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A SOLVABILITY CRITERION FOR FINITE GROUPS RELATED TO CHARACTER DEGREES

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Abstract. Let $m > 1$ be a fixed positive integer. In this paper, we consider finite groups each of whose nonlinear character degrees has exactly m prime divisors. We show that such groups are solvable whenever $m > 2$. Moreover, we prove that if G is a non-solvable group with this property, then $m = 2$ and G is an extension of A_7 or S_7 by a solvable group.

Keywords: non-solvable group; solvable group; character degree

MSC 2020: 20C15, 20D10

1. Introduction

Throughout this paper, G will be a finite group. Let $\mathrm{cd}(G)$ be the set of all irreducible character degrees of G and $\pi(n)$ be the set of prime numbers dividing n.

Isaacs and Passman studied finite groups G with $\mathrm{cd}(G) \setminus \{1\}$ consisting of primes (see [6]). Also, Manz in [8] and [9] characterizes finite groups G with the property that $|\pi(\chi(1))| = 1$ for every nonlinear irreducible character χ of G. He shows that if G is a non-solvable group whose character degrees are power primes, then $G = A \times S$, where A is an abelian group and S is either $\mathrm{PSL}(2,4)$ or $\mathrm{PSL}(2,8)$.

Let $m > 1$ be a fixed positive integer. Suppose that G is a finite group such that $|\pi(\chi(1))| = m$ for each nonlinear irreducible character χ of G. In this paper, we show that either (i) G is solvable, for some normal subgroup K of G we have that G/K is a Frobenius group with Frobenius kernel N/K which is an elementary abelian q-group for some prime q and a cyclic Frobenius complement, or (ii) G is non-solvable, $m = 2$ and G is an extension of A_7 or S_7 by a solvable group.

Consider that G is the non-split central extension of A_7 by $\mathbb{Z}_3$. Using GAP, see [4], "`G:=PerfectGroup(7560,1)`", we observe that

$$\mathrm{cd}(G) = \{1, 6, 10, 14, 15, 21, 24, 35\}.$$

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Thus, if G is a non-solvable group each of whose nonlinear character degrees has exactly two prime divisors, then it is not required that G is a split extension.

On the other hand, assume that G is a Frobenius group with an abelian kernel K and a cyclic complement H of order $p_1^{\alpha_1} \ldots p_m^{\alpha_m}$ for some prime number p_i, $1 \leqslant i \leqslant m$. We can check easily that $\mathrm{cd}(G) = \{1, p_1^{\alpha_1} \ldots p_m^{\alpha_m}\}$. Hence, for each positive integer m, there exists a solvable group each of whose nonlinear character degrees has exactly m prime divisors.

2. Main results

In this section we aim to present our main result. We can check that $\mathrm{cd}(\mathrm{A}_7) = \{1, 6, 10, 14, 15, 21, 35\}$ and $|\pi(\chi(1))| = 2$ for every nonlinear irreducible character χ of A_7.

Lemma 2.1. *Let S be a nonabelian simple group such that $S \ncong \mathrm{A}_7$. Then there exist two nonlinear irreducible characters χ and ψ of S which extend to $\mathrm{Aut}(S)$ such that either $|\pi(\chi(1))| = 1$ or $|\pi(\chi(1))| \neq |\pi(\psi(1))|$.*

P r o o f. According to the classification of finite simple groups, a nonabelian simple group is either an alternating group A_n for $n \geqslant 5$, a simple group of Lie type, or one of the 26 sporadic groups. Thus, we prove the lemma for three cases.

Case 1: Suppose that S is a nonabelian simple group of Lie type. For these group, we know that the Steinberg character χ of S extends to $\mathrm{Aut}(S)$ and $\chi(1)$ is a prime power, by [10].

Case 2: Assume that S is an alternating group A_n for $n \geqslant 5$. If $n = 5$ or 6, $\mathrm{cd}(\mathrm{A}_n)$ contains a prime number. For $n \geqslant 8$, consider that the irreducible character χ of the symmetric group S_n corresponds to the partition $(n-4, 1^4)$ and the irreducible character ψ corresponds to the partition $(n-1, 1)$. The restrictions of ψ and χ to A_n are irreducible, since the Young diagram corresponding to the partitions is not symmetric, by [7]. Observe that

$$\chi(1) = \frac{(n-1)(n-2)(n-3)(n-4)}{2^3.3} \quad \text{and} \quad \psi(1) = n-1$$

and we can check that $|\pi(\chi(1))| \neq |\pi(\psi(1))|$.

Case 3: Suppose that S is a sporadic simple group. In Table I, using ATLAS, see [3], we provide two nonlinear irreducible characters χ and ψ of S which extend to $\mathrm{Aut}(S)$ such that $|\pi(\chi(1))| \neq |\pi(\psi(1))|$. □

J_1	$\chi_2(1) = 23.7$	$\chi_9(1) = 2^3.3.5$
J_2	$\chi_{10}(1) = 2.5.9$	$\chi_6(1) = 2^2.3^2$
J_3	$\chi_6(1) = 2^3.3^4$	$\chi_{13}(1) = 5.17.19$
J_4	$\chi_2(1) = 31.43$	$\chi_{11}(1) = 2^3.3^2.23.29.37$
M_{11}	$\chi_2(1) = 2.5$	$\chi_5(1) = 11$
M_{12}	$\chi_{11}(1) = 2.3.11$	$\chi_7(1) = 2.3^3$
M_{22}	$\chi_3(1) = 3^2.5$	$\chi_8(1) = 2.3.5.7$
M_{23}	$\chi_2(1) = 2.11$	$\chi_5(1) = 2.5.23$
M_{24}	$\chi_2(1) = 23$	$\chi_3(1) = 3^2.5$
HS	$\chi_7(1) = 5^2.7$	$\chi_4(1) = 2.7.11$
He	$\chi_{13}(1) = 2^4.3.5.17$	$\chi_6(1) = 2^3.5.17$
Ru	$\chi_2(1) = 2.3^3.7$	$\chi_5(1) = 3^3.29$
HN	$\chi_4(1) = 2^3.5.19$	$\chi_8(1) = 2.3^4.5.11$
Suz	$\chi_3(1) = 2^2.7.13$	$\chi_{20}(1) = 2^3.5.7.11.13$
M^{cL}	$\chi_3(1) = 3.7.11$	$\chi_2(1) = 2.11$
O'N	$\chi_2(1) = 2^6.3^2.19$	$\chi_{11}(1) = 2^2.3^2.7.11.19$
Co_1	$\chi_2(1) = 2^2.3.23$	$\chi_3(1) = 13.23$
Co_2	$\chi_2(1) = 23$	$\chi_3(1) = 11.23$
Co_3	$\chi_2(1) = 23$	$\chi_3(1) = 11.23$
Fi_{22}	$\chi_2(1) = 2.3.13$	$\chi_5(1) = 2.5.11.13$
Fi_{23}	$\chi_2(1) = 2.7.23$	$\chi_3(1) = 2^2.3.13.23$
Fi'_{24}	$\chi_2(1) = 23.29.13$	$\chi_6(1) = 5^2.7^3.11.17$
Ly	$\chi_7(1) = 2^8.7.67$	$\chi_{50}(1) = 3.5^6.31.37$
TH	$\chi_2(1) = 2^3.31$	$\chi_3(1) = 7.19.31$
B	$\chi_2(1) = 3.31.47$	$\chi_3(1) = 3^3.5.23.31$
M	$\chi_2(1) = 47.59.71$	$\chi_{11}(1) = 2^2.31.41.59.71$

Table 1.

Proposition 2.1 ([2], Lemma 5)**.** *Let G be a group and $M = S_1 \times \ldots \times S_k$ a minimal normal subgroup of G, where every S_i is isomorphic to a nonabelian simple group S. If $\theta \in \mathrm{Irr}(S)$ extends to $\mathrm{Aut}(S)$, then $\theta \times \ldots \times \theta \in \mathrm{Irr}(M)$ extends to G.*

Theorem 2.1. *Let $m > 1$ be a fixed positive integer. Suppose that G is a finite group each of whose nonlinear character degrees has exactly m prime divisors. Then one of the following situations occurs:*

(i) *G is a solvable group.*

(ii) *$m = 2$ and $G/M \cong \mathrm{A}_7$ or S_7 in which M is the soluble radical of G.*

Proof. Assume that G is non-solvable and M is the soluble radical of G. Thus, every minimal normal subgroup N/M of G/M is nonabelian and $N/M \cong S_1 \times$

$S_2 \times \ldots \times S_t$, where $S_i \cong S$ for a nonabelian simple group S. If $S \ncong \mathrm{A}_7$, by Lemma 2.1, there exist two nonlinear irreducible characters χ and ψ of S which extend to $\mathrm{Aut}(S)$ such that either $|\pi(\chi(1))| = 1$ or $|\pi(\chi(1))| \neq |\pi(\psi(1))|$. Furthermore, by Proposition 2.2, $\theta_1 = \chi \times \ldots \times \chi \in \mathrm{Irr}(N/M)$ and $\theta_2 = \psi \times \ldots \times \psi \in \mathrm{Irr}(N/M)$ extend to G/M and so $\theta_1(1), \theta_2(1) \in \mathrm{cd}(G/M)$. Therefore, either $|\pi(\theta_1(1))| = 1$ or $|\pi(\theta_1(1))| \neq |\pi(\theta_2(1))|$, which is a contradiction.

We now show that $S \cong \mathrm{A}_7$ implies that $N/M \cong \mathrm{A}_7$. Suppose on the contrary that N/M has more than one simple factor. Choose $\chi, \psi \in \mathrm{Irr}(\mathrm{A}_7)$ such that χ and ψ extend to $\mathrm{Aut}(\mathrm{A}_7)$, where $\chi(1) = 6$ and $\psi(1) = 14$. We know that $\theta = \chi \times \chi \times \ldots \times \chi$, $\varphi = \chi \times \psi \times 1 \times \ldots \times 1 \in \mathrm{Irr}(N/M)$. Then, by Proposition 2.2, $\theta(1) \in \mathrm{cd}(G/M)$ and by Clifford's Theorem and Corollary 11.29 in [5], $b\varphi(1) \in \mathrm{cd}(G/M)$ for a divisor b of $|G/M : N/M|$. It follows that $|\pi(\theta(1))| \neq |\pi(b\varphi(1))|$, which is a contradiction. Thus, each minimal normal subgroup of G/M is isomorphic to A_7 and $m = 2$.

Similarly, G/M has no normal subgroup isomorphic to $\mathrm{A}_7 \times \mathrm{A}_7$. Therefore $N/M \cong \mathrm{A}_7$ is the unique minimal normal subgroup of G/M. Hence, we can deduce $\mathrm{A}_7 \leqslant G/M \leqslant \mathrm{Aut}(\mathrm{A}_7)$ and so $G/M \cong \mathrm{A}_7$ or S_7. $\square$

Lemma 2.2 ([1], Lemma 3.1)**.** *Let G be a finite nonabelian solvable group with $G' \leqslant \mathrm{O}^p(G)$ for all primes p. Suppose that $K \lhd G$ and K is maximal such that G/K is nonabelian. Then G/K is a Frobenius group with Frobenius kernel N/K, an elementary abelian q-group for a prime q, and a cyclic Frobenius complement. Let f denote the order of the Frobenius complement and assume further that K is chosen so that f is minimal. Then for each linear character λ of N, either λ^G is irreducible or λ extends to G. In particular, if $\chi \in \mathrm{Irr}(G)$ lies over a linear character of N, then χ must have degree 1 or f.*

Theorem 2.2. *Suppose that G is a finite solvable group such that $|\pi(\chi(1))| = m > 1$ for all nonlinear irreducible characters χ of G. Then G satisfies Lemma 2.4.*

P r o o f. If $G/\mathrm{O}^p(G)$ is a nonabelian group for some $p \in \pi(G)$, then $|\pi(\chi(1))| = 1$ for a nonlinear irreducible character χ of G, which is a contradiction. Thus, $G/\mathrm{O}^p(G)$ is abelian for each $p \in \pi(G)$ and so G satisfies Lemma 2.4. $\square$

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