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Abstract. In this paper we consider the φ-Laplacian problem with Dirichlet boundary condition,

$$-\mathrm{div}\Big(\varphi(|\nabla u|)\frac{\nabla u}{|\nabla u|}\Big) = \lambda g(\cdot)\varphi(u) \qquad \text{in } \Omega,\ \lambda \in \mathbb{R} \ \text{ and } \ u|_{\partial\Omega} = 0.$$

The term φ is a real odd and increasing homeomorphism, g is a nonnegative function in $L^\infty(\Omega)$ and $\Omega \subseteq \mathbb{R}^N$ is a bounded domain. In these notes an analysis of the asymptotic behavior of sequences of eigenvalues of the differential equation is provided. We assume conditions which guarantee the existence of stationary solutions of the system. Under these rather stringent hypotheses we prove that any extremal is both a minimizer and an eigenfunction of the φ-Laplacian. It turns out that if, in addition, a suitable Δ_2-condition holds then any number greater than or equal to the minimum of the Rayleigh quotient is an eigenvalue of the differential equation.

Keywords: Orlicz–Sobolev space; φ-Laplacian; eigenvalue; Rayleigh quotient

Classification: 35P20, 35P30, 35J60

1. Introduction

This paper is part of a vast program devoted to the study of solutions of differential operators in the divergence form. We consider the φ-Laplacian problem with Dirichlet boundary condition

$$\begin{cases} -\mathrm{div}\Big(\varphi(|\nabla u|)\dfrac{\nabla u}{|\nabla u|}\Big) = \lambda\, g(x)\varphi(u), & x \in \Omega,\ \lambda \in \mathbb{R}, \\ u = 0 \quad \text{on } \partial\Omega. \end{cases} \tag{1.1}$$

The domain $\Omega \subseteq \mathbb{R}^N$ is bounded and satisfies a segment condition, $g \not\equiv 0$ is a nonnegative function in $L^\infty(\Omega)$ and $\varphi\colon \mathbb{R} \to \mathbb{R}$ is an odd, increasing and not necessarily differentiable homeomorphism.

The space of solutions of (1.1) has been characterized in the article [3], only in the case $\Omega = \mathbb{R}^N$. In the same article a classical Lagrange multipliers rule is employed to prove that under additional, stringent restrictions on g and φ, nontrivial

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solutions of the Laplace operator exist and are nonnegative. Regularity (Hölder continuity), positivity and vanishing at infinity of the solutions have subsequently been proved in [4]. In the article [2] the asymptotic properties of blow-up (large) solutions of (1.1) have been treated, when the right-hand side satisfies particular growth conditions and the left-hand side contains an additional nonlinear term in $|\nabla u|$. In all these references we assume a Lieberman-like condition, see [22, (1.1)], but the hypothesis of differentiability on φ is dropped. This is in striking contrast with the classical case.

Several sources in the literature address the eigenvalue problem associated with (1.1) in the particular case of the degenerate p-Laplacian equation with Dirichlet boundary condition

$$(1.2)\qquad \begin{cases} -\operatorname{div}(|\nabla u|^{p-2}\nabla u) = \overline{\lambda} g(x)|u|^{p-2}u, & x\in\Omega,\ \overline{\lambda}\in\mathbb{R}, \\ u=0 \quad \text{on } \partial\Omega \end{cases}$$

for which $\varphi(t)=|t|^{p-2}t$. Here, $1<p<\infty$ and the function g is locally integrable or belongs to some Lebesgue space. In the case $g\equiv 1$ it is known from [13], [21], [23] that an infinite sequence $\{\overline{\lambda}_n\}$ of eigenvalues exists if the associated sequence of eigenfunctions belongs to a closed subspace of the Sobolev space $W^{1,p}(\Omega)$. The supremum of the set of eigenvalues is ∞ and the infimum is equal to the first eigenvalue $\overline{\lambda}_1$ of (1.2), also called the *principal frequency*.

In the nonhomogeneous case where $q\colon\overline{\Omega}\to(1,\infty)$ is continuous ($\overline{\Omega}$ is the adherence of Ω) and $\varphi(t)=|t|^{q(x)-2}t$ for $t\neq 0$ and $\varphi(0)=0$, the eigenvalue problem was analyzed in [11]. Using Ljusternik–Schnirelmann critical point methods the authors proved that, also in this case, an infinite sequence of eigenvalues exists and that the supremum of the set of all nonnegative eigenvalues is ∞. Only under additional hypotheses the infimum of this set is positive, as in the homogeneous case $q(x)=p$. Eigenvalue problems involving quasilinear nonhomogeneous operators in other spaces were studied in [14] but in a different context. Additional resonance problems and existence of weak solutions under Landesman–Lazer conditions are tackled in [9], [10]. A characterization of the spectra and of the eigenfunctions in the Neumann case are addressed in [8]. Further approaches can be found in [25], [27].

In this work we determine the asymptotic behavior of subsequences of eigenvalues of problem (1.1) under stringent restrictions. The latter ensure the existence of minimizing sequences formed by nontrivial and nonnegative solutions of (1.1). We first take a sequence $\{\lambda_n\}$ of eigenvalues and assume that the corresponding sequence of eigenfunctions is uniformly *bounded* (in the precise sense of the norm). In Theorem 3.1 we prove that $\underline{\lambda}=\liminf\limits_{n}\lambda_n$ is itself an eigenvalue of the equation. Next, we define the Rayleigh quotient R globally on a punctured

Orlicz–Sobolev space and note that it is Fréchet differentiable there, provided a suitable Δ_2-condition be fulfilled. This last property allows to compute minima and maxima (local and global) and stationary solutions (extremals) of the eigenvalue problem (1.1). We demonstrate that any minimizing sequence $\{v_n\}$ gives always birth to a minimizing sequence $\{u_n\}$ of nontrivial and nonnegative eigenfunctions of (1.1) with the same asymptotic behavior. If the sequence $\{u_n\}$ converges to an extremal $\overline{u}$ of R then $\underline{\lambda} = \Lambda_1$. In this case the stationary solution $\overline{u}$ is both a minimizer and an eigenfunction of the problem with associated eigenvalue Λ_1.

In Theorem 4.2 we prove the existence of a minimizer provided the Rayleigh quotient be unbounded on sequences of eigenfunctions which tend to zero or infinity in norm. (This asymptotic condition is a sort of coercivity.) In addition, in this case we prove that any real number greater than or equal to the minimum value Λ_1 is an eigenvalue of the equation if a Δ_2-condition is fulfilled. These results impose stringent constraints on the system (1.1) as the Rayleigh quotient might not even admit any extremals at all. The problem whether further characterizations are possible when the coercivity condition is dropped, is still open. A description of the properties of the eigenvalues based solely on asymptotic assumptions on the associated sequence of eigenfunctions is seemingly a difficult problem.

2. Orlicz and Orlicz–Sobolev spaces

This is a brief survey on Orlicz–Sobolev spaces. For further details we refer the reader to [15], [19], [29] and, in the nonhomogeneous case of variable exponents, to [26], [27], [30].

Orlicz–Sobolev spaces somewhat generalize the classical Sobolev spaces $W^{1,p}(\Omega)$: the role played by the convex map $t \mapsto |t|^p/p$ is assumed now by a more general real map denominated an N-function. That is, a convex, even and continuous function $\Phi\colon \mathbb{R} \to [0,\infty)$ satisfying $\Phi(t) = 0$ if and only if $t = 0$ and such that

$$\frac{\Phi(t)}{t} \to 0 \quad \text{as } t \to 0 \qquad \text{and} \qquad \frac{\Phi(t)}{t} \to \infty \quad \text{as } t \to \infty.$$

Equivalently, the N-function Φ can be represented in the integral form

$$\Phi(t) = \int_0^t \varphi(s)\,\mathrm{d}s \tag{2.1}$$

where $\varphi\colon \mathbb{R} \to \mathbb{R}$ is an odd, nondecreasing, right-continuous function satisfying $\varphi(t) = 0$ if and only if $t = 0$ and $\varphi(t) \to \infty$ as $t \to \infty$. It is not hard to verify that such a map satisfies the estimate

$$\Big(\varphi(|\vec{u}|)\frac{\vec{u}}{|\vec{u}|}-\varphi(|\vec{v}|)\frac{\vec{v}}{|\vec{v}|}\Big)\cdot(\vec{u}-\vec{v})\geq 0 \tag{2.2}$$

for any pair of nonzero vectors $\vec{u},\vec{v}$ in $\mathbb{R}^N$ [6, Lemma 3.2]. If the inverse φ^{-1} exists then the integral

$$\overline{\Phi}(t)=\int_0^t \varphi^{-1}(s)\,\mathrm{d}s$$

is an N-function as well, called the conjugate (or complementary) of Φ. It is known that Young's inequality $st\leq \Phi(t)+\overline{\Phi}(s)$ holds for s and $t\in\mathbb{R}$ and that equality is attained if and only if $t=\varphi^{-1}(s)$. Therefore, $\overline{\Phi}(\varphi(t))=t\varphi(t)-\Phi(t)\leq t\varphi(t)$. Since $\Phi(2t)\geq\int_t^{2t}\varphi(s)\,\mathrm{d}s\geq\int_t^{2t}\varphi(t)\,\mathrm{d}s=t\varphi(t)$, we obtain the useful inequality

$$\overline{\Phi}(\varphi(t))\leq\Phi(2t) \tag{2.3}$$

which will be repeatedly employed in these notes.

The Sobolev conjugate N-function Φ_* of Φ is defined by

$$\Phi_*^{-1}(t)=\int_0^t\frac{\Phi^{-1}(s)}{s^{1+1/N}}\,\mathrm{d}s$$

where Φ^{-1} denotes the inverse function of $\Phi|_{[0,\infty)}$. It is known, see [7], that the Sobolev conjugate exists if and only if

$$\int_0^1\frac{\Phi^{-1}(s)}{s^{1+1/N}}\,\mathrm{d}s<\infty\qquad\text{and}\qquad\lim_{t\to\infty}\int_0^t\frac{\Phi^{-1}(s)}{s^{1+1/N}}\,\mathrm{d}s=\infty. \tag{2.4}$$

Let Φ be an N-function. The Orlicz class $\mathcal{L}_\Phi(\Omega)$ is the set of (equivalence classes of) real-valued measurable functions u such that $\Phi(u)\in L^1(\Omega)$. In general, $\mathcal{L}_\Phi(\Omega)$ is not a vector space, see [19]. The linear hull (span) $L_\Phi(\Omega)$ of the Orlicz class $\mathcal{L}_\Phi(\Omega)$ is called the Orlicz space generated by Φ. It is known that $L_\Phi(\Omega)$ is a complete space with respect to the Luxemburg norm,

$$\|u\|_{(\Phi)}=\inf\Big\{k>0\colon\int_\Omega\Phi\Big(\frac{|u|}{k}\Big)\,\mathrm{d}x\leq 1\Big\}.$$

The usual (norm) convergence in $L_\Phi(\Omega)$ is introduced as follows:

$$u_n\to u\quad\text{in }\ L_\Phi(\Omega)\qquad\text{if }\ \lim_{n\to\infty}\|u_n-u\|_{(\Phi)}=0.$$

The space $L_{\overline{\Phi}}(\Omega)$ is defined analogously (after replacing Φ by $\overline{\Phi}$ in the definitions above). It is known that if Φ and $\overline{\Phi}$ are complementary N-functions, then the *improved* Hölder's inequality

$$\int_\Omega |uv|\,\mathrm{d}x \le 2\|u\|_{(\Phi)}\|v\|_{(\overline{\Phi})} \tag{2.5}$$

holds for all $u \in L_\Phi(\Omega)$ and $v \in L_{\overline{\Phi}}(\Omega)$, see [19].

Let $E_\Phi(\Omega)$ be the closure (for the norm-topology) of $L^\infty(\Omega)$ in $L_\Phi(\Omega)$. The space $E_\Phi(\Omega)$ is separable and Banach for the inherited norm. In general, $E_\Phi(\Omega) \subseteq \mathcal{L}_\Phi(\Omega) \subseteq L_\Phi(\Omega)$ but it is known that $E_\Phi(\Omega) = L_\Phi(\Omega)$ if and only if Φ satisfies a Δ_2-condition at infinity. This means that for $r > 1$ there exists $\alpha(r) > 0$ such that

$$\Phi(rt) \le \alpha(r)\Phi(t) \qquad \text{for } t > T \tag{2.6}$$

where T is also positive. If $T = 0$ then Φ is said to satisfy a global Δ_2-condition and in this case we write $\Phi \in \Delta_2$. It is known that if Φ and $\overline{\Phi}$ satisfy a Δ_2-condition at infinity then the spaces $L_\Phi(\Omega)$ and $L_{\overline{\Phi}}(\Omega)$ are reflexive and separable, see [16]. It is also known that $L_\Phi(\Omega)$ can be identified with the dual space of $E_{\overline{\Phi}}(\Omega)$ and $L_{\overline{\Phi}}(\Omega)$ with the dual of $E_\Phi(\Omega)$, see [15].

Lemma 2.1 ([18])**.** *Let Φ be an N-function. Let $\{u_n\}$ be a sequence in $L_\Phi(\Omega)$ such that $\lim_{n\to\infty} u_n(x) = u(x)$ for a.e. $x \in \Omega$. Suppose that there exists $r \in E_\Phi(\Omega)$ such that $|u_n(x)| \le r(x)$ for a.e. $x \in \Omega$ and every n. Then $u \in E_\Phi(\Omega)$ and $u_n \to u$ in $L_\Phi(\Omega)$.*

The next proposition somewhat provides a converse to Lemma 2.1.

Proposition 2.1. *Let u be a function and $\{u_n\}$ be a sequence in $L_\Phi(\Omega)$ such that $u_n \to u$ in $L_\Phi(\Omega)$. Then there exists a subsequence $\{u_{n_k}\}$ and $h \in L_\Phi(\Omega)$ such that*

(a) *$u_{n_k}(x) \to u(x)$ for a.e. $x \in \Omega$;*
(b) *$|u_{n_k}(x)| \le h(x)$ for a.e. $x \in \Omega$.*

Proof: Since $\{u_n\}$ is a Cauchy sequence in $L_\Phi(\Omega)$ there exists a subsequence $\{u_{n_k}\}$ such that $\|u_{n_{k+1}} - u_{n_k}\|_{(\Phi)} < \varepsilon/2^k$ for any integer k. Let us define $f_m = \sum_{k=1}^m |u_{n_{k+1}} - u_{n_k}| \in L_\Phi(\Omega)$. Hence,

$$\Big\|\frac{f_m}{\varepsilon}\Big\|_{(\Phi)} \le \frac{1}{\varepsilon}\sum_{k=1}^m \|u_{n_{k+1}} - u_{n_k}\|_{(\Phi)} < 1.$$

Define $f(x) = \lim_{m\to\infty} f_m(x)$. Fatou's lemma yields

$$\int_\Omega \Phi\Big(\frac{|f(x)|}{\varepsilon}\Big)\mathrm{d}x \le \liminf_{m\to\infty} \int_\Omega \Phi\Big(\frac{|f_m(x)|}{\varepsilon}\Big)\,\mathrm{d}x \le 1$$

and hence $f \in L_\Phi(\Omega)$. On the other hand, note that if $n_m > n_k$ then

$$\begin{aligned}|u_{n_m}(x) - u_{n_k}(x)| &\le |u_{n_m}(x) - u_{n_{m-1}}(x)| + \cdots + |u_{n_{k+1}}(x) - u_{n_k}(x)| \\ &\le f(x) - f_{n_{k-1}}(x)\end{aligned}$$

and hence $\{u_{n_k}\}$ is a Cauchy sequence which converges a.e. in Ω to a function $\overline{u}$. Taking $m \to \infty$ in the inequality above produces $|\overline{u}(x) - u_{n_k}(x)| \le f(x)$ for a.e. $x \in \Omega$. Notice that $\Phi(|\overline{u}(x) - u_{n_k}(x)|/\varepsilon) \to 0$ for a.e. $x \in \Omega$ as $k \to \infty$ and also $\Phi(|\overline{u}(x) - u_{n_k}(x)|/\varepsilon) \le \Phi(f(x)/\varepsilon) \in L^1(\Omega)$. The dominated convergence thus yields

$$\lim_{k\to\infty} \int_\Omega \Phi\Big(\frac{|\overline{u}(x) - u_{n_k}(x)|}{\varepsilon}\Big)\,\mathrm{d}x = 0$$

and since ε is arbitrary, $\|\overline{u} - u_{n_k}\|_{(\Phi)} \to 0$ as $k \to \infty$. The unicity of the limit implies $u = \overline{u}$ and by the triangle inequality, $|u_{n_k}(x)| \le |u_{n_k}(x) - \overline{u}(x)| + |\overline{u}(x)| \le f(x) + |\overline{u}(x)|$. Finally, we set $h(x) = f(x) + |\overline{u}(x)|$ and the proposition is proved. $\square$

2.1 Orlicz–Sobolev spaces. The Orlicz–Sobolev space $W^1L_\Phi(\Omega)$ ($W^1E_\Phi(\Omega)$) is the vector subspace of functions in $L_\Phi(\Omega)$ ($E_\Phi(\Omega)$) with first distributional derivatives in $L_\Phi(\Omega)$ ($E_\Phi(\Omega)$, respectively). The spaces $W^1L_\Phi(\Omega)$ and $W^1E_\Phi(\Omega)$ are Banach when endowed with the norm

$$\|u\|_{1,\Phi} = \|u\|_{(\Phi)} + \|\nabla u\|_{(\Phi)}$$

where we have employed the notation $\|\nabla u\|_{(\Phi)} = \sum_{i=1}^N \|\partial u_{x_i}\|_{(\Phi)}$. Usually, $W^1L_\Phi(\Omega)$ and $W^1E_\Phi(\Omega)$ are identified with subspaces of the products ΠL_Φ and ΠE_Φ (we omit Ω to lighten notations). The natural imbedding of $W^1E_\Phi(\Omega)$ into ΠE_Φ proves that $W^1E_\Phi(\Omega)$ is separable since $E_\Phi(\Omega)$ is itself separable. The space $W^1L_\Phi(\Omega)$ is not separable in general and is closed for the weak-$*$ topology $\sigma = \sigma(\Pi L_\Phi, \Pi E_{\overline{\Phi}})$, see [15, page 167].

Let $\mathcal{D}(\Omega)$ denote the space of infinitely-differentiable functions with compact support in Ω. We define the spaces $W_0^1L_\Phi(\Omega)$ and $W_0^1E_\Phi(\Omega)$ to be respectively the σ-closure and the norm-closure of $\mathcal{D}(\Omega)$ in $W^1L_\Phi(\Omega)$:

$$W_0^1L_\Phi(\Omega) = \overline{\mathcal{D}(\Omega)}^{\,\sigma} \qquad \text{and} \qquad W_0^1E_\Phi(\Omega) = \overline{\mathcal{D}(\Omega)}^{\,\|\cdot\|_{1,\Phi}}.$$

The space $W_0^1L_\Phi(\Omega)$ is Banach with the norm $\|\cdot\|_{1,\Phi}$ inherited from $W^1L_\Phi(\Omega)$. The following Poincare's inequality [15]

$$\int_\Omega \Phi(u)\,\mathrm{d}x \le \int_\Omega \Phi(d\,|\nabla u|)\,\mathrm{d}x \tag{2.7}$$

(where d is twice the diameter of Ω), ensures that the norm

$$\|u\|_{\circ} = \|\nabla u\|_{(\Phi)}$$

is an equivalent norm on $W_0^1 L_\Phi(\Omega)$. To ease notations in this manuscript, we will write

$$Y = W_0^1 L_\Phi(\Omega) \qquad \text{and} \qquad Y_0 = W_0^1 E_\Phi(\Omega).$$

The domain Ω satisfies a segment condition, see [1, Section 3.21], if every $x \in \partial\Omega$ has a neighborhood $\mathscr{U}_x$ and a nonzero vector $y_x \in \mathbb{R}^N$ such that if $z \in \overline{\Omega} \cap \mathscr{U}_x$ then $z + ty_x \in \Omega$ for $0 < t < 1$. It is well-known, see [15], that if Ω satisfies a segment condition then $Y_0 = Y \cap \Pi E_\Phi$. Let us consider the following Banach spaces of distributions

$$Z_0 = \{\theta \in \mathcal{D}'(\Omega) \colon \theta = \theta_0 - \sum_{i=1}^{N} \frac{\partial \theta_i}{\partial x_i} \text{ with } \theta_0, \theta_i \in E_{\overline{\Phi}}(\Omega)\}$$

and

$$Z = \{\theta \in \mathcal{D}'(\Omega) \colon \theta = \theta_0 - \sum_{i=1}^{N} \frac{\partial \theta_i}{\partial x_i} \text{ with } \theta_0, \theta_i \in L_{\overline{\Phi}}(\Omega)\}$$

endowed with the quotient norms. If Ω satisfies a segment condition then the dual of Z_0 can be identified (algebraically and topologically) with Y under the natural pairing $\langle \cdot, \cdot \rangle \colon Y \times Z_0 \to \mathbb{R}$,

$$\langle u, \theta \rangle = \int_\Omega u\theta_0 \, \mathrm{d}x + \sum_{i=1}^{N} \int_\Omega \frac{\partial u}{\partial x_i} \theta_i \, \mathrm{d}x, \tag{2.8}$$

see [17]. Therefore Y can be viewed as the dual of a separable Banach space [16, Sections 2.1 and 2.2]. Similarly the dual of Y_0 can be identified with Z. The tuple $(Y, Y_0; Z, Z_0)$ is called a *complementary system*, see [15].

Lemma 2.2 ([5, Corollary III.26]). *Any bounded sequence in the dual of a separable Banach space admits a subsequence which converges in the weak-$*$ topology.*

3. The asymptotic limit of the eigenvalues

In this paper, the function φ defining the φ-Laplacian in (1.1) is to be a real, odd, increasing and not-necessarily differentiable homeomorphism of the real line and Φ is the N-function generated by φ via (2.1). In the sequel, if $r \in (0, N)$ then $r^* = Nr/(N-r)$ and $\overline{r} = r/(r-1)$ will denote the Sobolev and Hölder conjugate exponents, respectively. To guarantee some regularity on the functionals associated with our problem, we will hereafter assume the following hypotheses:

($\mathcal{H}_1$) the domain Ω satisfies a segment condition;

($\mathcal{H}_2$) the function g on the right-hand side of (1.1) is nontrivial and nonnegative in Ω;

($\mathcal{H}_3$) there exist two numbers $1 < \mathfrak{p} \le \mathfrak{q} < N$ such that

$$\lim_{t\to 0^+} \frac{t\,\varphi(t)}{\Phi(t)} = \mathfrak{p} \qquad \text{and} \qquad \lim_{t\to\infty} \frac{t\,\varphi(t)}{\Phi(t)} = \mathfrak{q}. \tag{3.1}$$

Remark 3.1. The three requirements above imply the following properties.

(i) Since $t \mapsto t\varphi(t)/\Phi(t)$ is continuous, conditions (3.1) imply that there exist two positive numbers p_Φ and q_Φ, which depend only on $\mathfrak{p}$ and $\mathfrak{q}$, such that

$$p_\Phi \le \frac{t\,\varphi(t)}{\Phi(t)} \le q_\Phi \qquad \text{for } t \neq 0. \tag{3.2}$$

In particular, $\Phi \in \Delta_2$. It is known [12, Lemma 2.5] that $p_\Phi > 1$ if and only if $\overline{\Phi} \in \Delta_2$.

(ii) The requirement $\mathfrak{q} < N$ in ($\mathcal{H}_3$) ensures that estimates (2.4) are met [18, Proposition 6.1], i.e., the N-function Φ_* exists. In turn, Theorem 3.2 in [7] guarantees that the imbedding $Y \hookrightarrow E_P(\Omega)$ is compact for any N-function P which verifies

$$\lim_{t\to\infty} \frac{P(t)}{\Phi_*(kt)} = 0 \qquad \text{for all } k > 0.$$

For example, the N-function $P = \Phi$ satisfies this condition [14, Proposition 2.1] and hence the embedding $Y \hookrightarrow L_\Phi(\Omega)$ is compact.

Since $\Phi \in \Delta_2$ we have $L_\Phi(\Omega) = \mathcal{L}_\Phi(\Omega)$. In addition, by (2.3) the functions $\varphi(u)$ and $\varphi(|\nabla u|)$ belong to $L_{\overline{\Phi}}(\Omega)$ provided $u \in W^1L_\Phi(\Omega)$. Hölder's inequality (2.5) thus implies that the following definition is consistent (i.e., the integrals on both sides of the equality are finite).

Definition 3.1. A function $u \in Y$ is called a *solution of* (1.1) if there exists $\lambda \in \mathbb{R}$ such that

$$\int_\Omega \varphi(|\nabla u|)\frac{\nabla u}{|\nabla u|} \cdot \nabla v \,\mathrm{d}x = \lambda \int_\Omega g(x)\varphi(u)v \,\mathrm{d}x$$

for all $v \in Y$. If u is a solution of (1.1) and $u \not\equiv 0$ we call λ the *eigenvalue of* (1.1) *with associated eigenfunction* u and vice-versa.

Hypothesis ($\mathcal{H}_2$) means $g \ge 0$ and the positive part $g^+ \not\equiv 0$ in Ω. The latter implies existence of nonnegative solutions of (1.1), see Theorem 4.1 below. The core result in this section is provided in the following theorem.

Theorem 3.1. *Let* $\{u_n\} \subseteq Y$ *be a sequence of eigenfunctions of* (1.1) *and let* $\{\lambda_n\}$ *denote the corresponding sequence of eigenvalues. If* $\sup_n \|u_n\|_\circ < \infty$ *then*

$$\underline{\lambda} = \liminf_{n\to\infty} \lambda_n \tag{3.3}$$

is itself an eigenvalue of (1.1).

Proof: Inequality (2.2) (with $\vec{u} = \nabla u$ and $\vec{v} = \nabla u_n$) yields the estimate

$$\int_\Omega \varphi(|\nabla u|)\frac{\nabla u}{|\nabla u|} \cdot (\nabla u - \nabla u_n)\,\mathrm{d}x \geq \lambda_n \int_\Omega g(x)\varphi(u_n)(u - u_n)\,\mathrm{d}x \tag{3.4}$$

for any integer n and $u \in Y$. Note that $\{u_n\}$ is bounded in Y which is the dual of a separable space. Lemma 2.2 implies $u_n \overset{*}{\rightharpoonup} \zeta$ in $\sigma(Y, Z_0)$ where $\zeta \in Y$. Item (ii) in Remark 3.1 and Proposition 2.1 yield, up to a subsequence, $u_n(x) \to \zeta(x)$ for a.e. $x \in \Omega$ and $u_n(x) \leq h(x)$ for a.e. $x \in \Omega$ with $h \in L_\Phi(\Omega)$. It follows that $\langle u_n, \theta\rangle \to \langle \zeta, \theta\rangle$ for all $\theta \in Z_0$ where $\langle\cdot,\cdot\rangle$ is the pairing (2.8). The remark above Definition 3.1 implies $\varphi(|\nabla u|) \in E_{\overline{\Phi}}(\Omega)$. Thus, if we choose $\theta_0 = 0$ and

$$\theta_i = \frac{\varphi(|\nabla u|)}{|\nabla u|}\frac{\partial u}{\partial x_i}$$

for all indices $i = 1, \ldots, N$, then the dual convergence $\langle u_n, \theta\rangle \to \langle \zeta, \theta\rangle$ yields

$$\int_\Omega \varphi(|\nabla u|)\frac{\nabla u}{|\nabla u|} \cdot \nabla u_n\,\mathrm{d}x \to \int_\Omega \varphi(|\nabla u|)\frac{\nabla u}{|\nabla u|} \cdot \nabla\zeta\,\mathrm{d}x$$

for any $u \in Y$. On the other hand, $\varphi(u_n) \to \varphi(\zeta)$ a.e. in Ω. Inequality (2.3) implies $\varphi(u_n) \leq \varphi(h) \in L_{\overline{\Phi}}(\Omega)$ and then Lemma 2.1 and Hölder's inequality (2.5) produce

$$\int_\Omega g(x)\varphi(u_n)u\,\mathrm{d}x \to \int_\Omega g(x)\varphi(\zeta)u\,\mathrm{d}x \quad \text{and} \quad \int_\Omega g(x)\varphi(u_n)\,u_n\,\mathrm{d}x \to \int_\Omega g(x)\varphi(\zeta)\zeta\,\mathrm{d}x.$$

If we let $n \to \infty$ in (3.4)

$$\int_\Omega \varphi(|\nabla u|)\frac{\nabla u}{|\nabla u|} \cdot (\nabla u - \nabla\zeta)\,\mathrm{d}x \geq \underline{\lambda}\int_\Omega g(x)\,\varphi(\zeta)\,(u - \zeta)\,\mathrm{d}x$$

for any $u \in Y$, where $\underline{\lambda} = \liminf_{n\to\infty} \lambda_n$. We write $u = \zeta + t\,v$ where t is a nonzero real number and $v \in Y$. If $t > 0$ then

$$\int_\Omega \varphi(|\nabla(\zeta + tv)|)\frac{\nabla(\zeta + t\,v)}{|\nabla(\zeta + tv)|} \cdot \nabla v\,\mathrm{d}x \geq \underline{\lambda}\int_\Omega g(x)\varphi(\zeta)v\,\mathrm{d}x$$

and the reversed inequality is obtained for $t < 0$. Since t is arbitrary and as φ is increasing, taking the limit $t \to 0$ yields

$$\int_\Omega \varphi(|\nabla\zeta|)\frac{\nabla\zeta}{|\nabla\zeta|}\cdot\nabla v\,\mathrm{d}x = \underline{\lambda}\int_\Omega g(x)\varphi(\zeta)v\,\mathrm{d}x$$

for all $v \in Y$. The conclusion follows. $\square$

4. Connection with the Rayleigh quotient

Let us denote by $\mathbb{R}_+$ the set of nonnegative real numbers. Definition 3.1 motivates the introduction of the operators $J, G\colon Y \to \mathbb{R}_+ \cup \{\infty\}$

$$J(u) = \int_\Omega \Phi(|\nabla u|)\,\mathrm{d}x \qquad \text{and} \qquad G(u) = \int_\Omega g(x)\Phi(u)\,\mathrm{d}x.$$

It is known that J is finite since $\Phi \in \Delta_2$ (the converse is also true). The compact embedding in item (ii) of Remark 3.1 ensures that G is finite as well [28, Remark 3.1]. It is also known [14, Lemma 3.4] that if $\overline{\Phi} \in \Delta_2$ then J is of class C^1 with Fréchet derivative

$$J'(u)(v) = \int_\Omega \varphi(|\nabla u|)\frac{\nabla u}{|\nabla u|}\cdot\nabla v\,\mathrm{d}x, \qquad u, v \in Y.$$

On the contrary, G is always of class C^1 [17, page 898] with Fréchet derivative

$$G'(u)(v) = \int_\Omega g(x)\,\varphi(u)v\,\mathrm{d}x, \qquad u, v \in Y.$$

Integration of (3.2) and the definition of the Luxemburg norm yield the useful estimates

$$\min\{\|u\|_{\mathrm{o}}^{p_\Phi}, \|u\|_{\mathrm{o}}^{q_\Phi}\} \le J(u) \le \max\{\|u\|_{\mathrm{o}}^{p_\Phi}, \|u\|_{\mathrm{o}}^{q_\Phi}\} \qquad \text{for any } u \in Y. \tag{4.1}$$

Lemma 4.1 ([28, Lemma 3.2]). *Let $u_n \overset{*}{\rightharpoonup} u$ in Y for the weak-$*$ topology $\sigma(Y, Z_0)$. Then:*

(a) *the operator J is $\sigma(Y, Z_0)$-lower-semi-continuous: $J(u) \le \liminf J(u_n)$;*
(b) *the operator G is $\sigma(Y, Z_0)$-continuous: $G(u_n) \to G(u)$.*

The next theorem is proved in [17]. In the unbounded case $\Omega = \mathbb{R}^N$ this result remains true only if additional conditions on the right-hand side of (1.1) are met, see [3].

Theorem 4.1. *Let μ be a positive number. The optimization problem*

$$\inf\{J(u)\colon\ u \in Y,\ G(u) = \mu\} \tag{4.2}$$

has a nontrivial solution $u_\mu \in Y$. Moreover, define the nonzero number

$$\lambda_\mu = \frac{\int_\Omega \varphi(|\nabla u_\mu|)|\nabla u_\mu|\,\mathrm{d}x}{\int_\Omega g(x)\,\varphi(u_\mu)u_\mu\,\mathrm{d}x}. \tag{4.3}$$

Then u_μ is a nonnegative eigenfunction of problem (1.1) *with associated eigenvalue $\lambda = \lambda_\mu$.*

Observe that bounds (3.2) imply

$$\int_\Omega g(x)\varphi(u_\mu)u_\mu\,\mathrm{d}x \geq p_\Phi \int_\Omega g(x)\Phi(u_\mu)\,\mathrm{d}x = p_\Phi\,\mu > 0$$

and therefore the quantity λ_μ is well defined.

4.1 Rayleigh quotient. In the sequel we denote by $Y^* = Y\backslash\{0\}$. Define the Rayleigh quotient $R\colon Y^* \to \mathbb{R}$ by the formula $R(u) = J(u)/G(u)$. Inequality (2.7) and the Δ_2-condition on Φ imply

$$R(u) \geq \frac{1}{\alpha(d)\,\|g\|_\infty} > 0$$

where $\alpha(d)$ is the constant in (2.6), d is twice the diameter of Ω and $\|\cdot\|_\infty$ is the norm in $L^\infty(\Omega)$. In particular, the number $\Lambda_1 = \inf\{R(u)\colon u \in Y^*\}$ is positive.

Definition 4.1. A nonzero sequence $\{v_n\} \subseteq Y$ with

$$\lim_{n\to\infty} R(v_n) = \Lambda_1$$

is called *R-minimizing*. A sequence $\{v_n\} \subseteq Y$ is *bounded* if $\sup_n \|v_n\|_\circ < \infty$. The sequence is *asymptotic to zero* if $\|v_n\|_\circ \to 0$ and is *asymptotic to infinity* if $\|v_n\|_\circ \to \infty$.

Lemma 4.2. *Let $\{v_n\} \subseteq Y$ be an R-minimizing sequence. There exists a sequence $\{u_n\} \subseteq Y$ satisfying the following properties:*

- (a) *the element u_n is a nontrivial and nonnegative eigenfunction of* (1.1) *for every integer n;*
- (b) *the sequence $\{u_n\}$ is R-minimizing and the associated sequence of eigenvalues $\{\lambda_n\}$ is bounded;*
- (c) *the sequence $\{u_n\}$ is either asymptotic to zero, asymptotic to infinity or bounded if and only if $\{v_n\}$ is asymptotic to zero, asymptotic to infinity or bounded, respectively.*

PROOF: For each nonnegative integer n we define the number $\mu_n = G(v_n)$. Since Λ_1 is positive it must necessarily be that $\mu_n > 0$ as well for n sufficiently large.

Then we let $u_n \in Y$ be the (nontrivial, nonnegative) solution to (4.2) with $\mu = \mu_n$. Formula (4.3) provides an eigenvalue $\lambda_n \in \mathbb{R}$ with corresponding eigenfunction u_n.

On the other hand, by definition we have $J(u_n) \leq J(v_n)$ and $G(u_n) = G(v_n) = \mu_n > 0$ for n sufficiently large. Hence, there exists a real sequence $\varepsilon_n \to 0$ such that $\Lambda_1 \leq R(u_n) \leq R(v_n) \leq \Lambda_1 + \varepsilon_n$ and this proves that $\{u_n\}$ is R-minimizing. Note that this chain of inequalities yields

$$\frac{\Lambda_1}{R(v_n)} \leq \frac{R(u_n)}{R(v_n)} = \frac{J(u_n)}{J(v_n)} \qquad \text{and} \qquad \frac{1}{\Lambda_1 + \varepsilon_n} \leq \frac{1}{R(v_n)}$$

for every positive integer n. Merging together the latter estimates produces

$$\frac{\Lambda_1}{\Lambda_1 + \varepsilon_n} J(v_n) \leq J(u_n).$$

Since $J(u_n) \leq J(v_n)$ the statement (b) is thus a simple consequence of estimates (4.1). On the other hand inequalities (3.2) yield

$$\frac{p_\varphi}{q_\varphi} R(u_n) \leq \lambda_n \leq \frac{q_\varphi}{p_\varphi} R(u_n)$$

and hence the sequence of eigenvalues is bounded. □

4.2 The energy operator. Let Λ be a positive number. The functional $T_\Lambda : Y \to \mathbb{R}$,

$$T_\Lambda(u) = J(u) - \Lambda\, G(u) \tag{4.4}$$

is called the energy operator associated with equation (1.1). Define

$$r_0(\Lambda) = \inf\{T_\Lambda(u) \colon u \in Y\}.$$

By analogy, a nonzero sequence $\{v_n\} \subseteq Y$ satisfying $\lim_{n\to\infty} T_\Lambda(v_n) = r_0(\Lambda)$ is called T_Λ-minimizing. Minor modifications to the proof of Lemma 4.2 yield the following result.

Lemma 4.3. *Let $\{v_n\} \subseteq Y$ be a T_Λ-minimizing sequence. If $\Lambda > \Lambda_1$ then there exists a sequence $\{u_n\} \subseteq Y$ satisfying the following properties:*

(a) *the element u_n is a nontrivial and nonnegative eigenfunction of* (1.1) *for every integer n;*
(b) *the sequence $\{u_n\}$ is T_Λ-minimizing;*
(c) *the sequence $\{u_n\}$ is either asymptotic to zero, asymptotic to infinity or bounded if and only if $\{v_n\}$ is asymptotic to zero, asymptotic to infinity or bounded, respectively.*

PROOF: For each n we define the number $\mu_n = G(v_n)$. Notice that $\mu_n \geq 0$ for all n since Φ is even and increasing. It is obvious as well that $G(v_n) = 0$ only for a finite number of integers n because otherwise there would exist an infinite subsequence $\{v_{n_k}\}$ for which $r_0(\Lambda) = \lim_{k\to\infty} T_\Lambda(v_{n_k}) = \lim_{k\to\infty} J(v_{n_k}) \geq 0$. However, since $\Lambda > \Lambda_1$, there exists $\overline{v} \in Y$ such that $\Lambda_1 < R(\overline{v}) < \Lambda$ and this implies $T_\Lambda(\overline{v}) < 0$. Therefore, we can assume that μ_n is strictly positive for all n. We define $u_n \in Y$ to be the (nontrivial, nonnegative) solution to (4.2) with $\mu = \mu_n$.

On the other hand, there exists a real sequence $\varepsilon_n \to 0$ such that $r_0(\Lambda) \leq T_\Lambda(u_n) \leq T_\Lambda(v_n) \leq r_0(\Lambda) + \varepsilon_n$ and then $\{u_n\}$ is T_Λ-minimizing. Moreover, since $J(u_n) \leq J(v_n)$, estimates (4.1) imply that if $\{v_n\}$ is asymptotic to zero then so is $\{u_n\}$ and if the latter is asymptotic to infinity then so is $\{v_n\}$. Further, simple rearrangement of the terms in the inequalities above yields in this case

$$\frac{r_0(\Lambda)}{r_0(\Lambda) + \varepsilon_n} J(v_n) \leq J(u_n) - \Lambda G(u_n) \frac{\varepsilon_n}{r_0(\Lambda) + \varepsilon_n} \leq J(u_n)$$

for any natural n. It is thus clear that if $\{v_n\}$ is asymptotic to infinity then so is $\{u_n\}$ and if the latter is asymptotic to zero then so is the former sequence. $\square$

4.3 Extremals. Let $\overline{\Phi} \in \Delta_2$. It is easy to see that the energy functional (4.4) is Fréchet differentiable with Fréchet derivative $T'_\Lambda(u)(v) = J'(u)(v) - \Lambda\, G'(u)(v)$ for all $u, v \in Y$. The differentiability of the functionals J and G ensures that the Rayleigh quotient is Fréchet differentiable as well at any $u \in Y$ such that $G(u) \neq 0$. The Gâteaux variations of J and G exist in any directions and thus

$$\delta R = \frac{J + \delta J}{G + \delta G} - \frac{J}{G} = \frac{J + \delta J}{G}\Big(1 + \frac{\delta G}{G}\Big)^{-1} - \frac{J}{G}$$

exists as well in any direction. A first-order Taylor expansion for the term $(1 + \delta G/G)^{-1}$ produces

$$\delta R = \frac{\delta J}{G} - \frac{J \delta G}{G^2} \tag{4.5}$$

(again, to first order). Formula (4.5) hence yields the Fréchet derivative R' of the Rayleigh quotient. (The Fréchet-differentiability of R is easily obtained as well from the product and chain rules, see [20, Chapter XIII, Section 3].) Therefore, $R' = 0$ if and only if $J' - RG' = 0$.

Consider an R-minimizing sequence $\{u_n\} \subseteq Y$ of eigenfunctions of (1.1) and let $\{\lambda_n\}$ denote the associated sequence of eigenvalues. Suppose that $\{u_n\}$ converges to an extremal or stationary function $\overline{u} \in Y^*$ of the Rayleigh quotient (i.e. $R'(\overline{u})(v) = 0$ for all $v \in Y$). Since $G(u_n) > 0$ for n sufficiently large, formula

(4.5) yields

$$R'(u_n)(v) = \big(\lambda_n - R(u_n)\big)\frac{G'(u_n)(v)}{G(u_n)} \tag{4.6}$$

where $v \in Y$, provided $\overline{\Phi} \in \Delta_2$. Since $\overline{u}$ is nontrivial the continuity of G implies $G(\overline{u}) > 0$ and it is clear that $G'(\overline{u})(\cdot) \not\equiv 0$ on Y. Otherwise, hypotheses (3.2) would imply

$$0 = G'(\overline{u})(\overline{u}) = \int_\Omega g(x)\varphi(\overline{u})\overline{u}\,\mathrm{d}x \geq p_\Phi\, G(\overline{u}) > 0.$$

Taking $n \to \infty$ in (4.6) produces $\underline{\lambda} = \Lambda_1$ (by the continuity of the derivative) and then Λ_1 is an eigenvalue of (1.1) with associated eigenfunction $\overline{u}$. It is evident as well that $\overline{u}$ minimizes the Rayleigh quotient: $\Lambda_1 = R(\overline{u})$. Hence, the existence of extremals yields a characterization of the quantity (3.3). This condition is automatically verified if an asymptotic hypothesis on the Rayleigh quotient is fulfilled.

Theorem 4.2. *Let $\overline{\Phi} \in \Delta_2$. Suppose that the following asymptotic condition*

$$\lim_{n\to\infty} R(u_n) = \infty \tag{4.7}$$

is fulfilled on any subsequence $\{u_n\} \subseteq Y$ of eigenfunctions of (1.1) *which is either asymptotic to zero or infinity. Then there exists a global minimum $\overline{u} \in Y^*$ of the Rayleigh quotient. In this case, Λ_1 is an eigenvalue of the system with corresponding eigenfunction $\overline{u}$. Moreover,*

(a) *any $\Lambda > \Lambda_1$ is an eigenvalue of* (1.1)*;*
(b) *there exists $0 < \lambda_0 \leq \Lambda_1$ such that $(0, \lambda_0)$ contains no eigenvalue of* (1.1).

Proof: We first take an R-minimizing sequence $\{u_n\}$ in Y^*. Without loss of generality, by Lemma 4.2 we can assume that u_n is a nontrivial and nonnegative eigenfunction of (1.1) for any natural number n. It is clear that $\sup_n \|u_n\|_\circ < \infty$ (otherwise condition (4.7) yields a contradiction). Since Y is the dual of a separable space, Lemma 2.2 implies that there exists $\overline{u} \in Y$ such that $u_n \overset{*}{\rightharpoonup} \overline{u}$ in the weak-$*$ topology $\sigma(Y, Z_0)$. Lemma 4.1 thus yields $\liminf_{n\to\infty} J(u_n) \geq J(\overline{u})$ and $\lim_{n\to\infty} G(u_n) = G(\overline{u})$. Therefore, $R(\overline{u}) = \Lambda_1$ provided $\overline{u} \not\equiv 0$. Assume, on the contrary, that $\overline{u} \equiv 0$. The definition of the limit gives

$$(\Lambda_1 - \varepsilon)G(u_n) \leq J(u_n) \leq (\Lambda_1 + \varepsilon)G(u_n)$$

where $\varepsilon < \Lambda_1$ is a positive number. Since $G(u_n) \to 0$ as $n \to \infty$ we deduce $J(u_n) \to 0$ as well. Then inequality (4.1) yields $\|u_n\|_\circ \to 0$ as $n \to \infty$. But then condition (4.7) implies $\lim_{n\to\infty} R(u_n) = \infty$ and thus we obtain a contradiction, since $\{u_n\}$ is R-minimizing. Since $\overline{u}$ is a local minimum of R the derivative vanishes there and then

$$\frac{J'(\overline{u})(v)}{G'(\overline{u})(v)} = \frac{J(\overline{u})}{G(\overline{u})} = \Lambda_1$$

for every $v \in Y$ and thus Λ_1 is an eigenvalue of (1.1).

To prove the second part of the theorem, we take a number $\Lambda > \Lambda_1$ and consider $r_0(\Lambda) = \inf_Y T_\Lambda$ and choose a T_Λ-minimizing sequence $\{u_n\}$. Lemma 4.3 allows us to assume that u_n is a nontrivial and nonnegative eigenfunction of problem (1.1) for every nonzero integer n. It is again clear that $\{u_n\}$ is uniformly bounded in Y. Indeed,

$$T_\Lambda(u_n) = J(u_n)\Big(1 - \frac{\Lambda}{R(u_n)}\Big).$$

Thus if the sequence $\{u_n\}$ were unbounded, inequalities (4.1) and condition (4.7) would imply $T_\Lambda(u_n) \to \infty$ as $n \to \infty$ and this is impossible. Lemma 2.2 implies that there exists $u_\Lambda \in Y$ such that $u_n \overset{*}{\rightharpoonup} u_\Lambda$ in $\sigma(Y, Z_0)$. By Lemma 4.1, the functional T_Λ is $\sigma(Y, Z_0)$-lower-semi-continuous and then $T_\Lambda(u_\Lambda) \le \liminf T_\Lambda(u_n) = r_0(\Lambda)$. Thus u_Λ is a global minimum of T_Λ and hence a stationary solution of that functional, i.e. $T'_\Lambda(u_\Lambda)(v) = 0$ for all $v \in Y$ and thus u_Λ is an eigenfunction with associated eigenvalue Λ. Since $\Lambda > \Lambda_1$ it follows that there exists $v_\Lambda \in Y$ such that $R(v_\Lambda) < \Lambda$. That is, $T_\Lambda(v_\Lambda) < 0$ and hence $\inf\{T_\Lambda(u) \colon u \in Y\} < 0$. Since $T_\Lambda(0) = 0$ we have $u_\Lambda \not\equiv 0$.

Next, note that the function $\mathfrak{r} \colon Y^* \to \mathbb{R} \cup \{\infty\}$ given by $\mathfrak{r}(u) = (J'(u)(u))/(G'(u)(u))$ is well defined since J and G are differentiable. In this case again, Poincaré's inequality (2.7) and bound (3.2) imply

$$\mathfrak{r}(u) \ge 1\alpha(d)\,\|g\|_\infty\, q_\Phi > 0$$

where $\alpha(d)$ is the constant in (2.6) and d is twice the diameter of Ω. Hence the number

$$\lambda_0 := \inf\{\mathfrak{r}(u) \colon u \in Y^*\}$$

is strictly positive. Let us assume that $\lambda \in (0, \lambda_0)$ is an eigenvalue of (1.1) with the associated eigenfunction $u_\lambda \in Y^*$. Then

$$\int_\Omega \varphi(|\nabla u_\lambda|)|\nabla u_\lambda| \,\mathrm{d}x = \lambda \int_\Omega g(x)\varphi(u_\lambda)u_\lambda \,\mathrm{d}x.$$

Since the left-hand side of the latter equality is positive and as $\lambda < \lambda_0$ we have

$$J'(u_\lambda)(u_\lambda) \ge \lambda_0\, G'(u_\lambda)(u_\lambda) > \lambda\, G'(u_\lambda)(u_\lambda) = J'(u_\lambda)(u_\lambda).$$

This is a contradiction. In particular $\lambda_0 \le \Lambda_1$. □

The next example shows the importance of hypothesis $(\mathcal{H}_3)$ in Section 3.

Example 4.1. Consider the 2-Laplacian boundary-eigenvalue problem

$$\begin{aligned} -u'' &= \overline{\lambda} u, \qquad && x \in [0,\pi],\ \overline{\lambda} \in \mathbb{R}, \\ u(0) &= u(\pi) = 0 \end{aligned}$$

on the Sobolev space $W_0^{1,2}([0,\pi])$. In this case it is easily checked that $\|u\|_\circ = \|u'\|_{L^2[0,\pi]}/\sqrt{2}$ for any $u \in W_0^{1,2}([0,\pi])$ and the Rayleigh quotient corresponds to

$$R(u) = \frac{\int_0^\pi (u'(x))^2 \,\mathrm{d}x}{\int_0^\pi (u(x))^2 \,\mathrm{d}x} = \frac{\|u'\|^2_{L^2[0,\pi]}}{\|u\|^2_{L^2[0,\pi]}}.$$

A simple calculation shows that the sequence of eigenfunctions of this problem is given by $u_n = A_n \sin(nx)$ where n is any integer and A_n an arbitrary real constant. If $\{\overline{\lambda}_n\}$ denotes the sequence of eigenvalues associated with $\{u_n\}$ then integration by parts proves that $R(u_n) = \overline{\lambda}_n = n^2$ for any integer n (and hence $\Lambda_1 = 1$). Notice that in this case the definition of the norm yields

$$\|u_n\|_\circ = n|A_n|\frac{\sqrt{\pi}}{2}$$

for any integer n. Therefore, the sequence $\{u_n\}$ is asymptotic to zero or infinity provided $n|A_n|$ tends to zero or infinity, respectively, as $n \to \infty$. In any case condition (4.7) is met but the existence of the minimizer does not follow from Theorem 4.2 since the spectrum is discrete in the one-dimensional case, see [24].

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