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A NOTE ON THE SIZE RAMSEY NUMBERS  
FOR MATCHINGS VERSUS CYCLES

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*Abstract.* For graphs  $G$ ,  $F_1$ ,  $F_2$ , we write  $G \rightarrow (F_1, F_2)$  if for every red-blue colouring of the edge set of  $G$  we have a red copy of  $F_1$  or a blue copy of  $F_2$  in  $G$ . The size Ramsey number  $\hat{r}(F_1, F_2)$  is the minimum number of edges of a graph  $G$  such that  $G \rightarrow (F_1, F_2)$ . Erdős and Faudree proved that for the cycle  $C_n$  of length  $n$  and for  $t \geq 2$  matchings  $tK_2$ , the size Ramsey number  $\hat{r}(tK_2, C_n) < n + (4t + 3)\sqrt{n}$ . We improve their upper bound for  $t = 2$  and  $t = 3$  by showing that  $\hat{r}(2K_2, C_n) \leq n + 2\sqrt{3n} + 9$  for  $n \geq 12$  and  $\hat{r}(3K_2, C_n) < n + 6\sqrt{n} + 9$  for  $n \geq 25$ .

*Keywords:* size Ramsey number; matching; cycle

*MSC 2020:* 05C55, 05C35

## 1. INTRODUCTION

Ramsey theory studies problems which can be grouped under the common theme that every large system contains a highly organized subsystem. Ramsey-type theorems have roots in various branches of mathematics and the theory developed from them has influenced areas such as set theory, number theory, ergodic theory, geometry and theoretical computer science.

The size Ramsey number was introduced by Erdős et al. (see [3]) who investigated the size Ramsey number for various graphs. Size Ramsey numbers for all pairs of connected graphs having at most four vertices were found by Faudree and Sheehan (see [4]). Bounds on the size Ramsey number for trees were presented by Ke in [5], paths and stars were investigated by Lortz and Mengersen in [6]. Modifications of

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the size Ramsey number have been studied extensively, too (see [1] and [7] for results on the size multipartite Ramsey numbers).

We denote the edge set of a graph  $G$  by  $E(G)$  and the number of edges in  $G$  by  $|E(G)|$ . For an edge set  $E_1 \subseteq E(G)$ , the edge-induced subgraph of  $G$  consists of the edges in  $E_1$  and the vertices incident to edges in  $E_1$ .

A *cycle* of length  $n$  (an  $n$ -cycle)  $C_n = v_1v_2 \dots v_nv_1$  is a graph with  $n$  vertices  $v_1, v_2, \dots, v_n$  and  $n$  edges  $v_1v_2, v_2v_3, \dots, v_{n-1}v_n$  and  $v_nv_1$ . Similarly, a *path*  $v_1v_2 \dots v_n$  of length  $n - 1$  contains  $n - 1$  edges  $v_1v_2, v_2v_3, \dots, v_{n-1}v_n$ . For  $t \geq 1$ , the graph which consists of  $t$  *independent edges* (*matchings*) is denoted by  $tK_2$  (it is a 1-regular graph having  $2t$  vertices).

For simple graphs  $G, F_1, F_2$ , we write  $G \rightarrow (F_1, F_2)$  if for each 2-colouring (say red and blue) of  $E(G)$  we necessarily get a red copy of  $F_1$  or a blue copy of  $F_2$  in  $G$ . The *size Ramsey number*  $\hat{r}(F_1, F_2)$  is the minimum number of edges in a graph  $G$  such that  $G \rightarrow (F_1, F_2)$ .

Erdős and Faudree (see [2]) mentioned that the difficulty in calculating  $\hat{r}(tK_2, C_n)$  is surprising. They proved that for a fixed  $t \geq 2$ , there exist positive constants  $c_1, c_2$ , such that  $n + c_1\sqrt{n} < \hat{r}(tK_2, C_n) < n + c_2\sqrt{n}$ . Their upper bound depends on  $t$  and it has the form

$$\hat{r}(tK_2, C_n) < n + (4t + 3)\sqrt{n}.$$

We considerably improve this bound for  $t = 2$  and  $t = 3$  by showing that  $\hat{r}(2K_2, C_n) \leq n + 2\sqrt{3n} + 9$  for  $n \geq 12$  and  $\hat{r}(3K_2, C_n) < n + 6\sqrt{n} + 9$  for  $n \geq 25$ .

## 2. RESULTS

Let us present upper bounds on the size Ramsey numbers  $\hat{r}(2K_2, C_n)$  and  $\hat{r}(3K_2, C_n)$ .

**Theorem 2.1.** *Let  $n \geq 12$ . Then  $\hat{r}(2K_2, C_n) \leq n + 2\sqrt{3n} + 9$ .*

**Proof.** Let  $k$  be an integer where  $k\varphi + 2 \leq n < (k + 1)\varphi + 2$ , and  $\varphi = \lfloor \sqrt{\frac{1}{3}n} \rfloor$ . Then we can write

$$n = k\varphi + 2 + p,$$

where  $0 \leq p \leq \varphi - 1$ . Let  $t_1, t_2 \in \mathbb{Z}$ , where  $1 \leq t_1 + 1 < t_2 \leq k + 2$  and  $(t_1, t_2) \neq (0, k + 2)$ . Let  $G$  be a graph having  $n' = (k + 3)\varphi$  vertices  $v_0, v_1, v_2, \dots, v_{n'-1}$  and

$$\begin{aligned} E(G) = & \{v_i v_{i+1} : i = 0, 1, 2, \dots, n' - 1\} \\ & \cup \{v_j v_{j+2\varphi} : j = 0, \varphi, 2\varphi, \dots, (k + 2)\varphi\} \\ & \cup \{v_{t_1\varphi} v_{(t_1+1)\varphi-p}, v_{t_2\varphi} v_{(t_2+1)\varphi-p}\}, \end{aligned}$$

the indices are taken modulo  $n'$ . Since  $n \geq 12$ , we have  $\varphi \geq 2$  and  $v_{t_1\varphi}v_{(t_1+1)\varphi-p}$ ,  $v_{t_2\varphi}v_{(t_2+1)\varphi-p} \notin \{v_i v_{i+1} : i = 0, 1, 2, \dots, n' - 1\}$ . Thus

$$|E(G)| = (k+3)\varphi + (k+3) + 2 = n - p + 3\varphi + k + 3.$$

It can be checked that  $k \leq 3\varphi + 6$  (where  $k = 3\varphi + 6$  only if  $\frac{1}{3}(n+1)$  is a square). Therefore

$$|E(G)| \leq n + 6\varphi + 9 \leq n + 2\sqrt{3n} + 9.$$

It remains to show that  $G$  contains a red  $2K_2$  or a blue  $C_n$ . Assume that  $G$  does not contain a red  $2K_2$ . We show that  $G$  contains a blue  $C_n$ . Let us consider two cases.

*Case 1.* A graph induced by red edges is a subgraph of a star. Let  $v_i$  be the center of this star. Without loss of generality we can suppose that  $1 \leq i \leq \varphi$ . We have a blue path  $v_{2\varphi}v_{2\varphi+1} \dots v_{(k+3)\varphi-1}v_0$  of length  $(k+1)\varphi$  and a blue cycle  $C' = v_0v_{2\varphi}v_{2\varphi+1} \dots v_{(k+3)\varphi-1}v_0$  of length  $(k+1)\varphi + 1$ . Then we can replace the path  $v_{t_2\varphi}v_{t_2\varphi} \dots v_{(t_2+1)\varphi}$  having length  $\varphi$  in  $C'$  by the blue path  $v_{t_2\varphi}v_{(t_2+1)\varphi-p}v_{(t_2+1)\varphi-p+1} \dots v_{(t_2+1)\varphi}$  of length  $p+1$  to obtain a blue cycle  $C$  of length  $k\varphi + p + 2 = n$ .

*Case 2.* A graph induced by red edges is a 3-cycle. The graph  $G$  contains 3-cycles only if  $p = \varphi - 2$  (cycles  $v_{t_i\varphi}v_{t_i\varphi+1}v_{t_i\varphi+2}v_{t_i\varphi}$  for  $i = 1, 2$ ). Without loss of generality we can suppose that  $t_1 = 0$  and the 3-cycle  $v_0v_1v_2v_0$  is red. Then  $G$  contains blue cycles  $C'$  and  $C$  described in the previous case. The proof is complete.  $\square$

**Theorem 2.2.** *Let  $n \geq 25$ . Then  $\hat{r}(3K_2, C_n) < n + 6\sqrt{n} + 9$ .*

*Proof.* Let  $k$  be an integer such that  $k\omega + 3 \leq n < (k+1)\omega + 3$ , where  $\omega = \lfloor \sqrt{n} \rfloor$ . Then we can write

$$(2.1) \quad n = k\omega + 3 + p,$$

where  $0 \leq p \leq \omega - 1$ . Let  $t_1, t_2, t_3 \in \mathbb{Z}$ , where  $1 \leq t_1 + 1 < t_2 < t_3 - 1 \leq k + 1$  and  $(t_1, t_3) \neq (0, k + 2)$ . Let  $G$  be a graph having  $n' = (k+3)\omega$  vertices  $v_0, v_1, v_2, \dots, v_{n'-1}$  and

$$\begin{aligned} E(G) = & \{v_i v_{i+1} : i = 0, 1, 2, \dots, n' - 1\} \cup \{v_j v_{j+\omega} : j = 0, \omega, 2\omega, \dots, n' - \omega\} \\ & \cup \left\{ v_r v_{r+\omega} : r = \left\lfloor \frac{\omega}{2} \right\rfloor, \left\lfloor \frac{3\omega}{2} \right\rfloor, \left\lfloor \frac{5\omega}{2} \right\rfloor, \dots, n' - \left\lfloor \frac{\omega}{2} \right\rfloor \right\} \\ & \cup \{v_s v_{s+2\omega-1} : s = 0, \omega, 2\omega, \dots, n' - \omega\} \cup \{v_{t_i\omega} v_{(t_i+1)\omega-p} : i = 1, 2, 3\}, \end{aligned}$$

the indices are taken modulo  $n'$ . Then

$$(2.2) \quad |E(G)| = (k+3)\omega + 3(k+3) + 3 = (n-3-p) + 3\omega + 3(k+3) + 3 \\ \leq n + 3\omega + 3k + 9.$$

It can be checked that  $k \leq \lceil \sqrt{n} \rceil$ . Note that  $k = \lceil \sqrt{n} \rceil$  only if  $n$  has the form  $n = b^2 - b + 3 + p$  (where  $0 \leq p \leq b - 4$ ). Then  $\omega = b - 1$  and  $k = b$ . We obtain

$$(2.3) \quad \omega + k = 2\left(b - \frac{1}{2}\right) < 2\sqrt{n},$$

since  $n = (b - \frac{1}{2})^2 + \frac{11}{4} + p$ . From (2.2) and (2.3) we get  $|E(G)| < n + 6\sqrt{n} + 9$ .

If  $k = \omega$ , then from (2.1) we know that  $\sqrt{n}$  is not an integer ( $\omega < \sqrt{n}$ ) and by (2.2), we obtain  $|E(G)| < n + 6\sqrt{n} + 9$ . If  $k < \omega$ , again by (2.2),  $|E(G)| < n + 6\sqrt{n} + 9$ .

It remains to show that  $G$  has a red  $3K_2$  or a blue  $C_n$ . Assume that  $G$  does not contain a red  $3K_2$ . We show that  $G$  contains a blue  $C_n$ . Let us consider a few cases.

*Case 1.* A graph induced by red edges is a subgraph of two stars. Let  $v_i$  and  $v_j$  be the centers of these stars. Without loss of generality we can suppose that  $1 \leq i \leq \omega$  and  $1 \leq j - i \leq \frac{1}{2}n'$ . If  $j \leq 2\omega - 2$ , then  $G$  has a blue path  $v_0 v_{2\omega-1} v_{2\omega}$  of length 2 and also a blue cycle  $C' = v_0 v_{2\omega-1} v_{2\omega} \dots v_{(k+3)\omega-1} v_0$  of length  $(k+1)\omega + 2$ . We can replace the path  $v_{t_3\omega} v_{t_3\omega+1} \dots v_{(t_3+1)\omega}$  of length  $\omega$  in  $C'$  by the blue path

$$v_{t_3\omega} v_{(t_3+1)\omega-p} v_{(t_3+1)\omega-p+1} \dots v_{(t_3+1)\omega}$$

of length  $p+1$  to obtain a blue cycle of length  $k\omega + p + 3 = n$ .

Let  $j \geq 2\omega - 1$ . If  $i \neq \omega$ , then  $G$  contains a blue path  $v_0 v_\omega v_{\omega+1} \dots v_{\lfloor 3\omega/2 \rfloor}$  of length  $\lfloor \frac{1}{2}\omega \rfloor + 1$ , and if  $i = \omega$ , then  $G$  contains a blue path  $v_0 v_1 \dots v_{\lfloor \omega/2 \rfloor} v_{\lfloor 3\omega/2 \rfloor}$  of length  $\lfloor \frac{1}{2}\omega \rfloor + 1$ .

Note that  $\lfloor \frac{1}{2}(c-1)\omega \rfloor < j \leq \lfloor \frac{1}{2}(c+1)\omega \rfloor$  for some even  $c \geq 4$ . If  $j \neq \lfloor \frac{1}{2}(c+1)\omega \rfloor$ , then  $v_{\lfloor (c-1)\omega/2 \rfloor} v_{\lfloor (c+1)\omega/2 \rfloor} v_{\lfloor (c+1)\omega/2 \rfloor + 1} \dots v_{(c/2+1)\omega}$  is a blue path having length  $\lfloor \frac{1}{2}\omega \rfloor + 1$ , and if  $j = \lfloor \frac{1}{2}(c+1)\omega \rfloor$ , then  $v_{\lfloor (c-1)\omega/2 \rfloor} v_{\lfloor (c-1)\omega/2 \rfloor + 1} \dots v_{(c/2)\omega} v_{(c/2+1)\omega}$  is a blue path of length  $\lfloor \frac{1}{2}\omega \rfloor + 1$ .

$G$  also contains a blue path  $v_{\lfloor 3\omega/2 \rfloor} v_{\lfloor 3\omega/2 \rfloor + 1} \dots v_{\lfloor (c-1)\omega/2 \rfloor}$  having length  $(\frac{1}{2}c - 2)\omega$  and a blue path  $v_{(c/2+1)\omega} v_{(c/2+1)\omega+1} \dots v_{(k+3)\omega-1} v_0$  having length  $(k - \frac{1}{2}c + 2)\omega$ .

Thus  $G$  contains a blue cycle  $C''$  having length

$$\left(\left\lfloor \frac{\omega}{2} \right\rfloor + 1\right) + \left(\left\lceil \frac{\omega}{2} \right\rceil + 1\right) + \left(\frac{c}{2} - 2\right)\omega + \left(k - \frac{c}{2} + 2\right)\omega = k\omega + \omega + 2.$$

From the definition of the edges  $v_{t_i\omega} v_{(t_i+1)\omega-p}$ , it follows that the cycle  $C''$  contains a path  $v_{t_i\omega} v_{t_i\omega+1} \dots v_{(t_i+1)\omega}$  of length  $\omega$  for some  $i \in \{1, 2, 3\}$ . That path can be

replaced by the blue path  $v_{t_i\omega}v_{(t_i+1)\omega-p}v_{(t_i+1)\omega-p+1}\cdots v_{(t_i+1)\omega}$  of length  $p+1$ , which implies that  $G$  has a blue cycle of length  $k\omega + p + 3 = n$ .

*Case 2.* A graph induced by red edges contains a 3-cycle.  $G$  contains 3-cycles only if  $p = 1$  (cycles  $v_{t_i\omega}v_{(t_i+1)\omega-1}v_{(t_i+1)\omega}v_{t_i\omega}$  for  $i = 1, 2, 3$ ) or  $p = \omega - 2$  (cycles  $v_{t_i\omega}v_{t_i\omega+1}v_{t_i\omega+2}v_{t_i\omega}$  for  $i = 1, 2, 3$ ).

Without loss of generality we can suppose that  $t_1 = 0$  and the 3-cycle  $v_0 v_{\omega-1} v_{\omega} v_0$  is red for  $p = 1$ , and the 3-cycle  $v_0 v_1 v_2 v_0$  is red for  $p = \omega - 2$ .

*Case 2.1.* A graph induced by red edges is a 3-cycle and a subgraph of a star. Let  $v_j$  be the center of this star. If  $j \leq 2\omega - 2$ , then  $G$  has a blue cycle  $C' = v_0 v_{2\omega-1} v_{2\omega} \cdots v_{(k+3)\omega-1} v_0$  of length  $(k+1)\omega + 2$ . Let us replace the path  $v_{t_3\omega} v_{t_3\omega+1} \cdots v_{(t_3+1)\omega}$  of length  $\omega$  in  $C'$  by the blue path

$$v_{t_3\omega} v_{(t_3+1)\omega-p} v_{(t_3+1)\omega-p+1} \cdots v_{(t_3+1)\omega}$$

of length  $p+1$  to obtain a blue cycle of length  $k\omega + p + 3 = n$ .

If  $j \geq 2\omega - 1$ , then  $G$  contains a blue path  $v_0 v_1 \cdots v_{\lfloor \frac{1}{2}\omega \rfloor} v_{\lfloor \frac{3\omega}{2} \rfloor}$  of length  $\lfloor \frac{1}{2}\omega \rfloor + 1$  for  $p = 1$ , and a blue path  $v_0 v_{\omega} v_{\omega+1} \cdots v_{\lfloor \frac{3\omega}{2} \rfloor}$  of length  $\lfloor \frac{1}{2}\omega \rfloor + 1$  for  $p = \omega - 2$ . It can be shown as in the last three paragraphs of Case 1 that  $G$  has a blue cycle of length  $n$ .

*Case 2.2.* A graph induced by red edges consists of two 3-cycles. Without loss of generality we can suppose that  $v_{z\omega} v_{z\omega+1} v_{z\omega+2} v_{z\omega}$  or  $v_{z\omega} v_{(z-1)\omega} v_{z\omega-1} v_{z\omega}$  for some  $z \geq 2$  is the other red 3-cycle. Thus  $G$  contains a blue path

$$v_{\lfloor \frac{3\omega}{2} \rfloor} v_{\lfloor \frac{3\omega}{2} \rfloor + 1} \cdots v_{\lfloor (z-1/2)\omega \rfloor} v_{\lfloor (z+1/2)\omega \rfloor} v_{\lfloor (z+1/2)\omega \rfloor + 1} \cdots v_{(k+3)\omega-1} v_0$$

of length  $k\omega + \lceil \frac{1}{2}\omega \rceil + 1$ . Note that we also have a blue path of length  $\lfloor \frac{1}{2}\omega \rfloor + 1$  between  $v_0$  and  $v_{\lfloor \frac{3\omega}{2} \rfloor}$ , which means that  $G$  contains a blue cycle  $C''$  of length  $k\omega + \omega + 2$ . This cycle contains a path  $v_{t_i\omega} v_{t_i\omega+1} \cdots v_{(t_i+1)\omega}$  for some  $i \in \{1, 2, 3\}$ . That path can be replaced by the blue path  $v_{t_i\omega} v_{(t_i+1)\omega-p} v_{(t_i+1)\omega-p+1} \cdots v_{(t_i+1)\omega}$ , so  $G$  has a blue cycle of length  $k\omega + p + 3 = n$ .

*Case 3.* A graph induced by red edges consists of a 5-cycle. Since  $n \geq 25$ , every 5-cycle of  $G$  contains an edge  $v_{t_i\omega} v_{(t_i+1)\omega-p}$  for some  $i$ , where  $1 \leq i \leq 3$ . Without loss of generality we can suppose that  $t_i = 0$  and the red 5-cycle contains the edge  $v_0 v_{\omega-p}$ .

All 5-cycles (except for two 5-cycles) containing the edge  $v_0 v_{\omega-p}$  consist only of edges connecting some of the vertices  $v_0, v_1, \dots, v_{\lfloor \frac{3\omega}{2} \rfloor}$  or some of the vertices  $v_{(k+2)\omega}, v_{(k+2)\omega+1}, \dots, v_{\lfloor \frac{1}{2}\omega \rfloor}$ . Then we have a blue cycle  $v_0 v_{2\omega-1} v_{2\omega} \cdots v_{(k+3)\omega-1} v_0$  or a blue cycle  $v_{(k+2)\omega} v_{\omega-1} v_{\omega} \cdots v_{(k+2)\omega}$  (of length  $(k+1)\omega + 2$ ). These cycles contain the path  $v_{t_3\omega} v_{t_3\omega+1} \cdots v_{(t_3+1)\omega}$ . We replace this path by the blue path  $v_{t_3\omega} v_{(t_3+1)\omega-p} v_{(t_3+1)\omega-p+1} \cdots v_{(t_3+1)\omega}$  to obtain a blue cycle of length  $k\omega + p + 3 = n$ .

Those two exceptions are the 5-cycles

$$v_0 v_{(k+2)\omega} v_{\omega-1} v_{\omega-2} v_{\omega-3} v_0 \quad \text{and} \quad v_0 v_{2\omega-1} v_{2\omega} v_{\omega} v_{\omega-1} v_0$$

(note that these cycles exist only for particular values of  $p$ ).

If the 5-cycle  $v_0 v_{(k+2)\omega} v_{\omega-1} v_{\omega-2} v_{\omega-3} v_0$  is red, then the cycle

$$v_0 v_{2\omega-1} v_{2\omega} \dots v_{(k+3)\omega-1} v_0$$

is blue, and if the 5-cycle  $v_0 v_{2\omega-1} v_{2\omega} v_{\omega} v_{\omega-1} v_0$  is red, then we have a blue cycle  $v_{\lfloor \omega/2 \rfloor} v_{\lfloor 3\omega/2 \rfloor} v_{\lfloor 5\omega/2 \rfloor} v_{\lfloor 5\omega/2 \rfloor + 1} \dots v_{\lfloor \omega/2 \rfloor}$ . These blue cycles have length  $(k+1)\omega + 2$  and it is easy to obtain a blue cycle having length  $k\omega + p + 3 = n$ .  $\square$

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