = \sigma < \infty$. By Lemma 2.3, for any given $\varepsilon > 0$ there exists a set $E \subset (0,1)$ that has finite logarithmic measure such that for all $r = |z_0 - z| \in (0,1) \setminus E$, we have

$$\Bigl|\frac{f^{(j)}(z)}{f(z)}\Bigr| \leqslant \frac{1}{r^{j(\sigma+2+\varepsilon)}}, \quad j = 1, \ldots, k. \tag{3.1}$$

From (1.9) we can write

$$1 \leqslant \frac{1}{|A_0(z)|}\Bigl|\frac{f^{(k)}}{f}\Bigr| + \frac{|A_{k-1}(z)|}{|A_0(z)|}\Bigl|\frac{f^{(k-1)}}{f}\Bigr| + \ldots + \frac{|A_1(z)|}{|A_0(z)|}\Bigl|\frac{f'}{f}\Bigr|. \tag{3.2}$$

By the assumption (1.8), for $r \in F$ and any fixed $\mu > 0$, we have

$$\lim_{r\to 0} \frac{|A_j(z)|}{|A_0(z)| r^\mu} = 0, \quad j = 1, \ldots, k \tag{3.3}$$

and

$$\lim_{r\to 0} \frac{1}{|A_0(z)| r^\mu} = 0. \tag{3.4}$$

Using (3.1), (3.3) and (3.4) in (3.2), a contradiction follows as $r \to 0$ with $r = |z_0 - z| \in F \setminus E$. □

P r o o f of Theorem 1.2. Suppose that $f \not\equiv 0$ is a solution of (1.9) with $\sigma_n(f, z_0) = \sigma_n < \infty$, $n \geqslant 1$. If $n = 1$ we have (3.1) and if $n \geqslant 2$, by Lemma 2.3, for any given $\varepsilon > 0$ there exists a set $E \subset (0, 1)$ that has finite logarithmic measure such that for all $r = |z_0 - z| \in (0, 1) \setminus E$, we have

$$\left|\frac{f^{(j)}(z)}{f(z)}\right| \leqslant \left(\exp_{n-1} \frac{1}{r^{\sigma_n + \varepsilon}}\right)^j, \quad j = 1, \ldots, k. \tag{3.5}$$

By the assumption (1.11), for $r \in F$, we have

$$\lim_{r \to 0} \frac{|A_j(z)|}{|A_0(z)|} \exp_n \frac{\lambda}{r^\mu} = 0, \quad j = 1, \ldots, k \tag{3.6}$$

and

$$\lim_{r \to 0} \frac{1}{|A_0(z)|} \exp_n \frac{\lambda}{r^\mu} = 0. \tag{3.7}$$

Using (3.1) or (3.5), (3.6) and (3.7) in (3.2), a contradiction follows as $r \to 0$ on γ with $r = |z_0 - z| \in F \setminus E$. So, $\sigma_n(f, z_0) = \infty$ for $n \geqslant 1$. Now, by Lemma 2.1, and since $\sigma_n(f, z_0) = \infty$, we have

$$\left|\frac{f^{(j)}(z)}{f(z)}\right| \leqslant A\left(\frac{1}{r} T_{z_0}(\alpha r, f)\right)^{2k}, \quad j = 1, \ldots, k. \tag{3.8}$$

By the assumption (1.11), for $\varepsilon_1 > 0$, $\varepsilon_2 > 0$, we have

$$\frac{|A_j(z)|}{|A_0(z)|} \leqslant \frac{\varepsilon_1}{\exp_n(\lambda / r^\mu)}, \quad j = 1, \ldots, k \tag{3.9}$$

and

$$\frac{1}{|A_0(z)|} \leqslant \frac{\varepsilon_2}{\exp_n(\lambda / r^\mu)} \tag{3.10}$$

as $r \to 0$ on γ with $r = |z_0 - z| \in F$. Using (3.8)–(3.10) in (3.2), we obtain, for $r = |z_0 - z| \in F \setminus E$,

$$1 \leqslant \frac{M}{\exp_n(\lambda / r^\mu)} \left(\frac{1}{r} T_{z_0}(\alpha r, f)\right)^{2k}, \tag{3.11}$$

where $M > 0$ is a real constant. Set $R = \alpha r$. We signal here that E is of finite logarithmic measure if and only if αE is of finite logarithmic measure. So, from (3.11), we get

$$\exp_n \frac{\lambda \alpha^\mu}{R^\mu} \leqslant M\left(\frac{\alpha}{R} T_{z_0}(r, f)\right)^{2k}, \quad R \in F \setminus E. \tag{3.12}$$

From (3.12) we obtain

$$\sigma_{n+1}(f, z_0) = \limsup_{r \to 0} \frac{\log^+_{n+1} T_{z_0}(r, f)}{-\log R} \geqslant \mu.$$

□

Proof of Theorem 1.3. First, we have to prove that $f(z)$ is bounded in $S(\varepsilon)$, for $\varepsilon > 0$ small enough and for that we prove that $f^{(s)}(z)$ is also bounded in $S(\varepsilon)$. From Lemma 2.4 and Lemma 2.2, it follows that there exists a set $E \subset [0, 2\pi)$ that has linear measure zero, such that for all $j \in \{s+1, \ldots, k\}$

$$\left|\frac{f^{(j)}(z)}{f^{(s)}(z)}\right| \leqslant \frac{1}{r^{(j-s)(\sigma+2+\varepsilon)}}, \tag{3.13}$$

where $\arg(z_0 - z) \in I(0) \setminus E$ and $r = |z_0 - z| \in \Gamma_\theta$. If we suppose that $f^{(s)}(z)$ is unbounded on some ray $\arg(z_0 - z) = \varphi \in I(0) \setminus E$, then by Lemma 2.5 there exists an infinite sequence of points $z_m = z_0 - r_m e^{i\varphi}$, $m = 1, 2, \ldots$, with $r_m \to 0$, such that $f^{(k)}(z_m) \to \infty$ and

$$\left|\frac{f^{(q)}(z_m)}{f^{(s)}(z_m)}\right| \leqslant M_1, \tag{3.14}$$

where $M_1 > 0$, $q \in \{0, 1, \ldots, s-1\}$ and m large enough. From (1.9) we can write

$$\begin{aligned} 1 \leqslant {} & \frac{1}{|A_s(z)|}\left|\frac{f^{(k)}}{f^{(s)}}\right| + \frac{|A_{k-1}(z)|}{|A_s(z)|}\left|\frac{f^{(k-1)}}{f^{(s)}}\right| + \ldots + \frac{|A_{s+1}(z)|}{|A_s(z)|}\left|\frac{f^{(s+1)}}{f^{(s)}}\right| \\ & + \frac{|A_{s-1}(z)|}{|A_s(z)|}\left|\frac{f^{(s-1)}}{f^{(s)}}\right| + \ldots + \frac{|A_0(z)|}{|A_s(z)|}\left|\frac{f}{f^{(s)}}\right|. \end{aligned} \tag{3.15}$$

Combining now (1.13), (3.13)–(3.15) and letting $m \to \infty$ we obtain a contradiction. Therefore, $f^{(s)}(z)$ remains bounded on all rays $\arg(z_0 - z) = \varphi \in I(0) \setminus E$. By Lemma 2.6, we conclude that $f^{(s)}(z)$ is bounded, say $|f^{(s)}(z)| \leqslant M_2$, in the whole sector $S(\frac{1}{2}\varepsilon)$ for $\varepsilon > 0$ small enough.

By integrating s times along the line segment $[z_1, z]$ in $S(\frac{1}{2}\varepsilon)$, we have

$$\begin{aligned} f(z) = {} & f(z_1) + f'(z_1)(z - z_1) + \ldots + \frac{1}{(s-1)!} f^{(s-1)}(z_1)(z - z_1)^{s-1} \\ & + \int_{z_1}^{z} \ldots \int_{z_1}^{z} f^{(s)}(t)\,dt \ldots dt; \end{aligned}$$

and by an elementary triangle inequality estimate, we obtain

$$|f(z)| \leqslant |f(z_1)| + |f'(z_1)||z - z_1| + \ldots + \frac{1}{(s-1)!}|f^{(s-1)}(z_1)||z - z_1|^{s-1} + \frac{1}{(s)!} M|z - z_1|^s$$

and therefore, as $z \to z_0$, we get

$$|f(z)| \leqslant M_3 \tag{3.16}$$

for a certain constant $M_3 > 0$. Now, we begin to prove (1.15) for $m = s$. Using (1.9), we can write

$$(3.17)\quad |f^{(s)}(z)| \leqslant |f|\Big(\frac{1}{|A_s(z)|}\Big|\frac{f^{(k)}}{f}\Big| + \frac{|A_{k-1}(z)|}{|A_s(z)|}\Big|\frac{f^{(k-1)}}{f}\Big| + \ldots + \frac{|A_{s+1}(z)|}{|A_s(z)|}\Big|\frac{f^{(s+1)}}{f}\Big| + \frac{|A_{s-1}(z)|}{|A_s(z)|}\Big|\frac{f^{(s-1)}}{f}\Big| + \ldots + \frac{|A_1(z)|}{|A_s(z)|}\Big|\frac{f'}{f}\Big| + \frac{|A_0(z)|}{|A_s(z)|}\Big).$$

By the assumption (1.13), for any $\mu > 0$, for every $j \in \{0, 1, \ldots, s-1, s+1, \ldots, k-1\}$ and for $\varepsilon > 0$, there exists δ such that for $|z_0 - z| < \delta$ we have

$$(3.18)\quad \frac{|A_j(z)|}{|A_s(z)|} \leqslant \varepsilon |z_0 - z|^{\mu},$$

$$(3.19)\quad \frac{1}{|A_s(z)|} \leqslant \varepsilon |z_0 - z|^{\mu},$$

where $\arg(z_0 - z) = \theta \in I(0)$ and $|z_0 - z| = r \in \Gamma_\theta$. Substituting (3.13), (3.16), (3.18) and (3.19) into (3.17), we obtain that for any $\mu > 0$, we have

$$|f^{(s)}(z)| \leqslant M_4 \frac{|z_0 - z|^{\mu}}{r^{k(\sigma+2+\varepsilon)}} \quad \text{as } r \to 0.$$

We conclude that for any fixed $\alpha > 0$

$$(3.20)\quad \lim_{z \to z_0} \frac{|f^{(s)}(z)|}{r^{\alpha}} = 0,$$

with $r = |z_0 - z| \in \Gamma_\theta$ and $\arg(z_0 - z) = \varphi \in I(\frac{1}{2}\varepsilon) \setminus E$.

Proof of equation (1.15) for $m > s$. Consider $z = z_0 - re^{i\theta} \in S(\varepsilon)$ and $C(z)$ the circle centered at z of radius ϱ small enough such that $C(z)$ is contained in $S(\frac{1}{2}\varepsilon)$, we may take $\varrho = r\sin(\frac{1}{2}\varepsilon)$. By the Cauchy formula applied to the function $f^{(s)}(z)$ we have

$$(3.21)\quad f^{(m)}(z) = \frac{(m-s)!}{2\pi} \int_{C(z)} \frac{f^{(s)}(\zeta)}{(z-\zeta)^{m-s+1}}\,d\zeta,$$

and using (3.20), we get

$$|f^{(m)}(z)| \leqslant \frac{(m-s)!}{2\pi} \int_0^{2\pi} \frac{|z_0 - z|^{\mu}}{\varrho^{m-s+1}}\,\varrho\,d\theta \leqslant \frac{(m-s)!}{\sin^{m-s}(\frac{1}{2}\varepsilon)} \frac{|z_0 - z|^{\mu}}{r^{m-s}}.$$

We conclude that, for any fixed $\alpha > 0$ and $z \in S(\varepsilon)$ with $r = |z_0 - z| \in \Gamma_\theta$, we have

$$\lim_{z \to z_0} \frac{|f^{(m)}(z)|}{|z_0 - z|^{\alpha}} = 0.$$

Until now, we have proved the second assertion for $m \geqslant s$. We start to prove the first assertion for $j = s-1$. Set

$$a_s = \int_0^\infty f^{(s)}(z_0 - t\mathrm{e}^{\mathrm{i}\theta})\mathrm{e}^{\mathrm{i}\theta}\,\mathrm{d}t.$$

By (3.20), it is easy to see that $\int_0^\infty f^{(s)}(z_0 - t\mathrm{e}^{\mathrm{i}\theta})\mathrm{e}^{\mathrm{i}\theta}\,\mathrm{d}t$ converges. Moreover, a_s is independent of θ, because by (3.20), the integral of $f^{(s)}(\zeta)$ over the arc $z_0 - r\mathrm{e}^{\mathrm{i}\theta}$, $\theta \in (\varphi, \varphi) \subset I(\frac{1}{2}\varepsilon)$, we get

$$\left|\int_\varphi^\varphi f^{(s)}(z_0 - r\mathrm{e}^{\mathrm{i}\theta})\mathrm{i}r\mathrm{e}^{\mathrm{i}\theta}\,\mathrm{d}\theta\right| \leqslant Mr^{\alpha+1}|\varphi - \varphi| \to 0, \quad r \to 0,\ M > 0.$$

Define now $b_{s-1} = f^{(s-1)}(\infty) + a_s$, and suppose that $b_{s-1} \neq 0$. Let $z = z_0 - r\mathrm{e}^{\mathrm{i}\theta}$ be an arbitrary point in $S(\varepsilon)$. Then, since

$$f^{(s-1)}(z) - b_{s-1} = \int_\infty^z f^{(s)}(\zeta)\,\mathrm{d}\zeta - \int_0^\infty f^{(s)}(z_0 - t\mathrm{e}^{\mathrm{i}\theta})\mathrm{e}^{\mathrm{i}\theta}\,\mathrm{d}t,$$

we may apply (3.20) and Lemma 2.7, and we get

$$\begin{aligned}(3.22)\qquad |f^{(s-1)}(z) - b_{s-1}| &= \left|\int_\infty^z f^{(s)}(\zeta)\,\mathrm{d}\zeta - \int_0^\infty f^{(s)}(z_0 - t\mathrm{e}^{\mathrm{i}\theta})\mathrm{e}^{\mathrm{i}\theta}\,\mathrm{d}t\right| \\ &= \left|\int_r^\infty f^{(s)}(z_0 - t\mathrm{e}^{\mathrm{i}\theta})\mathrm{e}^{\mathrm{i}\theta}\,\mathrm{d}t + \int_\infty^0 f^{(s)}(z_0 - t\mathrm{e}^{\mathrm{i}\theta})\mathrm{e}^{\mathrm{i}\theta}\,\mathrm{d}t\right| \\ &= \left|\int_r^0 f^{(s)}(z_0 - t\mathrm{e}^{\mathrm{i}\theta})\mathrm{e}^{\mathrm{i}\theta}\,\mathrm{d}t\right| \\ &\leqslant \int_0^r |f^{(s)}(z_0 - t\mathrm{e}^{\mathrm{i}\theta})|\,\mathrm{d}t \leqslant r^\mu \quad \text{as } r \to 0\end{aligned}$$

for any $\mu > 0$ and $z \in S(\varepsilon)$ with $r = |z_0 - z| \in \Gamma_\theta$. By Lemma 2.8, we have completed the proof in the case $b_{s-1} \neq 0$. If $b_{s-1} = 0$, we define $a_{s-1} = \int_0^\infty f^{(s-1)}(z_0 - t\mathrm{e}^{\mathrm{i}\theta})\mathrm{e}^{\mathrm{i}\theta}\,\mathrm{d}t$ and $b_{s-2} = f^{(s-2)}(\infty) + a_{s-1}$ and by applying Lemma 2.7 with (3.22) we obtain that, for every fixed $\mu > 0$,

$$|f^{(s-2)}(z) - b_{s-2}| \leqslant r^\mu \quad \text{as } r \to 0$$

for $z \in S(\varepsilon)$ with $r = |z_0 - z| \in \Gamma_\theta$. By the same method, if $b_{s-1} = b_{s-2} = \ldots = b_{j+1} = 0$ and $b_j \neq 0$, $j \in \{0, \ldots, s-1\}$, then for any fixed $\mu > 0$

$$|f^{(j)}(z) - b_j| \leqslant r^\mu \quad \text{as } r \to 0,$$

and

$$(3.23)\qquad |f^{(m)}(z)| \leqslant r^\mu \quad \text{as } r \to 0 \text{ for all } m \geqslant j+1$$

for $z \in S(\varepsilon)$ with $r = |z_0 - z| \in \Gamma_\theta$. Now it remains to show that the case $b_{s-1} = b_{s-2} = \ldots = b_0 = 0$ is not possible. In this case, we have, for any fixed $\mu > 0$

$$|f^{(m)}(z)| \leqslant r^\mu \quad \text{as } r \to 0 \tag{3.24}$$

for $z \in S(\varepsilon)$ with $r = |z_0 - z| \in \Gamma_\theta$, for every $m \geqslant 0$ and any $\mu > 0$, there exists $r_0(\mu, m) > 0$ such that if $|z_0 - z| = r < r_0$ then $|f^{(m)}(z)| \leqslant |z_0 - z|^\mu$. Now we take $z \in S(\varepsilon)$ such that $r = |z_0 - z| < r_1 = \min\limits_{m=0,\ldots,s} r_0(\mu, m)$; we remark here that if z is fixed then (3.24) is valid for only some $\mu > 0$ and not for all $\mu > 0$. From (1.9) we can write

$$\begin{aligned} \frac{|f^{(s)}(z)|}{|f(z)|} \leqslant \frac{1}{|A_s(z)|}\Big|\frac{f^{(k)}}{f}\Big| + \frac{|A_{k-1}(z)|}{|A_s(z)|}\Big|\frac{f^{(k-1)}}{f}\Big| + \ldots + \frac{|A_{s+1}(z)|}{|A_s(z)|}\Big|\frac{f^{(s+1)}}{f}\Big| \\ + \frac{|A_{s-1}(z)|}{|A_s(z)|}\Big|\frac{f^{(s-1)}}{f}\Big| + \ldots + \frac{|A_1(z)|}{|A_s(z)|}\Big|\frac{f'}{f}\Big| + \frac{|A_0(z)|}{|A_s(z)|}, \end{aligned} \tag{3.25}$$

and by using (1.13) and Lemma 2.2 in (3.25), we obtain

$$\frac{|f^{(s)}(z)|}{|f(z)|} \leqslant |z_0 - z|^\mu, \tag{3.26}$$

and by (3.24) for $m = 0$ in (3.25), we get

$$|f^{(s)}(z)| \leqslant |z_0 - z|^{2\mu} \tag{3.27}$$

for $|z_0 - z| < r_1$ and $\arg(z_0 - z) \in I(\varepsilon) \setminus E$, hence in $S(\varepsilon + \frac{1}{2}\varepsilon)$ by Lemma 2.6. Repeating the reasoning of (3.22)–(3.24) with (3.27), we obtain

$$|f(z)| \leqslant |z_0 - z|^{2\mu},$$

and by combining with (3.26), we get

$$|f^{(s)}(z)| \leqslant |z_0 - z|^{3\mu},$$

in $S(\varepsilon + \frac{1}{2}\varepsilon + \frac{1}{2^2}\varepsilon)$. Inductively, by the same reasoning, after $(T-1)$ steps, we obtain

$$|f^{(s)}(z)| \leqslant |z_0 - z|^{T\mu} \tag{3.28}$$

in

$$S\Big(\varepsilon + \frac{\varepsilon}{2} + \frac{\varepsilon}{2^2} + \ldots + \frac{\varepsilon}{2^{T-1}}\Big) = S\Big(2\varepsilon\Big(1 - \frac{1}{2^{T-1}}\Big)\Big)$$

with $|z_0 - z| < r_1$. Thus, we have proved, in this special case $b_{s-1} = b_{s-2} = \ldots = b_0 = 0$, that (3.28) is valid in $S(2\varepsilon)$ for all $T \in \mathbb{N}$, provided $|z_0 - z| < r_1$. Fix now a finite line segment $L \subset S(2\varepsilon)$ with $|z_0 - z| < \min(1, r_1)$. By taking $T \to \infty$ in (3.28), $f^{(s)}(z)$ vanishes identically on such a line segment. Therefore, f must be a polynomial. Since f is analytic in $\overline{\mathbb{C}} - \{z_0\}$, f has to be a constant. It is easy to see that the only constant solution of (1.9) is $f \equiv 0$, a contradiction. □

P r o o f of Theorem 1.4. We will use the same method of the proof of Theorem 1.3. The assumption (1.17) implies that for any $\varepsilon > 0$ there exists $\delta > 0$ such that for $r = |z_0 - z| < \delta$, we have

$$\frac{|A_j(z)|}{|A_s(z)|} \leqslant \varepsilon \exp\Big(-\frac{\lambda}{r^\alpha}\Big), \tag{3.29}$$

$$\frac{1}{|A_s(z)|} \leqslant \varepsilon \exp\Big(-\frac{\lambda}{r^\alpha}\Big). \tag{3.30}$$

By the same steps (3.13)–(3.15) with (3.29) and (3.30), we can prove that $f^{(s)}(z)$ is bounded in $S(\varepsilon)$, say

$$|f^{(s)}(z)| \leqslant M_1,$$

in the whole sector $S(\frac{1}{2}\varepsilon)$ for some $\varepsilon > 0$ small enough. As above, we can prove also that

$$|f(z)| \leqslant M_2.$$

By using (3.29)–(3.30) in (3.17), for $r = |z_0 - z| \in \Gamma_\theta$ and $\arg(z_0 - z) = \varphi \in I(\frac{1}{2}\varepsilon)\backslash E$, we get

$$|f^{(s)}(z)| \leqslant \exp\frac{-\lambda + \tau}{r^\alpha},$$

where $0 < \tau < \lambda$. For $m > s$, as above, by (3.21) we obtain

$$|f^{(m)}(z)| \leqslant \exp\frac{-\lambda + \tau}{r^\alpha}$$

for all $z \in S(\varepsilon)$ with $r = |z_0 - z| \in \Gamma_\theta,\ 0 < \tau < \lambda$. Puting a_s and b_{s-1} as above and by Lemma 2.9, we get

$$|f^{(s-1)}(z) - b_{s-1}| \leqslant \exp\frac{-\lambda + \tau}{r^\alpha}$$

as $r = |z_0 - z| \to 0$, where $0 < \tau < \lambda$. By the same method used in the proof of Theorem 1.3, we can prove the impossibility of the case $b_{s-1} = b_{s-2} = \ldots = b_0 = 0$. □

References

[1] *L. Bieberbach*: Theorie der gewöhnlichen Differentialgleichungen auf funktionentheoretischer Grundlage dargestellt. Die Grundlehren der Mathematischen Wissenschaften 66. Springer, Berlin, 1965. (In German.) zbl MR

[2] *H. Fettouch, S. Hamouda*: Growth of local solutions to linear differential equations around an isolated essential singularity. Electron. J. Differ. Equ. *2016* (2016), Paper No. 226, 10 pages. zbl MR

[3] *S. Hamouda*: Finite and infinite order solutions of a class of higher order linear differential equations. Aust. J. Math. Anal. Appl. *9* (2012), Article No. 10, 9 pages. zbl MR

[4] *S. Hamouda*: Properties of solutions to linear differential equations with analytic coefficients in the unit disc. Electron. J. Differ. Equ. *2012* (2012), Paper No. 177, 8 pages. zbl MR

[5] *S. Hamouda*: Iterated order of solutions of linear differential equations in the unit disc. Comput. Methods Funct. Theory *13* (2013), 545–555. zbl MR doi

[6] *S. Hamouda*: The possible orders of growth of solutions to certain linear differential equations near a singular point. J. Math. Anal. Appl. *458* (2018), 992–1008. zbl MR doi

[7] *W. K. Hayman*: Meromorphic Functions. Oxford Mathematical Monographs. Clarendon Press, Oxford, 1964. zbl MR

[8] *A. Ya. Khrystiyanyn, A. A. Kondratyuk*: On the Nevanlinna theory for meromorphic functions on annuli. I. Mat. Stud. *23* (2005), 19–30. zbl MR

[9] *L. Kinnunen*: Linear differential equations with solutions of finite iterated order. Southeast Asian Bull. Math. *22* (1998), 385–405. zbl MR

[10] *A. Kondratyuk, I. Laine*: Meromorphic functions in multiply connected domains. Fourier Series Methods in Complex Analysis (I. Laine, ed.). University of Joensuu 10. Department of Mathematics, University of Joensuu, Joensuu, 2006, pp. 9–111. zbl MR

[11] *R. Korhonen*: Nevanlinna theory in an annulus. Value Distribution Theory and Related Topics. Advances in Complex Analysis and Its Applications 3. Kluwer Academic Publishers, Boston, 2004, pp. 167–179. zbl MR doi

[12] *I. Laine*: Nevanlinna Theory and Complex Differential Equations. De Gruyter Studies in Mathematics 15. W. de Gruyter, Berlin, 1993. zbl MR doi

[13] *I. Laine, R. Yang*: Finite order solutions of complex linear differential equations. Electron. J. Differ. Equ. *2004* (2004), Paper No. 65, 8 pages. zbl MR

[14] *M. E. Lund, Z. Ye*: Logarithmic derivatives in annuli. J. Math. Anal. Appl. *356* (2009), 441–452. zbl MR doi

[15] *M. Tsuji*: Potential Theory in Modern Function Theory. Chelsea Publishing Company, New York, 1975. zbl MR

[16] *J. M. Whittaker*: The order of the derivative of a meromorphic function. J. Lond. Math. Soc. *11* (1936), 82–87. zbl MR

[17] *L. Yang*: Value Distribution Theory. Springer, Berlin, 1993. zbl MR doi

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