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ON OSCILLATORY FIRST ORDER NEUTRAL IMPULSIVE DIFFERENCE EQUATIONS

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Abstract. We have established sufficient conditions for oscillation of a class of first order neutral impulsive difference equations with deviating arguments and fixed moments of impulsive effect.

Keywords: oscillation; nonoscillation; impulsive difference equation; nonlinear neutral difference equation; delay

MSC 2010: 39A10, 39A12

1. Introduction

Consider a class of first order nonlinear neutral difference equations of the form

$$\Delta(y(n)+p(n)y(n-\tau))+q(n)F(y(n-\sigma))=0, \tag{1.1}$$

where p, q are real valued functions with discrete arguments such that $q(n)>0$, $|p(n)|<\infty$ for $n\in\mathbb{N}(n_0)=\{n_0,n_0+1,\ldots\}$, $F\in C(\mathbb{R},\mathbb{R})$ satisfying the property $xF(x)>0$ for $x\neq 0$ and Δ is the forward difference operator defined by $\Delta u(n)=u(n+1)-u(n)$. Let $m_1,m_2,m_3,\ldots$ be the moments of impulsive effect with the property

(A_0) $0<m_1<m_2<\ldots,\ \lim\limits_{j\to\infty} m_j=\infty$

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for the neutral equation (1.1) satisfying

$$\underline{\Delta}(y(m_j-1)+p(m_j-1)y(m_j-\tau-1))+r(m_j-1)F(y(m_j-\sigma-1))=0, \tag{1.2}$$

where $\tau,\sigma>0$ are integers, $r>0$ is a real valued function and $\underline{\Delta}$ is the difference operator defined by

$$\begin{aligned}&\underline{\Delta}(y(m_j-1)+p(m_j-1)y(m_j-\tau-1))\\&\qquad=y(m_j)+p(m_j)y(m_j-\tau)-(y(m_j-1)+p(m_j-1)y(m_j-\tau-1)).\end{aligned}$$

Many researchers have profound a good deal of research work on oscillatory and asymptotic behaviour of solutions of (1.1) (see for e.g. [6], [7], [9], [8]). Eventhough, (1.2) is another difference equation, still less attention has been given for its study. Moreover, there is no such work for (1.1) when the impulsive equation (1.2) joins to form an impulsive difference system of the form

$$(\mathrm{E}_1)\qquad\begin{cases}\Delta(y(n)+p(n)y(n-\tau))+q(n)F(y(n-\sigma))=0, & n\neq m_j,\ j\in\mathbb{N},\\ \underline{\Delta}(y(m_j-1)+p(m_j-1)y(m_j-\tau-1))\\ \qquad\qquad+r(m_j-1)F(y(m_j-\sigma-1))=0.\end{cases}$$

In this work, our objective is to study the oscillatory behaviour of solutions of system (E_1) when $|p(n)|<\infty$. For details about the impulsive differential/difference equations we refer the reader to the monograph [1] and some of the works [2], [3], [10]–[15] and the references cited therein.

In [4], Li et al. have established the oscillation criteria for third order difference equations with impulse of the form

$$(\mathrm{E}_2)\qquad\begin{cases}\Delta^3y(n)+p(n)y(n-\tau)=0, & n\neq n_k,\\ y(n_k)=a_ky(n_k-1), & k\in\mathbb{N},\\ \Delta y(n_k)=b_k\Delta y(n_k-1), & k\in\mathbb{N},\\ \Delta^2y(n_k)=c_k\Delta^2y(n_k-1), & k\in\mathbb{N}\end{cases}$$

and the same is extended in [5] for nonlinear third order difference equations of the form

$$(\mathrm{E}_2')\qquad\begin{cases}\Delta^3y(n)+p(n)f(y(n-\tau))=0, & n\neq n_k,\\ \Delta^iy(n_k)=g_{i,k}\Delta^iy(n_k-1), & i=0,1,2,\ k\in\mathbb{N},\end{cases}$$

where $a_{i,k}\leqslant g_{i,k}(u)/u\leqslant b_{i,k}$. Unlike the above method, our impulsive effect satisfies another neutral equation (1.2) subject to the difference equation (1.1). The present work for the impulsive difference system (E_1) is a different approach as compared to the existing works in the literature. We may note that in present years much effort has been given to the study of functional difference equations of neutral type. However, the impulsive difference equations of neutral type especially (E_1) is not well studied. Hence, in this work, an attempt is made to study the impulsive system (E_1).

Definition 1.1. By a solution of (E_1) we mean a real valued function $y(n)$ defined on $\mathbb{N}(n_0-\varrho)$ which satisfies (E_1) for $n \geqslant n_0$ with the initial conditions

$$y(i)=\varphi(i), \quad i=n_0-\varrho,\dots,n_0,$$

where $\varphi(i)$, $i=n_0-\varrho,\dots,n_0$ are given real constants and $\varrho=\max\{\tau,\sigma\}$.

Definition 1.2. A nontrivial solution $y(n)$ of (E_1) is said to be nonoscillatory if it is either eventually positive or eventually negative. Otherwise, the solution is called oscillatory. (E_1) is said to be oscillatory if all its solutions are oscillatory.

Definition 1.3. A solution $y(n)$ of (E_1) is said to be regular if it is defined on $\mathbb{N}(0)$ and $\sup\{|y(n)|\colon n \geqslant N > 0\} > 0$, where N is a positive integer. A regular solution $y(n)$ of (E_1) is said to be eventually positive (eventually negative) if there exists $n_0>0$ such that $y(n)>0$ ($y(n)<0$) for $n \geqslant n_0$.

2. Oscillation criteria

In this section, we discuss the oscillation properties of solutions of the impulsive system (E_1). Throughout our discussion we use the following notations:

$$\begin{cases} z(n)=y(n)+p(n)y(n-\tau), \\ z(m_j-1)=y(m_j-1)+p(m_j-1)y(m_j-\tau-1). \end{cases} \tag{2.1}$$

Theorem 2.1. *Let $-\infty < -\alpha \leqslant p(n) \leqslant -1$, $\alpha>0$. Assume that (A_0) and $\tau \geqslant \sigma$ hold. Furthermore, assume that*

(A_1) $F(-u)=-F(u)$, $u\in\mathbb{R}$,

(A_2) $F(uv)=F(u)F(v)$, $u,v\in\mathbb{R}$,

(A_3) *F is superlinear and*

$$\int_{\pm c}^{\pm\infty}\frac{\mathrm{d}x}{F(x)}<\infty,\ c>0, \quad \sum_{j=1}^{\infty}\int_{z(m_j-1)}^{z(m_j)}\frac{\mathrm{d}x}{F(x)}<\infty,$$

(A_4) $\sum\limits_{n=1}^{\infty}q(n)+\sum\limits_{j=1}^{\infty}r(m_j-1)=\infty$, $m_j>1$

hold. Then (E_1) is oscillatory.

P r o o f. On the contrary, let $y(n)$ be a regular solution of (E_1) such that $y(n) > 0$ or $y(n) < 0$ for $n \geqslant n_0$. Without loss of generality and due to (A_1), we may assume that $y(n) > 0$, $y(n-\tau) > 0$, $y(n-\sigma) > 0$ for $n \geqslant n_1 = n_0 + \varrho$. Using (2.1) in (E_1), we obtain

$$\Delta z(n) = -q(n)F(y(n-\sigma)) \leqslant 0, \quad n \neq m_j, \tag{2.2}$$

$$\underline{\Delta} z(m_j - 1) = -r(m_j - 1)F(y(m_j - \sigma - 1)) \leqslant 0, \quad j \in \mathbb{N} \tag{2.3}$$

for $n \geqslant n_1$. So, there exists $n_2 > n_1$ such that $z(n)$ is nonincreasing for $n \geqslant n_2$. We assert that $z(n) < 0$ for $n \geqslant n_2$. If not, let there exist $n_3 > n_2$ such that $z(n) \geqslant 0$ for $n \geqslant n_3$. As a result,

$$y(n) \geqslant -p(n)y(n-\tau) \geqslant y(n-\tau) \geqslant y(n-2\tau) \geqslant \ldots \geqslant y(n_3)$$

implies that $y(n)$ is bounded from below by a positive constant (say) B. Analogously,

$$y(m_j - 1) \geqslant y(m_j - \tau - 1) \geqslant y(m_j - 2\tau - 1) \geqslant \ldots \geqslant y(n_3)$$

due to nonimpulsive points $m_j - 1$, $m_j - \tau - 1, \ldots$, and so on. Summing (2.2) from n_3 to $n-1$ and then using (2.3), we obtain

$$\sum_{s=n_3}^{n-1} \Delta z(s) + \sum_{s=n_3}^{n-1} q(s)F(y(s-\sigma)) = 0,$$

that is,

$$z(n) - z(n_3) - \sum_{n_3 \leqslant m_j - 1 \leqslant n-1} \underline{\Delta} z(m_j - 1) + \sum_{s=n_3}^{n-1} q(s)F(y(s-\sigma)) = 0.$$

Therefore

$$z(n) = z(n_3) - \sum_{n_3 \leqslant m_j - 1 \leqslant n-1} r(m_j - 1)F(y(m_j - \sigma - 1)) - \sum_{s=n_3}^{n-1} q(s)F(y(s-\sigma))$$

implies that

$$z(n) \leqslant z(n_3) - F(B)\Bigg(\sum_{s=n_3}^{n-1} q(s) + \sum_{n_3 \leqslant m_j - 1 \leqslant n-1} r(m_j - 1)\Bigg) \to -\infty \quad \text{as } n \to \infty,$$

a contradiction to the fact that $z(n)>0$ for $n\geqslant n_3$. Hence, $z(n)<0$ for $n\geqslant n_2$. Therefore, we can find an $n_3>n_2$ such that

$$z(n)>p(n)y(n-\tau)\geqslant -\alpha y(n-\tau),$$
$$z(m_j-1)>p(m_j-1)y(m_j-\tau-1)\geqslant -\alpha y(m_j-\tau-1)$$

implies that $z(n+\tau-\sigma)\geqslant -\alpha y(n-\sigma)$ and $z(m_j+\tau-\sigma-1)\geqslant -\alpha y(m_j-\sigma-1)$ for $n\geqslant n_3$. Thus, (E_1) becomes

$$(\mathrm{E}_3)\qquad \begin{cases} \Delta z(n)+\dfrac{q(n)}{F(-\alpha)}F(z(n+\tau-\sigma))\leqslant 0, & n\neq m_j,\\ \underline{\Delta} z(m_j-1)+\dfrac{r(m_j-1)}{F(-\alpha)}F(z(m_j+\tau-\sigma-1))\leqslant 0, & j\in\mathbb{N}. \end{cases}$$

Since z is nonincreasing for $n\geqslant n_3$ and $m_j+\tau-\sigma-1$ are nonimpulsive points, then it follows that

$$\Delta z(n)+\frac{q(n)}{F(-\alpha)}F(z(n))\leqslant 0,\quad n\neq m_j,$$
$$\underline{\Delta} z(m_j-1)+\frac{r(m_j-1)}{F(-\alpha)}F(z(m_j-1))\leqslant 0,\quad j\in\mathbb{N},$$

that is,

$$\frac{\Delta z(n)}{F(z(n))}+\frac{q(n)}{F(-\alpha)}\geqslant 0,\quad n\neq m_j,$$
$$\frac{\underline{\Delta} z(m_j-1)}{F(z(m_j-1))}+\frac{r(m_j-1)}{F(-\alpha)}\geqslant 0,\quad j\in\mathbb{N}.$$

If $z(n+1)\leqslant u\leqslant z(n)$ and $z(m_j)\leqslant x\leqslant z(m_j-1)$, then the preceding inequalities reduce to

$$\int_{z(n)}^{z(n+1)}\frac{\mathrm{d}u}{F(u)}+\frac{q(n)}{F(-\alpha)}\geqslant 0,\quad n\neq m_j,$$
$$\int_{z(m_j-1)}^{z(m_j)}\frac{\mathrm{d}x}{F(x)}+\frac{r(m_j-1)}{F(-\alpha)}\geqslant 0,\quad j\in\mathbb{N}.$$

Therefore

$$\sum_{s=n_3}^{n}q(s)\leqslant -F(-\alpha)\sum_{s=n_3}^{n}\int_{z(s)}^{z(s+1)}\frac{\mathrm{d}u}{F(u)}=-F(-\alpha)\int_{z(n_3)}^{z(n+1)}\frac{\mathrm{d}u}{F(u)},$$
$$\sum_{j=1}^{\infty}r(m_j-1)\leqslant -F(-\alpha)\sum_{j=1}^{\infty}\int_{z(m_j-1)}^{z(m_j)}\frac{\mathrm{d}x}{F(x)},$$

that is,

$$\sum_{s=n_3}^{\infty} q(s) + \sum_{j=1}^{\infty} r(m_j - 1) < \infty$$

due to (A_3), a contradiction to (A_4). This completes the proof of the theorem. $\square$

Theorem 2.2. *Assume that all conditions of Theorem* 2.1 *hold except* (A_3). *Then every bounded solution of* (E_1) *oscillates.*

P r o o f. Proceeding as in the proof of Theorem 2.1, we obtain that $z(n) < 0$ for $n \geqslant n_2$. So, we can find an $n_3 > n_2$ and $C > 0$ such that $z(n) \leqslant -C$ for $n \geqslant n_3$. Consequently, (E_3) becomes

$$(E_4) \qquad \begin{cases} \Delta z(n) + F\Big(\dfrac{C}{\alpha}\Big) q(n) \leqslant 0, & n \neq m_j, \\ \underline{\Delta} z(m_j - 1) + F\Big(\dfrac{C}{\alpha}\Big) r(m_j - 1) \leqslant 0, & j \in \mathbb{N} \end{cases}$$

for $n \geqslant n_3$. Summing (E_4) from n_3 to $n-1$, we get

$$z(n) - z(n_3) - \sum_{n_3 \leqslant m_j - 1 \leqslant n-1} \underline{\Delta} z(m_j - 1) + F\Big(\frac{C}{\alpha}\Big) \sum_{s=n_3}^{n-1} q(s) \leqslant 0,$$

that is,

$$\begin{aligned} F\Big(\frac{C}{\alpha}\Big)\Big(\sum_{s=n_3}^{n-1} q(s) + \sum_{n_3 \leqslant m_j - 1 \leqslant n-1} r(m_j - 1)\Big) &\leqslant z(n_3) - z(n) \\ &\leqslant -z(n) < \infty \quad \text{as } n \to \infty, \end{aligned}$$

a contradiction to (A_4). Hence, the theorem is proved. $\square$

Theorem 2.3. *Let* $-1 \leqslant -\alpha \leqslant p(n) \leqslant 0$, $\alpha > 0$. *Assume that* (A_1), (A_2) *and* (A_4) *hold. Furthermore, assume that*

(A_5) *F is sublinear and*

$$\int_0^{\pm c} \frac{\mathrm{d}x}{F(x)} < \infty, \quad 0 < c < \infty,$$
$$\sum_{j=1}^{\infty} \int_{w(m_j - 1)}^{w(m_j)} \frac{\mathrm{d}x}{F(x)} < \infty, \quad \lim_{j \to \infty} w(m_j) < \infty$$

hold. Then every solution of (E_1) *oscillates.*

Proof. Proceeding as in Theorem 2.1, we obtain that $z(n)$ is nonincreasing for $n \geqslant n_2$. So, there exists $n_3 > n_2$ such that $z(n) > 0$ or < 0 for $n \geqslant n_3$. Assume that $z(n) > 0$ for $n \geqslant n_3$. Then $z(n) \leqslant y(n)$ for $n \geqslant n_3$. Consequently, (2.2) and (2.3) reduce to

$$(\mathrm{E}_5) \qquad \begin{cases} \Delta z(n) \leqslant -q(n)F(z(n-\sigma)), & n \neq m_j, \\ \underline{\Delta} z(m_j-1) \leqslant -r(m_j-1)F(z(m_j-\sigma-1)), & j \in \mathbb{N} \end{cases}$$

for $n \geqslant n_4 > n_3 + \sigma$ and due to nonincreasing $z(n)$,

$$\frac{\Delta z(n)}{F(z(n))} \leqslant -q(n), \quad n \neq m_j,$$
$$\frac{\underline{\Delta} z(m_j-1)}{F(z(m_j-1))} \leqslant -r(m_j-1), \quad j \in \mathbb{N}.$$

Since $\lim\limits_{n\to\infty} z(n) < \infty$ and $\lim\limits_{j\to\infty} z(m_j-1) < \infty$, then proceeding as in Theorem 2.1, we obtain a contradiction to (A_4). Indeed,

$$\sum_{s=n_4}^{n-1} q(s) \leqslant -\sum_{s=n_4}^{n-1} \frac{\Delta z(n)}{F(z(s))} \leqslant -\sum_{s=n_4}^{n-1} \int_{z(s)}^{z(s+1)} \frac{\mathrm{d}u}{F(u)} = -\int_{z(n_4)}^{z(n)} \frac{\mathrm{d}u}{F(u)}$$

and

$$\sum_{j=1}^{\infty} r(m_j-1) \leqslant -\sum_{j=1}^{\infty} \frac{\underline{\Delta} z(m_j-1)}{F(z(m_j-1))} \leqslant -\sum_{j=1}^{\infty} \int_{z(m_j-1)}^{z(m_j)} \frac{\mathrm{d}w}{F(w)},$$

where $z(s+1) < x < z(s)$ and $z(m_j) < w < z(m_j-1)$. Hence, $z(n) < 0$ for $n \geqslant n_3$. From (2.1) it follows that

$$y(n) < -p(n)y(n-\tau) \leqslant y(n-\tau) \leqslant y(n-2\tau) \leqslant \ldots \leqslant y(n_3),$$
$$y(m_j-1) < -p(m_j-1)y(m_j-\tau-1) \leqslant y(m_j-\tau-1)$$
$$\leqslant y(m_j-2\tau-1) \leqslant \ldots \leqslant y(n_3)$$

due to the nonimpulsive points $m_j-1, m_j-\tau-1, \ldots$ and so on. Indeed, the above observation reveals that $y(n)$ is bounded for $n \geqslant n_3$. The rest of the proof follows from Theorem 2.2. Hence, the proof of the theorem is completed. □

Theorem 2.4. *Let* $-1 \leqslant -\alpha \leqslant p(n) \leqslant 0$, $\alpha > 0$. *Assume that* (A_1), (A_2) *and* (A_4) *hold. If*

(A_6) *there exists* $\mu > 0$ *such that* $|F(u)| \geqslant \mu|u|$, $u \in \mathbb{R}$

and

(A_7) $\limsup\limits_{j\to\infty}\Big(\sum\limits_{n=m_j-\sigma}^{m_j-1} q(n) + \sum\limits_{i=1}^{\infty} r(m_i-1)\Big) > 1/\mu$, $\sigma \geqslant 1$

hold, then (E_1) *is oscillatory.*

P r o o f. Let $y(n)$ be a regular nonoscillatory solution of (E_1) such that $y(n) > 0$, $y(n-\tau) > 0$, $y(n-\sigma) > 0$ for $n \geqslant n_1 = n_0 + \sigma$. Proceeding as in Theorem 2.3, we get a contradiction to (A_4) when $z(n) < 0$ for $n \geqslant n_3$.

Assume that $z(n) > 0$ for $n \geqslant n_3$. Therefore, (E_5) holds for $n \geqslant n_4 = n_3 + \sigma$. Summing (E_5) from $m_j - \sigma$ to $m_j - 1$, $m_j \geqslant n_3 + \sigma$, we obtain

$$z(m_j) - z(m_j - \sigma) - \sum_{m_j - \sigma \leqslant m_i - 1 \leqslant m_j - 1} \underline{\Delta} z(m_i - 1) + \sum_{s=m_j-\sigma}^{m_j - 1} q(s) F(z(s-\sigma)) \leqslant 0,$$

that is,

$$-z(m_j - \sigma) + \sum_{m_j - \sigma \leqslant m_i - 1 \leqslant m_j - 1} r(m_i - 1) F(z(m_i - \sigma - 1)) + \sum_{s=m_j-\sigma}^{m_j - 1} q(s) F(z(s-\sigma)) \leqslant 0.$$

Using the fact that z is nonincreasing, the last inequality yields

$$-z(m_j - \sigma) + \mu z(m_j - \sigma) \sum_{s=m_j-\sigma}^{m_j - 1} q(s) + \mu z(m_j - \sigma) \sum_{m_j - \sigma \leqslant m_i - 1 \leqslant m_j - 1} r(m_i - 1) \leqslant 0$$

due to (A_6). Consequently, for $j \in \mathbb{N}$

$$\limsup_{j \to \infty} \Bigg(\sum_{s=m_j-\sigma}^{m_j - 1} q(s) + \sum_{m_j - \sigma \leqslant m_i - 1 \leqslant m_j - 1} r(m_i - 1) \Bigg) \leqslant \frac{1}{\mu}$$

which contradicts (A_7). Thus, the proof of the theorem is completed. □

Theorem 2.5. *Let* $p(n) \leqslant -1$ *and* $\tau - \sigma > 0$. *Assume that* (A_1), (A_2), (A_4) *and* (A_6) *hold. For* $\tau - \sigma > 1$, *if*

(A_8) $\limsup\limits_{j\to\infty} \Big(\sum\limits_{s=m_j+\sigma-\tau}^{m_j-1} -q(s)/p(s+\tau-\sigma) + \sum\limits_{i=1}^{\infty} -r(m_i-1)/p(m_i+\tau-\sigma-1) \Big) >$ $1/\mu$,

then (E_1) *is oscillatory.*

P r o o f. Proceeding as in Theorem 2.1 we have a contradiction to (A_4) when $z(n) > 0$ for $n \geqslant n_2$. Assume that $z(n) < 0$ for $n \geqslant n_2$. Consequently, there exists $n_3 > n_2$ such that

$$\begin{cases} z(n) > p(n) y(n-\tau), \\ z(m_j - 1) > p(m_j - 1) y(m_j - \tau - 1), \end{cases}$$

that is,

$$y(n-\sigma) > \frac{z(n+\tau-\sigma)}{p(n+\tau-\sigma)} \quad \text{and} \quad y(m_j-\sigma-1) > \frac{z(m_j+\tau-\sigma-1)}{p(m_j+\tau-\sigma-1)}$$

for $n \geqslant n_3$. Hence, (E_1) reduces to

$$(E_6) \qquad \begin{cases} \Delta z(n) + \mu q(n)\dfrac{z(n+\tau-\sigma)}{p(n+\tau-\sigma)} \leqslant 0, & n \neq m_j, \\ \underline{\Delta} z(m_j-1) + \mu r(m_j-1)\dfrac{z(m_j+\tau-\sigma-1)}{p(m_j+\tau-\sigma-1)} \leqslant 0, & j \in \mathbb{N} \end{cases}$$

due to (A_6). Summing (E_6) from $m_j+\sigma-\tau$ to m_j-1, $m_j \geqslant n_3+\tau-\sigma$, we have

$$z(m_j) - z(m_j+\sigma-\tau) - \sum_{m_j+\sigma-\tau \leqslant m_i-1 \leqslant m_j-1} \underline{\Delta} z(m_i-1) + \mu \sum_{s=m_j+\sigma-\tau}^{m_j-1} q(s)\frac{z(s+\tau-\sigma)}{p(s+\tau-\sigma)} \leqslant 0,$$

that is,

$$z(m_j) + \mu \sum_{m_j+\sigma-\tau \leqslant m_i-1 \leqslant m_j-1} r(m_i-1)\frac{z(m_i+\tau-\sigma-1)}{p(m_i+\tau-\sigma-1)} + \mu \sum_{s=m_j+\sigma-\tau}^{m_j-1} q(s)\frac{z(s+\tau-\sigma)}{p(s+\tau-\sigma)} \leqslant 0.$$

Since z is nonincreasing and $m_j+\sigma-\tau \leqslant m_j-1$, $m_j+\sigma-\tau \leqslant s$, then the preceding inequality becomes

$$z(m_j)\left(1+\mu \sum_{m_j+\sigma-\tau \leqslant m_i-1 \leqslant m_j-1} \frac{r(m_i-1)}{p(m_i+\tau-\sigma-1)} + \mu \sum_{s=m_j+\sigma-\tau}^{m_j-1} \frac{q(s)}{p(s+\tau-\sigma)}\right) \leqslant 0,$$

that is,

$$\sum_{s=m_j+\sigma-\tau}^{m_j-1} \frac{-q(s)}{p(s+\tau-\sigma)} + \sum_{m_j+\sigma-\tau \leqslant m_i-1 \leqslant m_j-1} \frac{-r(m_i-1)}{p(m_i+\tau-\sigma-1)} \leqslant \frac{1}{\mu},$$

a contradiction to (A_8). Hence, the theorem is proved. □

Theorem 2.6. *Let* $0 \leqslant p(n) \leqslant \beta < \infty$ *for* $\tau \leqslant \sigma$. *Assume that* (A_1) *and* (A_2) *hold. Furthermore, assume that*

(A_9) F *is sublinear and*

$$\int_0^{\pm c} \frac{\mathrm{d}x}{F(x)} < \infty, \quad 0 < c < \infty,$$

$$\sum_{j=1}^{\infty} \Bigl(\int_{w(m_j-1)}^{w(m_j)} \frac{\mathrm{d}x}{F(x)} + F(\beta) \int_{w(m_j-\tau-1)}^{w(m_j-\tau)} \frac{\mathrm{d}x}{F(x)} \Bigr) < \infty, \quad \lim_{j\to\infty} w(m_j) < \infty,$$

(A_{10}) *there exists* $\lambda > 0$ *such that* $F(u) + F(v) \geqslant \lambda F(u+v)$, $u, v \in \mathbb{R}_+$ *and*

(A_{11}) $\sum\limits_{n=\tau}^{\infty} Q(n) + \sum\limits_{j=1}^{\infty} R(m_j - 1) = \infty$, *where* $Q(n) = \min\{q(n), q(n-\tau)\}$, $R(m_j-1) = \min\{r(m_j-1), r(m_j-\tau-1)\}$, $n \geqslant \tau$, $m_j \geqslant \tau + 1$, $j \in \mathbb{N}$.

Then every solution of (E_1) *oscillates.*

P r o o f. On the contrary, we proceed as in Theorem 2.1 to obtain that $z(n)$ is nonincreasing for $n \geqslant n_2$. So there exists $n_3 > n_2$ such that $z(n) > 0$ for $n \geqslant n_3$. It is easy to verify that

$$\begin{aligned} &\Delta z(n) + q(n)F(z(n-\sigma)) + F(\beta)\Delta z(n-\tau) \\ &\qquad + F(\beta)q(n-\tau)F(z(n-\sigma-\tau)) \leqslant 0, && n \neq m_j, \\ &\underline{\Delta} z(m_j-1) + r(m_j-1)F(z(m_j-\sigma-1)) + F(\beta)\underline{\Delta} z(m_j-\tau-1) \\ &\qquad + F(\beta)r(m_j-\tau-1)F(z(m_j-\sigma-\tau-1)) \leqslant 0, && j \in \mathbb{N}. \end{aligned}$$

Applying (A_{10}) and (A_2) in the preceding two inequalities, we obtain

$$\Delta z(n) + F(\beta)\Delta z(n-\tau) + \lambda Q(n) F(z(n-\sigma)) \leqslant 0,$$
$$\underline{\Delta} z(m_j-1) + F(\beta)\underline{\Delta} z(m_j-\tau-1) + \lambda R(m_j-1)F(z(m_j-\sigma-1)) \leqslant 0.$$

Using the fact that z is nonincreasing and $\tau \leqslant \sigma$, we can find an $n_4 > 0$ such that the above inequalities can be written as

$$(\mathrm{E}_7) \qquad \begin{cases} \dfrac{\Delta z(n)}{F(z(n))} + F(\beta)\dfrac{\Delta z(n-\tau)}{F(z(n-\tau))} + \lambda Q(n) \leqslant 0, & n \neq m_j, \\ \dfrac{\underline{\Delta} z(m_j-1)}{F(z(m_j-1))} + F(\beta)\dfrac{\underline{\Delta} z(m_j-\tau-1)}{F(z(m_j-\tau-1))} + \lambda R(m_j-1) \leqslant 0, & j \in \mathbb{N} \end{cases}$$

for $n \geqslant n_4$. We may note that $m_j - 1$ and $m_j - \tau - 1$, $j \in \mathbb{N}$ are nonimpulsive points exceeding n_4. If

$$z(n+1) \leqslant t \leqslant z(n),$$
$$z(n-\tau+1) \leqslant x \leqslant z(n-\tau),$$
$$z(m_j) \leqslant u \leqslant z(m_j-1),$$
$$z(m_j-\tau) \leqslant v \leqslant z(m_j-\tau-1),$$

then from (E_7) it is easy to verify that

$$\begin{aligned}
&\int_{z(n)}^{z(n+1)} \frac{\mathrm{d}t}{F(t)} + F(\beta)\int_{z(n-\tau)}^{z(n+1-\tau)} \frac{\mathrm{d}x}{F(x)} + \lambda Q(n) \leqslant 0, \quad n \neq m_j,\\
&\int_{z(m_j-1)}^{z(m_j)} \frac{\mathrm{d}u}{F(u)} + F(\beta)\int_{z(m_j-\tau-1)}^{z(m_j-\tau)} \frac{\mathrm{d}v}{F(v)} + \lambda R(m_j-1) \leqslant 0, \quad j \in \mathbb{N},
\end{aligned}$$

that is,

$$\begin{aligned}
&\sum_{s=n_4}^{n}\Big(\int_{z(s)}^{z(s+1)} \frac{\mathrm{d}t}{F(t)} + F(\beta)\int_{z(s-\tau)}^{z(s+1-\tau)} \frac{\mathrm{d}x}{F(x)}\Big) + \lambda\sum_{s=n_4}^{n} Q(s) \leqslant 0, \quad n \neq m_j,\\
&\sum_{j=1}^{\infty}\Big(\int_{z(m_j-1)}^{z(m_j)} \frac{\mathrm{d}u}{F(u)} + F(\beta)\int_{z(m_j-\tau-1)}^{z(m_j-\tau)} \frac{\mathrm{d}v}{F(v)}\Big) + \lambda\sum_{j=1}^{\infty} R(m_j-1) \leqslant 0, \quad j \in \mathbb{N}.
\end{aligned}$$

Consequently,

$$\begin{aligned}
\lambda\sum_{s=n_4}^{\infty} Q(s) &\leqslant -\lim_{n\to\infty}\Big(\int_{z(n_4)}^{z(n+1)} \frac{\mathrm{d}t}{F(t)} + F(\beta)\int_{z(n_4-\tau)}^{z(n+1-\tau)} \frac{\mathrm{d}x}{F(x)}\Big),\\
\lambda\sum_{j=1}^{\infty} R(m_j-1) &\leqslant -\sum_{j=1}^{\infty}\Big(\int_{z(m_j-1)}^{z(m_j)} \frac{\mathrm{d}u}{F(u)} + F(\beta)\int_{z(m_j-\tau-1)}^{z(m_j-\tau)} \frac{\mathrm{d}v}{F(v)}\Big)
\end{aligned}$$

implies that

$$\sum_{s=n_4}^{\infty} Q(s) + \sum_{j=1}^{\infty} R(m_j-1) < \infty,$$

a contradiction to (A_{11}). This completes the proof of the theorem. □

Theorem 2.7. *Let* $0 \leqslant p(n) \leqslant \beta \leqslant 1$ *and* $2\tau \leqslant \sigma$. *If* (A_1), (A_2), (A_6) *and*

(A_{12}) $\limsup_{j\to\infty}\Big(\sum_{n=m_j-\tau}^{m_j-1} Q(n) + \sum_{m_j-\tau\leqslant m_i-1\leqslant m_j-1} R(m_i-1)\Big) > (1+\beta)/\mu$

hold, then every solution of (E_1) *oscillates, where* $Q(n)$ *and* $R(m_j-1)$ *are defined in Theorem* 2.6.

P r o o f. Proceeding as in Theorem 2.6, we obtain that $z(n) > 0$ and $z(n)$ is nonincreasing for $n \geqslant n_3$. Using (A_6) in (E_1), we get

$$(E_8) \qquad \begin{cases} \Delta z(n) + \mu q(n) y(n-\sigma) \leqslant 0, & n \neq m_j,\\ \underline{\Delta} z(m_j-1) + \mu r(m_j-1) y(m_j-\sigma-1) \leqslant 0, & j \in \mathbb{N} \end{cases}$$

due to (2.1). Upon using (E_8), we obtain

$$\Delta z(n)+\mu q(n)y(n-\sigma)+\beta(\Delta z(n-\tau)+\mu q(n-\tau)y(n-\sigma-\tau))\leqslant 0,$$
$$\underline{\Delta} z(m_j-1)+\mu r(m_j-1)y(m_j-\sigma-1)$$
$$+\beta(\underline{\Delta} z(m_j-\tau-1)+\mu r(m_j-\tau-1)y(m_j-\sigma-\tau-1))\leqslant 0,$$

that is,

$$(\mathrm{E}_9)\quad \begin{cases}\Delta z(n)+\beta\Delta z(n-\tau)+\mu Q(n)z(n-\sigma)\leqslant 0, & n\neq m_j,\\ \underline{\Delta} z(m_j-1)+\beta\underline{\Delta} z(m_j-\tau-1)+\mu R(m_j-1)z(m_j-\sigma-1)\leqslant 0, & j\in\mathbb{N}\end{cases}$$

for $n\geqslant n_4>n_3$. Summing (E_9) from $m_j-\tau$ to m_j-1, it follows that

$$z(m_j)-z(m_j-\tau)+\beta z(m_j-\tau)-\beta z(m_j-2\tau)+\mu\sum_{s=m_j-\tau}^{m_j-1}Q(s)z(s-\sigma)$$
$$+\mu\sum_{m_j-\tau\leqslant m_i-1\leqslant m_j-1}R(m_i-1)z(m_i-\sigma-1)\leqslant 0.$$

Therefore,

$$(2.4)\qquad -z(m_j-\tau)-\beta z(m_j-2\tau)+\mu\sum_{s=m_j-\tau}^{m_j-1}Q(s)z(s-\sigma)$$
$$+\mu\sum_{m_j-\tau\leqslant m_i-1\leqslant m_j-1}R(m_i-1)z(m_i-\sigma-1)\leqslant 0.$$

Using the fact that z is nonincreasing and $m_j-1\leqslant m_i-1\leqslant m_j$, $s\leqslant m_j-1<m_j$ in (2.4), we get

$$-z(m_j-\tau)-\beta z(m_j-2\tau)$$
$$+\mu z(m_j-\sigma)\Bigg(\sum_{s=m_j-\tau}^{m_j-1}Q(s)+\sum_{m_j-\tau\leqslant m_i-1\leqslant m_j-1}R(m_i-1)\Bigg)\leqslant 0.$$

Hence,

$$z(m_j-2\tau)\Bigg(-1-\beta+\mu\sum_{s=m_j-\tau}^{m_j-1}Q(s)+\mu\sum_{m_j-\tau\leqslant m_i-1\leqslant m_j-1}R(m_i-1)\Bigg)\leqslant 0$$

implies that

$$\sum_{s=m_j-\tau}^{m_j-1}Q(s)+\sum_{m_j-\tau\leqslant m_i-1\leqslant m_j-1}R(m_i-1)\leqslant\frac{1+\beta}{\mu},$$

which contradicts (A_{12}). This completes the proof of the theorem. □

Theorem 2.8. *Let $-1 < p_1 \leqslant p(n) \leqslant p_2 \leqslant 0$. Assume that*

(A_{13}) $\sum\limits_{n=N}^{\infty} q(n) + \sum\limits_{j=1}^{\infty} r(m_j-1) < \infty$, $N > 0$

hold. Then (E_1) *has a bounded nonoscillatory solution.*

Proof. Let $X = l_{\infty}^{n_0}$ be the Banach space of real valued bounded functions $y(n)$ for $n \geqslant n_0$ with sup norm defined by $\|y\| = \sup\{|y(n)|\colon\ n \geqslant n_0\}$.

Let $K = \{y \in X\colon\ y(n) \geqslant 0 \text{ for } n \geqslant n_0\}$. For $y_1, y_2 \in X$ we define $y_1 \leqslant y_2$ if and only if $y_2 - y_1 \in K$. Thus, X is a partially ordered Banach space. Set

$$S = \{y \in X\colon\ C_1 \leqslant y(n) \leqslant C_2,\ n \geqslant n_0\},$$

where C_1 and C_2 are two positive constants such that

$$C_1 < \alpha < (1+p_1)C_2.$$

Let $x_0(n) = C_1$ for $n \geqslant n_0$. Then $x_0(n) \in S$ and $x_0(n) = \inf S$. In addition, if $\varphi \subset S^* \subset S$, then

$$S^* = \{y \in X\colon\ l_1 \leqslant y(n) \leqslant l_2,\ C_1 \leqslant l_1,\ l_2 \leqslant C_2,\ n \geqslant n_0\}.$$

Let $x_1(n) = l_2' = \sup\{l_2\colon\ C_1 \leqslant l_2 \leqslant C_2\}$. Then $x_1(n) \in S$ and $x_1(n) = \sup S^*$. From (H_1) it is possible to choose $n_1 > n_0$ such that

$$\sum_{n=n_1}^{\infty} q(n) + \sum_{n_1 \leqslant m_j-1 \leqslant n} r(m_j-1) < \frac{(1+p_1)C_2 - \alpha}{F(C_2)}, \quad n \geqslant n_1. \tag{2.5}$$

Define a map $T\colon\ S \to S$ by

$$Ty(n) = \begin{cases} Ty(n_1+\varrho), & n_1 \leqslant n \leqslant n_1+\varrho,\\ \alpha - p(n)y(n-\tau) + \sum\limits_{s=n}^{\infty} q(s)F(y(s-\sigma)) & \\ \quad + \sum\limits_{n_1 \leqslant m_j-1 \leqslant n} r(m_j-1)F(y(m_j-\sigma-1)), & n \geqslant n_1+\varrho. \end{cases}$$

For $y \in X$ and using (2.5), we have

$$\begin{aligned} Ty(n) &\leqslant \alpha - p(n)y(n-\tau) + F(C_2)\Bigg(\sum_{s=n}^{\infty} q(s) + \sum_{n_1 \leqslant m_j-1 \leqslant n} r(m_j-1)\Bigg)\\ &\leqslant \alpha - p_1C_2 + F(C_2)\frac{(1+p_1)C_2-\alpha}{F(C_2)} = \alpha - p_1C_2 + C_2 + p_1C_2 - \alpha = C_2, \end{aligned}$$

and

$$Ty(n) \geqslant \alpha \geqslant C_2$$

implies that $Ty \in S$. Let $y_1, y_2 \in S$ be such that $y_1 \leqslant y_2$. It is easy to verify that $Ty_1 \leqslant Ty_2$. Hence, by Knaster-Tarski fixed point theorem, T has a unique $y \in S$ such that $Ty = y$. Therefore,

$$y(n) = \begin{cases} y(n_1+\varrho), & n_1 \leqslant n \leqslant n_1+\varrho, \\ \alpha - p(n)y(n-\tau) + \sum\limits_{s=n}^{\infty} q(s)F(y(s-\sigma)) & \\ \quad + \sum\limits_{j=1}^{\infty} r(m_j-1)F(y(m_j-\sigma-1)), & n \geqslant n_1+\varrho, \end{cases}$$

and it is easy to see that $y(n)$ is a nonoscillatory solution of (E_1). This completes the proof of the theorem. □

Example 2.1. Consider the impulsive difference equation of the form

$$(E_{10}) \quad \begin{cases} \Delta\Big(y(n) - \dfrac{1}{e}y(n-2)\Big) & \\ \qquad + (e^3-1)(e+1)e^{(2n-7)/3}y^{1/3}(n-2) = 0, & n \neq m_j,\ n > 2, \\ \underline{\Delta}\Big(y(m_j-1) - \dfrac{1}{e}y(m_j-3)\Big) & \\ \qquad + (e^3-1)(e+1)e^{(2m_j-9)/3}y^{1/3}(m_j-3) = 0, & j \in \mathbb{N}, \end{cases}$$

where $\tau = 2 = \sigma = 2$, $p(n) = -1/e$, $q(n) = (e^3-1)(e+1)e^{(2n-7)/3}$, $r(m_j-1) = (e^3-1)(e+1)e^{(2m_j-9)/3}$, $F(u) = u^{1/3}$, $m_j = 3j$ for $j \in \mathbb{N}$. Since

$$\sum_{n=1}^{\infty} q(n) = \sum_{n=1}^{\infty}(e^3-1)(e+1)e^{(2n-7)/3} = (e^3-1)(e+1)\sum_{n=1}^{\infty} e^{(2n-7)/3} = \infty,$$

then (A_4) holds true. Indeed, all conditions of Theorem 2.3 hold true. Hence, (E_{10}) is oscillatory.

Clearly, $y(n) = (-1)^n e^n$ is an oscillatory solution of the first equation of (E_{10}). It is easy to see that $(-1)^{m_j}e^{m_j}$ is an oscillatory solution of the second equation of (E_{10}).

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