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Abstract. We proceed with Kunen's research about existence of units (left, right, two-sided) in quasigroups with classical Bol–Moufang type identities, listed in paper Extra loops II, by F. Fenyves (1969). We consider those Bol–Moufang identities where it has not been decided yet whether a quasigroup fulfilling this identity has to possess a left or right identity. We also provide a table of all Moufang–Bol identities, indicating at each whether it describes the variety of groups, and whether it forces out the left unit or the right unit.

Keywords: quasigroup; Bol–Moufang type identity; right unit; left unit

Classification: 20N05

1. Introduction

We are exploring the existence of units in quasigroups with classical Bol–Moufang type identities. We use standard quasigroup definitions and concepts, see [2], [20], [22].

Definition 1 ([3], [4], [8])**.** A groupoid $(Q,\cdot)$ is called a quasigroup if on the set Q there exist operations "$\backslash$" and "$/$" such that in algebra $(Q,\cdot,\backslash,/)$ identities

$$x\cdot(x\backslash y)=y, \tag{1}$$

$$(y/x)\cdot x=y, \tag{2}$$

$$x\backslash(x\cdot y)=y, \tag{3}$$

$$(y\cdot x)/x=y, \tag{4}$$

are fulfilled.

Definition 2. An element $i\in Q$ is an *idempotent* of $(Q,\cdot)$ if and only if $i\cdot i=i$.
An element $f\in Q$ is a *left unit* of $(Q,\cdot)$ if and only if $f\cdot x=x$ for all $x\in Q$.
An element $e\in Q$ is a *right unit* of $(Q,\cdot)$ if and only if $x\cdot e=x$ for all $x\in Q$.
An element $e\in Q$ is a *(two–sided) unit* of $(Q,\cdot)$ if and only if it is both a left and a right unit.
An element $m\in Q$ is a *middle unit* of $(Q,\cdot)$ if and only if $x\cdot x=m$ for all $x\in Q$.

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Definition 3. A quasigroup is a left (right) loop if it has a left (right) unit. A quasigroup is a loop if it has both a left and a right unit.

An identity based on a single binary operation is of Bol–Moufang type if "both sides consist of the same three different letters taken in the same order, but one of them occurs twice on each side", see [9].

We use the list and denotations of 60 Bol–Moufang type identities given in [14]. See Table 1 below.

Remark 1. There exist other ("more general") definitions of Bol–Moufang type identities and, therefore, other lists and classifications of such identities, see [1], [6].

Quasigroups and loops, in which a Bol–Moufang type identity is true, are central and classical objects of Quasigroup Theory. We recall works of R. Moufang, G. Bol, R. H. Bruck, V. D. Belousov, K. Kunen, S. Gagola and many other mathematicians are devoted to the study of quasigroups and loops with Bol–Moufang type identities.

Reformulating the title of Gagola's article "How and why Moufang loops behave like groups", see [10], we can say that quasigroups with Bol–Moufang identities "behave like groups". This is one of the reasons why we study these quasigroup classes. Notice information on right and left unit elements in quasigroups with any Bol and any Moufang identity is known [2], [11], [16], [13], [23].

In some cases we have used Prover9 and Mace4, see [19], for finding the proofs and constructing counterexamples. The closest article to our paper is Kunen's article [17]. Notice Professor Kunen in his researches have used some versions of Prover and Mace.

2. Results

2.1 (12)**-parastrophe and identities.** We recall (12)-parastrophe of a groupoid $(G, \cdot)$ is the groupoid $(G, *)$ in which the operation "$*$" is obtained by the following rule:

$$x * y = y \cdot x. \tag{5}$$

The Cayley table of the groupoid $(G, *)$ is the mirror image of the Cayley table of the groupoid $(G, \cdot)$ relative to the main diagonal. Notice, for any binary quasigroup there exist five its parastrophes, see [2], [20], [22].

Suppose that an identity F is true in the groupoid $(G, \cdot)$. Then we can obtain a (12)-parastrophic identity F^* of the identity F replacing the operation "$\cdot$" with the operation "$*$" and changing the order of variables using rule (5).

Remark 2. In quasigroup case, similarly to (12)-parastrophe identity other parastrophe identities can be defined. See [21] for details.

It is clear that an identity F is true in a groupoid $(G, \cdot)$ if and only if in groupoid $(Q, *)$ the identity F^* is true. Pairs of Bol–Moufang identities and their 12-parastrophes are given in [17] for quasigroups and in [12] for groupoids. In this paper these pairs are given in Table 2.2.

Remark 3. It is easy to see that any group satisfies any of the identities F_1–F_{60}. Therefore, the cyclic group Z_3 is a counter-example to proposition that "there exists a quasigroup with an identity from the list of identities F_1–F_{60} and which has a middle unit".

Lemma 1. *If a quasigroup $(Q, \cdot)$ has a left (right) unit element, then* (12)-*parastrophe of quasigroup $(Q, \cdot)$ has a right (left) unit element.*

Proof: It is easy to see. □

We will frequently use the following observation:

Lemma 2. *In a quasigroup $(Q, \cdot)$ for every $a \in Q$, $Q = \{ax \colon x \in Q\} = \{xa \colon x \in Q\}$.*

Lemma 3. *In a quasigroup $(Q, \cdot)$, if the identity of associativity is true, then this quasigroup is a group.*

Proof: This is a standard proof, see, for example, [18].

We substitute $x = y/y$ in the identity of associativity $x \cdot yz = xy \cdot z$. Then we have $(y/y) \cdot yz = yz$. By Lemma 2, the element y/y is a left unit of quasigroup $(Q, \cdot)$.

We substitute $z = y\backslash y$ in the identity of associativity $x \cdot yz = xy \cdot z$. Then we have $xy = xy \cdot (y\backslash y)$. By Lemma 2, the element $y\backslash y$ is a right unit of quasigroup $(Q, \cdot)$. □

Theorem 1. *A quasigroup $(Q, \cdot)$ with identity F_1 $xy \cdot zx = (xy \cdot z)x$ is a group.*

Proof: Using Lemma 2 we can rewrite identity F_1 in the form $t \cdot zx = tz \cdot x$, where $t = xy$. We obtain identity of associativity. By Lemma 3, the quasigroup $(Q, \cdot)$ is a group. □

Theorem 2.

(i) *A quasigroup $(Q, \cdot)$ with identity F_4 $xy \cdot zx = x(yz \cdot x)$ is a loop.*
(ii) *A quasigroup $(Q, \cdot)$ with identity F_6 $(xy \cdot z)x = x(y \cdot zx)$ (middle extra identity) is a loop.*
(iii) *A quasigroup $(Q, \cdot)$ with identity F_{13} $xy \cdot xz = x(yx \cdot z)$ is a loop.*

(iv) *A quasigroup* $(Q, \cdot)$ *with identity* F_{17} $(xy \cdot x)z = x(y \cdot xz)$ *(left Moufang) is a loop.*

PROOF: (i) See [11], [23]. (ii) See [17]. (iii) See [15]. (iv) See [13], [16], [23]. □

It is well known that there exist non-associative Moufang loops, see [5], and there exist non-associative extra loops, see [15]. Examples of non-associative loops with identities F_{13} and F_{22} are given in [15].

Theorem 3. *A quasigroup* $(Q, \cdot)$ *with identity* F_{32} $yx \cdot xz = (y \cdot xx)z$ *is a group.*

PROOF: We prove that a quasigroup $(Q, \cdot)$ with identity F_{32} has a left unit. In identity F_{32} $yx \cdot xz = (y \cdot xx)z$ we change $y \to x/(zz)$, $x \to z$, $z \to y$ and obtain

$$((x/zz)zz)y = ((x/zz)z) \cdot zy, \tag{6}$$

and after application of identity (2) we have

$$xy = ((x/zz)z) \cdot zy. \tag{7}$$

If we denote term zy by the letter t, then $y = z\backslash t$. And we can rewrite equality (7) in the form

$$x(z\backslash t) = ((x/zz)z) \cdot t. \tag{8}$$

If we change $x = z$ in equality (8), then we have

$$x(x\backslash t) \overset{(1)}{=} t = ((x/xx)x) \cdot t. \tag{9}$$

We proved that the quasigroup $(Q, \cdot)$ with identity F_{32} has left unit $f = ((x/xx)x)$.

If we substitute $x = e$ in identity F_{32}, then we have

$$y \cdot ez = yz, \qquad ez = z, \qquad e = f. \tag{10}$$

Since in the quasigroup $(Q, \cdot)$ there exists a left unit, then there exists a right unit too.

If in identity F_{32} we substitute $z = e$, then we have

$$yx \cdot x = y \cdot xx. \tag{11}$$

If we apply identity (11) to the right side of identity F_{32}, then we have

$$yx \cdot xz = (yx \cdot x)z. \tag{12}$$

In order to prove that a quasigroup with identity F_{32} is a group we can denote term yx in identity (12) by the letter t and use Lemmas 2 and 3. □

Theorem 4. *A quasigroup $(Q,\cdot)$ with identity F_{42} $(xx\cdot yz=(xx\cdot y)z)$ is a left loop.*

Proof: We prove that a quasigroup $(Q,\cdot)$ with identity F_{42} has a left identity element. In identity F_{42} we substitute $y=e_{xx}=xx\backslash xx$. Then we have:

$$xx\cdot e_{xx}z=(xx\cdot e_{xx})z, \qquad xx\cdot e_{xx}z=xx\cdot z, \qquad e_{xx}z=z. \tag{13}$$

□

Example 1. The following example demonstrates a left loop with identity F_{42} has no right identity element.

$\cdot$	0	1	2
0	1	2	0
1	0	1	2
2	2	0	1

2.2 Table. In Table 2.2, sign "+" in the column denoted by the letter **f** means that a quasigroup with the corresponding identity has a left unit, sign "+" in the column denoted by the letter **e** means that a quasigroup with the corresponding identity has a right unit, sign "+" in the column denoted by the word **Loop** means that a quasigruop with the corresponding identity is a loop, sign "+" in the column denoted by the word **Group** means that a quasigruop with the corresponding identity is a group.

Remark 4. By opinion of the authors, quasigroups and groupoids with some identities listed in Table 2.2 deserve further study.

Name	**Identity**	**f**	**e**	**Loop**	**Group**
$F_1=(F_3)^*$	$xy\cdot zx=(xy\cdot z)x$	+	+	+	+
$F_3=(F_1)^*$	$xy\cdot zx=x(y\cdot zx)$	+	+	+	+
$F_5=(F_{10})^*$	$(xy\cdot z)x=(x\cdot yz)x$	+	+	+	+
$F_{10}=(F_5)^*$	$x(y\cdot zx)=x(yz\cdot x)$	+	+	+	+
$F_{11}=(F_{24})^*$	$xy\cdot xz=(xy\cdot x)z$	+	+	+	+
$F_{12}=(F_{23})^*$	$xy\cdot xz=(x\cdot yx)z$	+	+	+	+
$F_{14}=(F_{21})^*$	$xy\cdot xz=x(y\cdot xz)$	+	+	+	+
$F_{18}=(F_{28})^*$	$(x\cdot yx)z=x(yx\cdot z)$	+	+	+	+
$F_{20}=(F_{25})^*$	$x(yx\cdot z)=x(y\cdot xz)$	+	+	+	+

$F_{21} = (F_{14})^*$	$yx \cdot zx = (yx \cdot z)x$	+	+	+	+
$F_{23} = (F_{12})^*$	$yx \cdot zx = y(xz \cdot x)$	+	+	+	+
$F_{24} = (F_{11})^*$	$yx \cdot zx = y(x \cdot zx)$	+	+	+	+
$F_{25} = (F_{20})^*$	$(yx \cdot z)x = (y \cdot xz)x$	+	+	+	+
$F_{28} = (F_{18})^*$	$(y \cdot xz)x = y(xz \cdot x)$	+	+	+	+
$F_{31} = (F_{34})^*$	$yx \cdot xz = (yx \cdot x)z$	+	+	+	+
$F_{32} = (F_{33})^*$	$yx \cdot xz = (y \cdot xx)z$	+	+	+	+
$F_{33} = (F_{32})^*$	$yx \cdot xz = y(xx \cdot z)$	+	+	+	+
$F_{34} = (F_{31})^*$	$yx \cdot xz = y(x \cdot xz)$	+	+	+	+
$F_{47} = (F_{58})^*$	$(x \cdot xy)z = x(xy \cdot z)$	+	+	+	+
$F_{50} = (F_{55})^*$	$x(x \cdot yz) = x(xy \cdot z)$	+	+	+	+
$F_{55} = (F_{50})^*$	$(yz \cdot x)x = (y \cdot zx)x$	+	+	+	+
$F_{58} = (F_{47})^*$	$(y \cdot zx)x = y(zx \cdot x)$	+	+	+	+
$F_4 = (F_2)^*$	$xy \cdot zx = x(yz \cdot x)$	+	+	+	-
$F_2 = (F_4)^*$	$xy \cdot zx = (x \cdot yz)x$	+	+	+	-
$F_6 = (F_6)^*$	$(xy \cdot z)x = x(y \cdot zx)$	+	+	+	-
$F_{13} = (F_{22})^*$	$xy \cdot xz = x(yx \cdot z)$	+	+	+	-
$F_{17} = (F_{27})^*$	$(xy \cdot x)z = x(y \cdot xz)$	+	+	+	-
$F_{22} = (F_{13})^*$	$yx \cdot zx = (y \cdot xz)x$	+	+	+	-
$F_{27} = (F_{17})^*$	$(yx \cdot z)x = y(x \cdot zx)$	+	+	+	-
$F_{38} = (F_{38})^*$	$(y \cdot xx)z = y(xx \cdot z)$	+	+	+	-
$F_{41} = (F_{53})^*$	$xx \cdot yz = (x \cdot xy)z$	+	+	+	-
$F_{53} = (F_{41})^*$	$yz \cdot xx = y(zx \cdot x)$	+	+	+	-
$F_7 = (F_8)^*$	$(xy \cdot z)x = x(yz \cdot x)$	+	-	-	-
$F_{16} = (F_{29})^*$	$(xy \cdot x)z = x(yx \cdot z)$	+	-	-	-
$F_{26} = (F_{19})^*$	$(yx \cdot z)x = y(xz \cdot x)$	+	-	-	-
$F_{36} = (F_{39})^*$	$(yx \cdot x)z = y(xx \cdot z)$	+	-	-	-
$F_{40} = (F_{35})^*$	$y(xx \cdot z) = y(x \cdot xz)$	+	-	-	-
$F_{42} = (F_{54})^*$	$xx \cdot yz = (xx \cdot y)z$	+	-	-	-

$F_{43} = (F_{51})^*$	$xx \cdot yz = x(x \cdot yz)$	+	-	-	-
$F_{44} = (F_{52})^*$	$xx \cdot yz = x(xy \cdot z)$	+	-	-	-
$F_{45} = (F_{60})^*$	$(x \cdot xy)z = (xx \cdot y)z$	+	-	-	-
$F_{48} = (F_{57})^*$	$(xx \cdot y)z = x(x \cdot yz)$	+	-	-	-
$F_{49} = (F_{59})^*$	$(xx \cdot y)z = x(xy \cdot z)$	+	-	-	-
$F_8 = (F_7)^*$	$(x \cdot yz)x = x(y \cdot zx)$	-	+	-	-
$F_{19} = (F_{26})^*$	$(x \cdot yx)z = x(y \cdot xz)$	-	+	-	-
$F_{29} = (F_{16})^*$	$(y \cdot xz)x = y(x \cdot zx)$	-	+	-	-
$F_{35} = (F_{40})^*$	$(yx \cdot x)z = (y \cdot xx)z$	-	+	-	-
$F_{39} = (F_{36})^*$	$(y \cdot xx)z = y(x \cdot xz)$	-	+	-	-
$F_{51} = (F_{43})^*$	$yz \cdot xx = (yz \cdot x)x$	-	+	-	-
$F_{52} = (F_{44})^*$	$yz \cdot xx = (y \cdot zx)x$	-	+	-	-
$F_{54} = (F_{42})^*$	$yz \cdot xx = y(z \cdot xx)$	-	+	-	-
$F_{57} = (F_{48})^*$	$(yz \cdot x)x = y(z \cdot xx)$	-	+	-	-
$F_{59} = (F_{49})^*$	$(y \cdot zx)x = y(z \cdot xx)$	-	+	-	-
$F_{60} = (F_{45})^*$	$y(zx \cdot x) = y(z \cdot xx)$	-	+	-	-
$F_9 = (F_9)^*$	$(x \cdot yz)x = x(yz \cdot x)$	-	-	-	-
$F_{15} = (F_{30})^*$	$(xy \cdot x)z = (x \cdot yx)z$	-	-	-	-
$F_{30} = (F_{15})^*$	$y(xz \cdot x) = y(x \cdot zx)$	-	-	-	-
$F_{37} = (F_{37})^*$	$(yx \cdot x)z = y(x \cdot xz)$	-	-	-	-
$F_{46} = (F_{56})^*$	$(x \cdot xy)z = x(x \cdot yz)$	-	-	-	-
$F_{56} = (F_{46})^*$	$(yz \cdot x)x = y(zx \cdot x)$	-	-	-	-

Table 1: Units in quasigroups with Bol–Moufang identities

Remark 5. Some of the identities F_1–F_{60} have individual names: identities F_2, F_4 are middle Moufang identities, see [2]; identities F_6, F_{13}, and F_{22} are named as extra identities, see [17]; identity F_{17} is a left Moufang identity, see [2]; identity F_{27} is a right Moufang identity, see [2]; identities F_{39}, F_{41}, F_{46}, F_{48} are called

LC identities, see [14]; identities F_{36}, F_{53}, F_{56}, F_{57} are RC identities, see [14]; identity F_{19} is a left Bol identity, see [2]; identity F_{26} is a right Bol identity, see [2]; identity F_{37} is a C identity, see [14], [7].

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