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## INCREASING SEQUENCES OF SECTORIAL FORMS

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*Abstract.* We prove convergence results for ‘increasing’ sequences of sectorial forms. We treat both the case of closed forms and the case of non-closable forms.

*Keywords:* sectorial form; strong resolvent convergence

*MSC 2020:* 47A07

## 1. INTRODUCTION

In this paper we treat a convergence theorem for increasing sequences of sectorial forms in a complex Hilbert space. More precisely, we will deal with a sequence  $(a_n)$  of sectorial forms with vertex 0 and angle  $\theta \in [0, \pi/2)$ , with  $\text{dom}(a_n) \supseteq \text{dom}(a_{n+1})$  and such that  $a_{n+1} - a_n$  is sectorial with vertex 0 and angle  $\theta$  for all  $n \in \mathbb{N}$ . (This is what we mean by ‘increasing sequence’. The setup implies that the real parts of the forms constitute an increasing sequence of symmetric forms, in the usual sense.) We do not assume the forms to be densely defined, and hence one does not obtain an operator  $A_n$  associated with  $a_n$ , but rather a linear relation. These notions will be explained in more detail in Section 2 of the paper. The aim is to show that the linear relations converge to a limit linear relation  $A$  in strong resolvent sense; see Theorem 4.1 for the case of closed forms and Theorem 5.1 for the case of non-closable forms.

The history of this kind of convergence results starts with the treatment of increasing sequences of closed accretive *symmetric* forms, by Kato [7], Theorem VIII.3.13, and Simon [12], Theorem 3.1; see also Kato [8], Theorem VIII.3.13a. It was Simon in [12], Section 4, who advocated the use of non-densely defined (closed accretive symmetric) forms; the notions of non-densely defined *sectorial* forms and their associated linear relations were developed in [5] and [9]. The result for sectorial forms

sketched above, for the case of closed forms, is due to Batty and ter Elst, see [5], Theorem 2.2, where it is formulated in a different guise, with series of sectorial forms. Previously, a related kind of sequences of sectorial forms had been treated by Ouhabaz in [10], Theorem 5.

Our proof of Theorem 4.1 is inspired by [10], proof of Theorem 5. Whereas in [5], proof of Theorem 2.2, the convergence in strong resolvent sense is proved directly for the sequence under consideration, we use the existing convergence result for the case of increasing sequences of symmetric forms and then use Kato's holomorphic families of type (a) together with Vitali's convergence theorem. In Remark 4.2 we show that Ouhabaz' result in [10], Theorem 5, can be obtained as a corollary of Theorem 4.1.

Finally, we also treat the case in which the sequence consists of non-closable forms. Again, the result we prove is due to Batty and ter Elst; see [5], Theorem 3.2. It is remarkable that Ouhabaz' procedure can also be adapted to this case and yields a proof that is barely more complicated than the proof for the case of closed forms.

In Section 2 we explain our notation concerning linear relations associated with non-densely defined sectorial forms.

In Section 3 we discuss the correspondence between the order of non-densely defined closed accretive symmetric forms and the inverses of the corresponding linear relations. Our treatment is motivated by [8], Lemma VI.2.30.

Section 4 is devoted to the main result for the case of closed forms.

In Section 5 we treat the case of non-closable forms.

## 2. PRELIMINARIES ON SECTORIAL FORMS, LINEAR RELATIONS AND DEGENERATE SEMIGROUPS

Let  $H$  be a complex Hilbert space. A sectorial form  $a$  in  $H$  with vertex 0 and angle  $\theta \in [0, \pi/2)$  is a sesquilinear map  $a: \text{dom}(a) \times \text{dom}(a) \rightarrow \mathbb{C}$ , where the domain  $\text{dom}(a)$  is a subspace of  $H$  and

$$a(u) := a(u, u) \in \overline{\Sigma_\theta} \quad (u \in \text{dom}(a))$$

with  $\Sigma_\theta := \{z \in \mathbb{C} \setminus \{0\}; |\text{Arg } z| < \theta\}$  if  $0 < \theta < \pi/2$ , and  $\Sigma_0 := (0, \infty)$ . We define

$$\begin{aligned} a^*(u, v) &:= \overline{a(v, u)} \quad (u, v \in \text{dom}(a^*) := \text{dom}(a)), \\ \text{Re } a &:= \frac{a + a^*}{2}, \quad \text{Im } a := \frac{a - a^*}{2i} \end{aligned}$$

and

$$\|u\|_a := (\text{Re } a(u) + \|u\|_H^2)^{1/2} \quad (u \in \text{dom}(a)).$$

The form  $a$  is called closed if  $(\text{dom}(a), \|\cdot\|_a)$  is complete.

Let  $a$  be a closed sectorial form. Then an  $m$ -sectorial operator  $A_0$  in  $H_0 := \overline{\text{dom}(a)}$  is associated with  $a$  via

$$A_0 := \{(u, y) \in \text{dom}(a) \times H_0; a(u, v) = (y|v)_{H_0} \ (v \in \text{dom}(a))\},$$

and  $A_0$  is extended to an  $m$ -sectorial linear relation in  $H$  by

$$(2.1) \quad \begin{aligned} A &:= A_0 \oplus (\{0\} \times H_0^\perp) \\ &:= \{(u, y) \in \text{dom}(a) \times H; a(u, v) = (y|v)_H \ (v \in \text{dom}(a))\}. \end{aligned}$$

The  $m$ -sectoriality of  $A$ , with vertex 0 and angle  $\theta$ , means that

$$(y|x) \in \overline{\Sigma_\theta} \quad ((x, y) \in A)$$

as well as

$$\text{ran}(I + A) = \{x + y; (x, y) \in A\} = H.$$

We point out that each  $m$ -sectorial linear relation  $A$  in  $H$  is of the form

$$A = A_0 \oplus (\{0\} \times H_0^\perp),$$

where  $H_0 := \overline{\text{dom}(A)}$ , and where  $A_0 := A \cap (H_0 \times H_0)$  is an  $m$ -sectorial operator in  $H_0$ ; see [5], first paragraph of Section 2.

Now assume additionally that the form  $a$  is symmetric, i.e.,  $a^* = a$  (or equivalently,  $a$  is sectorial with vertex 0 and angle 0). Expressed differently, we now assume that  $a$  is a closed accretive symmetric form. Then the operator  $A_0$  described above is a self-adjoint operator in  $H_0$ , and  $A$  is a self-adjoint linear relation, i.e.,

$$A^* := ((-A)^\perp)^{-1} = A,$$

where the orthogonal complement of the linear relation  $-A = \{(x, -y); (x, y) \in A\}$  is taken in  $H \oplus H$ ; see [4], Section 5.

Closing this section we mention that an  $m$ -sectorial linear relation  $A$  also gives rise to a bounded holomorphic degenerate strongly continuous semigroup. More precisely, the operator  $-A_0$  from above generates a bounded holomorphic  $C_0$ -semigroup  $T_0$  on  $H_0$ . Let  $P_0 \in \mathcal{L}(H)$  denote the orthogonal projection onto  $H_0$ . Then  $T(t) := T_0(t)P_0$  ( $t \geq 0$ ) defines a holomorphic degenerate strongly continuous semigroup  $T$  on  $H$ . ('Degenerate' refers to the circumstance that  $T(0)$  may be different from the identity  $I$ , and 'strongly continuous' to the property  $T(0) = s\text{-}\lim_{t \rightarrow 0^+} T(t)$ .) The 'generator' of  $T$  is the linear relation  $-A$ .

We note that, for  $t > 0$ , the operator  $T(t)$  can be obtained as a contour integral

$$(2.2) \quad T(t) = \frac{1}{2\pi i} \int_{\Gamma} e^{-t\lambda} (\lambda I - A)^{-1} d\lambda$$

with a suitable (unbounded) path  $\Gamma$  in  $\mathbb{C}$ . This is traditional if  $H_0 = H$ . In the above general case it then follows from  $(\lambda I - A)^{-1} = (\lambda I_0 - A_0)^{-1} P_0$ , where  $I_0$  is the identity on  $H_0$  and (2.2) with  $T_0$  and  $A_0$  in place of  $T$  and  $A$ .

### 3. ON THE ORDER FOR SYMMETRIC FORMS AND SELF-ADJOINT LINEAR RELATIONS

Let  $H$  be a Hilbert space. Given two accretive symmetric forms  $a$  and  $b$  in  $H$ , we write  $a \leq b$  if  $\text{dom}(a) \supseteq \text{dom}(b)$  and  $a(u) \leq b(u)$  for all  $u \in \text{dom}(b)$ . A crucial fact needed in the proof of the form convergence theorem in the case of symmetric forms is the following result.

**Proposition 3.1.** *Let  $a$  and  $b$  be closed accretive symmetric forms in  $H$ . Let  $A$  and  $B$  be the associated self-adjoint linear relations, and assume that the inverses of  $A$  and  $B$  are operators belonging to  $\mathcal{L}(H)$ . Then  $a \leq b$  if and only if  $B^{-1} \leq A^{-1}$ .*

In [12], Proposition 1.1, the above result is stated for densely defined forms, but it appears that it is applied in the general situation nonetheless: in the proof of Theorem 4.1, which deals with the generalization to the non-densely defined case, it is claimed that ‘the proofs of Section 3 require no changes’. It is no surprise that the equivalence in Proposition 3.1 also holds for non-densely defined forms; in fact, proofs can be found in [9], Proposition 2.7 and in [6], Lemmas 3.2 and 3.3. Our proof of this equivalence is quite different from those proofs.

The key observation of our treatment is Proposition 3.3 below, to which the following elementary lemma is a preparation.

**Lemma 3.2.** *Let  $X$  be a normed space,  $H$  a Hilbert space,  $P \in \mathcal{L}(X, H)$ ,  $\eta \in X'$ ,  $c \geq 0$  with the property that*

$$(3.1) \quad |\eta x| \leq c \|Px\| \quad (x \in X).$$

*Then there exists  $z \in H$  such that  $\|z\| \leq c$  and  $\eta x = (Px|z)$  for all  $x \in X$ .*

**Proof.** Define  $\tilde{\eta}: \text{ran}(P) \rightarrow \mathbb{K}$  by  $\tilde{\eta}(Px) := \eta x$  for all  $x \in X$ ; note that (3.1) implies that  $\tilde{\eta}$  is well-defined and continuous on  $\text{ran}(P)$ ,  $\|\tilde{\eta}\| \leq c$ . The Riesz-Fréchet representation theorem implies that there exists  $z \in \overline{\text{ran}(P)}$  such that  $\|z\| \leq c$  and  $\eta x = \tilde{\eta}(Px) = (Px|z)$  for all  $x \in X$ .  $\square$

Assuming that  $G$  and  $H$  are Hilbert spaces and that  $C$  and  $D$  are linear relations in  $G \times H$ , we will say that  $D$  *dominates*  $C$  if for all  $(x, y) \in D$  there exists  $z \in H$  such that  $(x, z) \in C$  and  $\|z\| \leq \|y\|$ . If  $C$  and  $D$  are operators, this simply means that  $\text{dom}(D) \subseteq \text{dom}(C)$  and  $\|Cx\| \leq \|Dx\|$  for all  $x \in \text{dom}(D)$ . The following fundamental property concerning this notion is a more elaborate version of [8], Lemma VI.2.30.

**Proposition 3.3.** *Let  $G, H$  be Hilbert spaces, and let  $C, D$  be closed linear relations in  $G \times H$ . Then  $D$  dominates  $C$  if and only if  $C^\perp$  dominates  $D^\perp$ .*

*Proof.* It clearly suffices to show ‘ $\Rightarrow$ ’. Let  $(x, y) \in C^\perp$ .

Let  $(f, g) \in D$ . By hypothesis, there exists  $h \in H$  such that  $(f, h) \in C$  and  $\|h\| \leq \|g\|$ . Then  $(f, h) \perp (x, y)$ , hence

$$|(-f|x)| = |(h|y)| \leq \|h\| \|y\| \leq \|g\| \|y\|,$$

and with  $P: D \rightarrow H, (f, g) \mapsto g$  and  $\eta: D \rightarrow \mathbb{K}, (f, g) \mapsto (-f|x)$  it follows that  $|\eta(f, g)| \leq \|y\| \|P(f, g)\|$  (note that  $h$  has dropped out of these properties). Now we can apply Lemma 3.2 to obtain  $z \in H$  such that  $\|z\| \leq \|y\|$  and  $(-f|x) = (P(f, g)|z) = (g|z)$  for all  $(f, g) \in D$ , i.e.,  $(x, z) \in D^\perp$  and  $\|z\| \leq \|y\|$ .

Summarizing, we have shown that  $C^\perp$  dominates  $D^\perp$ . □

*Proof of Proposition 3.1.* Let  $A_0$  be the accretive self-adjoint operator in  $\overline{\text{dom}(a)}$  associated with  $a$ , and denote by  $P_a$  the orthogonal projection onto  $\overline{\text{dom}(a)}$ . The (accretive self-adjoint) square root of  $A^{-1}$  will be denoted by  $A^{-1/2}$ . We point out that then  $A_0^{-1/2}$ , the square root of  $A_0^{-1}$ , is the restriction of  $A^{-1/2}$  to  $\overline{\text{dom}(a)}$ , and we define

$$A^{1/2} := (A^{-1/2})^{-1} = \{(x, y) \in H \times H; (x, P_a y) \in A_0^{1/2}\}.$$

The corresponding notation and properties will also be used for  $b$ .

It is evident that  $a \leq b$  if and only if  $B_0^{1/2}$  dominates  $A_0^{1/2}$ . The latter, in turn, holds if and only if  $B^{1/2}$  dominates  $A^{1/2}$ . (Clearly,  $A_0^{1/2}$  dominates  $A^{1/2}$ , but also conversely: if  $(x, y) \in A^{1/2}$ , then  $(x, P_a y) \in A_0^{1/2}$  and  $\|P_a y\| \leq \|y\|$ . By the same token,  $B_0^{1/2}$  and  $B^{1/2}$  dominate each other. As it is easily seen that domination is transitive, one concludes the last assertion above.)

On the other hand,  $A^{-1} \geq B^{-1}$  if and only if  $A^{-1/2}$  dominates  $B^{-1/2}$ .

Finally we observe that

$$(A^{-1/2})^\perp = ((-A^{-1/2})^*)^{-1} = -(A^{-1/2})^{-1} = -A^{1/2}$$

by the self-adjointness of  $A^{-1/2}$ , and similarly  $(B^{-1/2})^\perp = -B^{-1/2}$ . Now, applying Proposition 3.3 we conclude that  $B^{1/2}$  dominates  $A^{1/2}$  if and only if  $A^{-1/2}$  dominates  $B^{-1/2}$ .

This proves the asserted equivalence. □

#### 4. MONOTONE CONVERGENCE OF SECTORIAL FORMS

In this section we prove the main result concerning closed forms. The basic idea is to use the well-established convergence result for symmetric forms and to ‘propagate’ it to the case of sectorial forms by holomorphy.

**Theorem 4.1.** *Let  $H$  be a complex Hilbert space, and let  $\theta \in [0, \pi/2)$ . Let  $(a_n)$  be a sequence of closed sectorial forms in  $H$  with vertex 0 and angle  $\theta$ , and assume that  $\text{dom}(a_n) \supseteq \text{dom}(a_{n+1})$  and that  $a_{n+1} - a_n$  is sectorial with vertex 0 and angle  $\theta$  for all  $n \in \mathbb{N}$ . For  $n \in \mathbb{N}$  let  $A_n$  be the  $m$ -sectorial linear relation associated with  $a_n$ . Define*

$$\text{dom}(a) := \left\{ u \in \bigcap_{n \in \mathbb{N}} \text{dom}(a_n); \sup_{n \in \mathbb{N}} \text{Re } a_n(u) < \infty \right\}.$$

Then for all  $u, v \in \text{dom}(a)$  the limit  $a(u, v) := \lim_{n \rightarrow \infty} a_n(u, v)$  exists, the form  $a$  thus defined is a closed sectorial form with vertex 0 and angle  $\theta$ , and  $A_n \rightarrow A$  ( $n \rightarrow \infty$ ) in strong resolvent sense, where  $A$  is the  $m$ -sectorial linear relation associated with  $a$ .

*P r o o f.* (i) Similarly to the well-known symmetric case one shows that  $a$  as defined above is a closed sectorial form; see [5], Lemma 2.1. For completeness we include the argument. The Cauchy-Schwarz inequality implies that  $\text{dom}(a)$  is a vector space, and from the sectoriality of the forms  $a_n - a_m$  ( $n > m$ ) and the polarization identity one concludes that  $\lim_{n \rightarrow \infty} a_n(u, v)$  exists for all  $u, v \in \text{dom}(a)$ . Clearly,  $a$  thus defined is a sectorial form. For the closedness of  $a$  we have to show that  $(\text{dom}(a), \|\cdot\|_a)$  is complete. Let  $(u_n)$  be a  $\|\cdot\|_a$ -Cauchy sequence in  $\text{dom}(a)$ . Then  $u := \lim u_n$  exists in  $H$ , and  $(u_n)$  is a  $\|\cdot\|_{a_m}$ -Cauchy sequence, hence  $u_n \rightarrow u$  in  $(\text{dom}(a_m), \|\cdot\|_{a_m})$  since  $a_m$  is closed for all  $m \in \mathbb{N}$ . For all  $n \in \mathbb{N}$  one has

$$\sup_{m \in \mathbb{N}} \text{Re } a_m(u - u_n) \leq \sup_{m \in \mathbb{N}, k \geq n} \text{Re } a_m(u_k - u_n) = \sup_{k \geq n} \text{Re } a(u_k - u_n) < \infty.$$

This inequality implies  $u \in \text{dom}(a)$ , and

$$\text{Re } a(u - u_n) \leq \sup_{k \geq n} \text{Re } a(u_k - u_n) \rightarrow 0 \quad (n \rightarrow \infty)$$

shows that  $u_n \rightarrow u$  in the  $\|\cdot\|_a$ -norm.

(ii) Assume that all the forms  $a_n$  are symmetric. In this case one has  $a_n \leq a_{n+1}$  for all  $n \in \mathbb{N}$ , and the assertion follows from [12], Theorem 4.1; recall that for this general case our Proposition 3.1 replaces [12], Proposition 1.1.

(iii) In the general case there exists  $C > 0$  such that

$$\begin{aligned} |\operatorname{Im} a_n(u)| &\leq C \operatorname{Re} a_n(u) \quad (u \in \operatorname{dom}(a_n)), \\ |\operatorname{Im}(a_{n+1}(u) - a_n(u))| &\leq C \operatorname{Re}(a_{n+1}(u) - a_n(u)) \quad (u \in \operatorname{dom}(a_{n+1})) \end{aligned}$$

for all  $n \in \mathbb{N}$ , and

$$|\operatorname{Im} a(u)| \leq C \operatorname{Re} a(u) \quad (u \in \operatorname{dom}(a)).$$

Then for  $z \in \Omega := \{z \in \mathbb{C}; |\operatorname{Re} z| < 1/C\}$  the forms

$$a_{n,z} := \operatorname{Re} a_n + z \operatorname{Im} a_n \quad (n \in \mathbb{N}), \quad a_z := \operatorname{Re} a + z \operatorname{Im} a$$

are closed sectorial forms in  $H$ . Indeed, for  $z \in \Omega$ ,  $u \in \operatorname{dom}(a_n)$  we have

$$\begin{aligned} \operatorname{Re} a_{n,z}(u) &= \operatorname{Re} a_n(u) + \operatorname{Re} z \operatorname{Im} a_n(u) \geq (1 - |\operatorname{Re} z| C) \operatorname{Re} a_n(u) \geq 0, \\ |\operatorname{Im} a_{n,z}(u)| &= |\operatorname{Im} z \operatorname{Im} a_n(u)| \leq |\operatorname{Im} z| C \operatorname{Re} a_n(u) \leq \frac{|\operatorname{Im} z| C}{1 - |\operatorname{Re} z| C} \operatorname{Re} a_{n,z}(u), \end{aligned}$$

and similarly for  $a$  instead of  $a_n$ . For  $z \in \Omega$ ,  $n \in \mathbb{N}$  let  $A_{n,z}$  be the  $m$ -sectorial linear relation associated with  $a_{n,z}$ , and let  $A_z$  be the  $m$ -sectorial linear relation associated with  $a_z$ .

For  $x \in (-1/C, 1/C)$ ,  $n \in \mathbb{N}$ ,  $u \in \operatorname{dom}(a_{n+1})$  we have

$$x \operatorname{Im}(a_n(u) - a_{n+1}(u)) \leq \frac{1}{C} |\operatorname{Im}(a_{n+1}(u) - a_n(u))| \leq \operatorname{Re}(a_{n+1}(u) - a_n(u)),$$

which implies

$$a_{n,x} = \operatorname{Re} a_n + x \operatorname{Im} a_n \leq \operatorname{Re} a_{n+1} + x \operatorname{Im} a_{n+1} = a_{n+1,x}.$$

Hence,  $(a_{n,x})_n$  is an increasing sequence of closed accretive symmetric forms. Note that for all  $x \in (-1/C, 1/C)$  the limit form of the sequence  $(a_{n,x})$  has the domain

$$\left\{ u \in \bigcap_{n \in \mathbb{N}} \operatorname{dom}(a_{n,x}); \sup_{n \in \mathbb{N}} a_{n,x}(u) < \infty \right\} = \operatorname{dom}(a) = \operatorname{dom}(a_x),$$

and that  $a_n(u) \rightarrow a(u)$  implies  $a_{n,x}(u) \rightarrow a_x(u)$  as  $n \rightarrow \infty$ , for all  $u \in \operatorname{dom}(a)$ . From the case treated in part (ii) above we conclude that

$$(4.1) \quad (I + A_{n,x})^{-1} \rightarrow (I + A_x)^{-1} \quad (n \rightarrow \infty)$$

strongly, for all  $x \in (-1/C, 1/C)$ .



For each  $n \in \mathbb{N}$ , the family  $(a_{n,z})_{z \in \Omega}$  is a holomorphic family of type (a) in the sense of Kato, and similarly for the family  $(a_z)_{z \in \Omega}$ . By Kato [8], Theorem VII.4.2, we also refer to the recent proof in [13], this implies that the mappings  $\Omega \ni z \mapsto (I + A_{n,z})^{-1} \in \mathcal{L}(H)$  ( $n \in \mathbb{N}$ ) and  $\Omega \ni z \mapsto (I + A_z)^{-1} \in \mathcal{L}(H)$  are holomorphic. (The quoted references only cover the case in which the families  $(a_{n,z})_{z \in \Omega}$  and  $(a_z)_{z \in \Omega}$  are defined on dense subspaces. For the general case one has to use the description (2.1) of the associated  $m$ -sectorial linear relations; see also Theorem 5.2.)

In view of the convergence (4.1) and the estimate  $\|(I + A_{n,z})^{-1}\| \leq 1$  for all  $n \in \mathbb{N}$ ,  $z \in \Omega$ , Vitali's convergence theorem shows that  $(I + A_{n,z})^{-1} \rightarrow (I + A_z)^{-1}$  ( $n \rightarrow \infty$ ) strongly, for all  $z \in \Omega$ . (We refer to [3], Theorem 2.1, for an elegant proof of Vitali's theorem.) In particular, setting  $z = i$  we obtain  $a_n = a_{n,i}$  for all  $n \in \mathbb{N}$  and  $a = a_i$ , and therefore  $(I + A_n)^{-1} \rightarrow (I + A)^{-1}$  ( $n \rightarrow \infty$ ) strongly.  $\square$

**Remark 4.2.** In [10], Theorem 5, Ouhabaz proved the following convergence theorem. Let  $a_n$  ( $n \in \mathbb{N}$ ) and  $a$  be densely defined closed sectorial forms with vertex 0 and angle  $\theta_0 \in [0, \pi/2)$ ,  $(\operatorname{Re} a_n)_n$  increasing to  $\operatorname{Re} a$  in the sense of Theorem 4.1, and suppose that  $\operatorname{Im} a(u) = \lim_{n \rightarrow \infty} \operatorname{Im} a_n(u)$  for all  $u \in \operatorname{dom}(a)$ . Assume that  $\operatorname{Im} a_n(u) \leq \operatorname{Im} a_{n+1}(u)$  for all  $u \in \operatorname{dom}(a_{n+1})$ ,  $n \in \mathbb{N}$ . Then  $(A_n)$  converges to  $A$  in strong resolvent sense, where  $A_n$  is associated with  $a_n$  for  $n \in \mathbb{N}$ , and  $A$  is associated with  $a$ .

We show that this result can be obtained as a corollary of Theorem 4.1 (or [5], Theorem 2.2). Obviously,

$$a_{n+1}(u) - a_n(u) \in \{z \in \mathbb{C}; 0 \leq \operatorname{Arg} z \leq \pi/2\}$$

for all  $u \in \operatorname{dom}(a_{n+1})$ ,  $n \in \mathbb{N}$ . Let  $\eta \in (0, \pi/2 - \theta_0)$ . Then replacing  $a_n$  by  $e^{-i\eta} a_n$  one easily checks that the sequence  $(e^{-i\eta} a_n)_n$  of forms satisfies the hypotheses of Theorem 4.1 above, with  $\theta := \max\{\theta_0 + \eta, \pi/2 - \eta\}$ . In order to check that the sequence  $(e^{-i\eta} a_n)_n$  converges to the form  $e^{-i\eta} a$  we note that

$$\begin{aligned} \cos \theta_0 |a_n(u)| &\leq \operatorname{Re} a_n(u) \leq |a_n(u)|, \\ \cos \theta |e^{-i\eta} a_n(u)| &\leq \operatorname{Re}(e^{-i\eta} a_n(u)) \leq |e^{-i\eta} a_n(u)| \end{aligned}$$

for all  $u \in \bigcap_{n \in \mathbb{N}} \operatorname{dom}(a_n)$ ,  $n \in \mathbb{N}$ , and this implies that

$$\left\{ u \in \bigcap_{n \in \mathbb{N}} \operatorname{dom}(a_n); \sup_{n \in \mathbb{N}} \operatorname{Re}(e^{-i\eta} a_n)(u) < \infty \right\} = \operatorname{dom}(a) = \operatorname{dom}(e^{-i\eta} a).$$

Analogously one can treat the alternative case in [10], Theorem 5, where the sequence  $(\operatorname{Im} a_n)$  is decreasing instead of increasing, i.e.,  $\operatorname{Im} a_n(u) \geq \operatorname{Im} a_{n+1}(u)$  for all  $u \in \operatorname{dom}(a_{n+1})$ ,  $n \in \mathbb{N}$ .

In view of the above, Theorem 4.1 implies that one can relax the hypotheses in Ouhabaz' result.

The remainder of this section is devoted to explaining the implications of the convergence in Theorem 4.1 for the associated degenerate strongly continuous semigroups. For all  $n \in \mathbb{N}$  let  $T_n$  be the holomorphic degenerate strongly continuous semigroup generated by  $-A_n$ , and let  $T$  be generated by  $-A$ . We are going to explain why

(i) for all  $u \in H$ ,  $t > 0$  one has

$$(4.2) \quad T(t)u = \lim_{n \rightarrow \infty} T_n(t)u$$

with uniform convergence on compact subsets of  $(0, \infty)$ ,

(ii) for all  $u \in \text{ran}(T(0)) = \overline{\text{dom}(a)}$ ,  $t \geq 0$  one has (4.2) with uniform convergence on compact subsets of  $[0, \infty)$ .

A preliminary observation for the proof of both the assertions is that the set  $\mathbb{C} \setminus \overline{\Sigma_\theta}$  is contained in the resolvent sets of  $A_n$  for all  $n \in \mathbb{N}$  and of  $A$  by our hypotheses. Furthermore, fixing  $\theta' \in (\theta, \pi/2)$ , the resolvents obey an estimate

$$(4.3) \quad \|(\lambda - A_n)^{-1}\| \leq \frac{c}{|\lambda|}$$

on  $\mathbb{C} \setminus (\Sigma_{\theta'} \cup \{0\})$  and the same for  $A$  with a constant  $c \geq 0$  independent of  $n$ . Finally, the strong resolvent convergence implies that  $(\lambda - A_n)^{-1} \rightarrow (\lambda - A)^{-1}$  strongly, uniformly for  $\lambda$  on compact subsets of  $\mathbb{C} \setminus (\Sigma_{\theta'} \cup \{0\})$ .

In order to show the convergence in (i) we now specify the path  $\Gamma$  mentioned in the formula (2.2) as the boundary of the set  $\Sigma_{\theta'} \cup B_{\mathbb{C}}(0, 1)$ , oriented ‘counterclockwise’. Then the assertion is an easy consequence of (2.2) for the semigroups  $T_n$  and  $T$ , the estimate (4.3) and the strong resolvent convergence formulated at the end of the previous paragraph; see [9], Theorem 2.3. Another source for (i) is [1], Theorem 5.2, where the erroneously asserted uniform convergence on  $[0, \tau]$  has to be replaced by uniform convergence on compact subsets of  $(0, \infty)$ .

For the convergence (ii) we refer to [1], Theorem 4.2 (a). Alternatively, one can prove (ii) by adapting the proof of the Trotter approximation theorem in [11], Section 3.4, to the case of degenerate strongly continuous semigroups.

## 5. THE CASE OF NON-CLOSABLE SECTORIAL FORMS

In this section we show that the following result, contained in [5], Theorem 3.2, can be obtained by the method presented in Section 4.

**Theorem 5.1.** *Let  $H$  be a complex Hilbert space, and let  $\theta \in [0, \pi/2)$ . Let  $(a_n)$  be a sequence of sectorial forms in  $H$  with vertex 0 and angle  $\theta$ , and assume that*

$\text{dom}(a_n) \supseteq \text{dom}(a_{n+1})$  and that  $a_{n+1} - a_n$  is sectorial with vertex 0 and angle  $\theta$ , for all  $n \in \mathbb{N}$ . For  $n \in \mathbb{N}$  let  $A_n$  be the  $m$ -sectorial linear relation associated with  $a_n$ .

Then there exists an  $m$ -sectorial linear relation  $A$  such that  $A_n \rightarrow A$  ( $n \rightarrow \infty$ ) in strong resolvent sense.

Note that the only difference between the hypotheses in this theorem and those in Theorem 4.1 is that here the forms  $a_n$  are no longer supposed to be closed; the “price” one has to pay is that in the conclusion there is no longer a description of the limit linear relation in terms of the forms  $a_n$ .

Before entering our proof of this result we have to explain how a linear relation is associated with a non-closable (short for “not necessarily closable”) form. Let  $a$  be a sectorial form in  $H$  with vertex 0 and angle  $\theta \in [0, \pi/2)$ . The norm  $\|\cdot\|_a$  on  $\text{dom}(a) \subseteq H$  is defined as in Section 2. Let  $V$  be the completion of  $(\text{dom}(a), \|\cdot\|_a)$ . Then the continuous extension  $j: V \rightarrow H$  of the embedding  $\text{dom}(a) \hookrightarrow H$  is not necessarily injective; in fact,  $a$  is called closable if  $j$  is injective.

The form  $a$  on  $\text{dom}(a)$  has a unique continuous extension  $\tilde{a}: V \times V \rightarrow \mathbb{C}$ , and this extension is  $j$ -coercive and sectorial with vertex 0 and angle  $\theta$ . (By definition,  $\tilde{a}$  being  $j$ -coercive, called ‘ $j$ -elliptic’ in [2] and [5], means that there exists  $\omega \in \mathbb{R}$  such that the form  $(u, v) \mapsto \tilde{a}(u, v) + \omega(ju|jv)_H$  is coercive. In the present case, this condition holds with  $\omega = 1$ .) The ( $m$ -sectorial) linear relation  $A$  associated with  $(a, j)$  is then given by

$$A = \{(x, y) \in H \times H; \exists u \in V: ju = x, \tilde{a}(u, v) = (y|jv)_H \ (v \in V)\}.$$

If  $a$  is symmetric, then the linear relation  $A$  is self-adjoint, and the (closed!) accretive form associated with  $A$  is the closure of the regular part  $a_r$  of  $a$ , defined in [12], Section 2. For these definitions and properties we refer to [2], Section 3, and [5], Section 3.

Next, we state a generalization of the result on holomorphic families of type (a), used in the proof of Theorem 4.1.

**Theorem 5.2.** *Let  $V$  and  $H$  be complex Hilbert spaces,  $j \in \mathcal{L}(V, H)$ , let  $\Omega \subseteq \mathbb{C}$  be open, and let  $\theta \in [0, \pi/2)$ . For each  $z \in \Omega$  let  $a_z: V \times V \rightarrow \mathbb{C}$  be a bounded  $j$ -coercive sectorial form with vertex 0 and angle  $\theta$ , and let  $A_z$  denote the  $m$ -sectorial linear relation associated with  $(a_z, j)$ . Assume that for all  $u, v \in V$  the function  $\Omega \ni z \mapsto a_z(u, v) \in \mathbb{C}$  is holomorphic.*

*Then the mapping  $\Omega \ni z \mapsto (I + A_z)^{-1} \in \mathcal{L}(H)$  is holomorphic.*

If  $\text{ran}(j)$  is dense in  $H$ , then the proof can be given in the same way as in [13], proof of Theorem 1.1. Otherwise we define  $H_0 := \overline{\text{ran}(j)}$  and apply the previous case to the

m-sectorial operators  $A_{z,0}$  associated with  $(a_z, j)$  in  $H_0$ . Then the description (2.1) of  $A_z$  yields the assertion. Indeed, the resolvent  $(I + A_z)^{-1}$  decomposes according to the orthogonal sum  $H_0 \oplus H_0^\perp$ , where the restriction to  $H_0$  is  $(I_{H_0} + A_{z,0})^{-1}$  and the restriction to  $H_0^\perp$  is the zero operator.

**P r o o f** of Theorem 5.1. (i) First we treat the case in which all the forms  $a_n$  are symmetric. In this case we refer to [12], Corollary 2.4, for the property that one has  $a_{n,r} \leq a_{n+1,r}$  for all  $n \in \mathbb{N}$ , where  $a_{n,r}$  denotes the regular part of  $a_n$ . (In the quoted reference the property needed here is only formulated for densely defined forms. However, it is immediate that the proof of [12], Theorem 2.2, which is the basis for the quoted property, works also for non-densely defined forms.) Then we conclude from [12], Theorem 4.1, that the sequence  $(A_n)$  converges (to some  $A$ ) in strong resolvent sense.

(ii) For the general case we define

$$a_{n,z} := \operatorname{Re} a_n + z \operatorname{Im} a_n \quad (n \in \mathbb{N}, z \in \mathbb{C})$$

and note that the arguments as in step (iii) of the proof of Theorem 4.1 yield a constant  $C > 0$  such that

$$|\operatorname{Im} a_{n,z}(u)| \leq \frac{|\operatorname{Im} z|C}{1 - |\operatorname{Re} z|C} \operatorname{Re} a_{n,z}(u)$$

for all  $z \in \Omega := \{z \in \mathbb{C}; |\operatorname{Re} z| < 1/C\}$ ,  $n \in \mathbb{N}$ ,  $u \in \operatorname{dom}(a_n)$ , and

$$(5.1) \quad a_{n,x} \leq a_{n+1,x} \quad (x \in (-1/C, 1/C), n \in \mathbb{N}).$$

In particular, for  $z \in \Omega$ ,  $n \in \mathbb{N}$  the form  $a_{n,z}$  is sectorial; let  $A_{n,z}$  be the linear relation associated with  $a_{n,z}$  (as explained above). From (5.1) and step (i) we conclude that, for  $x \in (-1/C, 1/C)$ , there exists an operator  $R_x \in \mathcal{L}(H)$  such that  $(I + A_{n,x})^{-1} \rightarrow R_x$  ( $n \rightarrow \infty$ ) strongly.

For  $n \in \mathbb{N}$ , let  $V_n$  be the completion of  $(\operatorname{dom}(a_n), \|\cdot\|_{a_n})$ ,  $j_n: V_n \rightarrow H$  the continuous extension of the embedding  $\operatorname{dom}(a_n) \hookrightarrow H$  as described above, let  $\tilde{a}_n$  be the continuous extension of  $a_n$  to  $V_n \times V_n$ , and define the  $j_n$ -coercive sectorial forms  $\tilde{a}_{n,z}$  by

$$\tilde{a}_{n,z} := \operatorname{Re} \tilde{a}_n + z \operatorname{Im} \tilde{a}_n.$$

Then  $\tilde{a}_{n,z}$  is the continuous extension of  $a_{n,z}$ , so  $A_{n,z}$  is associated with  $\tilde{a}_{n,z}$ . The application of Theorem 5.2 shows that  $\Omega \ni z \mapsto (I + A_{n,z})^{-1} \in \mathcal{L}(H)$  is holomorphic. As in step (iii) of the proof of Theorem 4.1, Vitali's convergence theorem implies the strong convergence of  $((I + A_n)^{-1} = (I + A_{n,i})^{-1})_{n \in \mathbb{N}}$  to an operator  $R \in \mathcal{L}(H)$ . It follows from Lemma 5.3, proved subsequently, that the linear relation  $A$  determined by  $R = (I + A)^{-1}$  is m-sectorial.  $\square$

**Lemma 5.3.** *Let  $H$  be a complex Hilbert space,  $(A_n)$  a sequence of  $m$ -sectorial relations in  $H$  with vertex 0 and angle  $\theta \in [0, \pi/2)$ , and assume that  $R := s\text{-}\lim(I + A_n)^{-1}$  exists. Then the linear relation  $A$  determined by  $R = (I + A)^{-1}$  is  $m$ -sectorial with vertex 0 and angle  $\theta$ .*

*Proof.* Note that  $A = R^{-1} - I = \{(y, x - y); (x, y) \in R\}$ . Clearly,  $\text{ran}(I + A) = \text{dom}(R) = H$ . Let  $(x, y) \in R$ . With  $y_n := (I + A_n)^{-1}x$ , i.e.,  $(y_n, x - y_n) \in A_n$ , we have  $(x - y_n | y_n) \in \overline{\Sigma_\theta}$  and  $y_n \rightarrow y$ , hence  $(x - y | y) \in \overline{\Sigma_\theta}$ .  $\square$

**Remark 5.4.** We did not derive Theorem 5.1 as a corollary of Theorem 4.1, but rather the proofs of the two results are “the same”. However, we comment on the following decisive difference. In Theorem 4.1 we had the limit form  $a$  at our disposal, and each of the linear relations  $A_z$  in step (iii) of the proof was associated with the form  $a_z = \text{Re } a + z \text{Im } a$ , whereas in the proof of Theorem 5.1, the linear relation  $A$  is obtained in an indirect way as a limit in strong resolvent sense.

An idea to prove Theorem 5.1 as a direct corollary of Theorem 4.1 would be as follows. Denote by  $\widehat{a}_n$  the closed sectorial form with  $\text{dom}(\widehat{a}_n) \subseteq H$  associated with  $A_n$ . As by hypothesis  $a_{n+1} - a_n$  is sectorial (uniformly in  $n$ ), it is tempting to think that  $\widehat{a}_{n+1} - \widehat{a}_n$  might be sectorial (with the same vertex and angle). However, this idea fails dramatically, as we will illustrate by Example 5.5 (a).

A noteworthy feature concerning the operators  $A_{n,z}$  associated with  $\text{Re } a_n + z \text{Im } a_n$ , for the non-closable forms  $a_n$  presented in Example 5.5, is illustrated in part (b) of this example: none of the families  $(A_{n,z})_{z \in \Omega}$  corresponds to a family  $(b + zc)_{z \in \Omega}$  with closed symmetric forms  $b$  and  $c$ .

**Example 5.5.** Let  $H = L_2(0, 1)$ .

(a) We present a sequence  $(a_n)$  of non-closable sectorial forms in  $H$ , with vertex 0 and angle  $\pi/4$ ,  $a_{n+1} - a_n$  symmetric and accretive for all  $n$ , where the associated closed forms  $\widehat{a}_n$  are all symmetric, and the sequence  $(\widehat{a}_n)$  is strictly *decreasing*.

For  $n \in \mathbb{N}$  we define  $\text{dom}(a_n) := C[0, 1]$ ,

$$a_n(u, v) := \int u\bar{v} + nu(0)\overline{v(0)} + i\left(u(0) \int \bar{v} + \left(\int u\right)\overline{v(0)}\right).$$

Then

$$(\text{Re } a_n)(u, v) = \int u\bar{v} + nu(0)\overline{v(0)},$$

$$(\text{Im } a_n)(u, v) = u(0) \int \bar{v} + \left(\int u\right)\overline{v(0)} \quad (u, v \in \text{dom}(a_n)).$$

Because of  $|\int u| \leq \|u\|_2$  and  $|u(0)|\|u\|_2 \leq \frac{1}{2}(\|u\|_2^2 + |u(0)|^2)$  we obtain

$$|\text{Im } a_n(u)| \leq \text{Re } a_n(u) \quad (u \in \text{dom}(a_n), n \in \mathbb{N}).$$

Also,  $a_{n+1}(u) - a_n(u) = |u(0)|^2 \geq 0$  for all  $u \in \text{dom}(a_{n+1})$ ,  $n \in \mathbb{N}$ .

For all  $n \in \mathbb{N}$ , the  $a_n$ -norm on  $C[0, 1]$  is equivalent to  $\|u\| := (\|u\|_2^2 + |u(0)|^2)^{1/2}$ . Therefore,  $V := L_2(0, 1) \oplus \mathbb{C}$  is a completion  $(C[0, 1], \|\cdot\|_{a_n})$ , and the continuous extension  $j: V \rightarrow L_2(0, 1)$  of the embedding  $C[0, 1] \hookrightarrow L_2(0, 1)$  is given by

$$j(u, \alpha) = u \quad ((u, \alpha) \in L_2(0, 1) \oplus \mathbb{C}).$$

The continuous extension  $\tilde{a}_n$  of  $a_n$  to  $V \times V$  is given by

$$\tilde{a}_n((u, \alpha), (v, \beta)) = \int u\bar{v} + n\alpha\bar{\beta} + i\left(\alpha \int \bar{v} + \left(\int u\right)\bar{\beta}\right).$$

For  $u, f \in L_2(0, 1)$  we have  $(u, f) \in A_n$  if and only if there exists  $\alpha \in \mathbb{C}$  such that for all  $(v, \beta) \in V$  one has

$$(5.2) \quad \int u\bar{v} + n\alpha\bar{\beta} + i\left(\alpha \int \bar{v} + \left(\int u\right)\bar{\beta}\right) = \tilde{a}_n((u, \alpha), (v, \beta)) = (f|j(v, \beta)) = \int f\bar{v}.$$

This property is equivalent to  $\alpha = -(i/n) \int u$  and  $f = u + (n^{-1} \int u)1$ . This shows that the operator  $A_n$  associated with the form  $a_n$  is given by  $A_n u = u + (n^{-1} \int u)1$  ( $u \in L_2(0, 1)$ ), and the associated closed form  $\hat{a}_n$  with  $\text{dom}(\hat{a}_n) \subseteq L_2(0, 1)$  is given by

$$\hat{a}_n(u, v) = \int u\bar{v} + \frac{1}{n} \int u \int \bar{v} \quad (u, v \in L_2(0, 1)).$$

Hence all the forms  $\hat{a}_n$  are symmetric,  $\hat{a}_{n+1} \leq \hat{a}_n$  for all  $n \in \mathbb{N}$ , and  $A_n \rightarrow I$  in  $\mathcal{L}(L_2(0, 1))$ .

(b) For the forms from part (a), using (4.2) with  $z$  in place of  $i$ , one can compute the operators  $A_{n,z}$  associated with the form  $a_{n,z} := \text{Re } a_n + z \text{Im } a_n$  for  $|\text{Re } z| < 1$ . The result is

$$A_{n,z}u = u - \frac{z^2}{n} \left( \int u \right) 1 \quad (u \in L_2(0, 1)),$$

with the associated closed form

$$\hat{a}_{n,z}(u, v) = \int u\bar{v} - \frac{z^2}{n} \int u \int \bar{v} \quad (u, v \in L_2(0, 1)).$$

In contrast, the forms  $(\hat{a}_n)_z := \text{Re } \hat{a}_n + z \text{Im } \hat{a}_n = \hat{a}_n$  do not depend on  $z$  since  $\hat{a}_n$  is symmetric. The only points  $z$  where the forms  $(\hat{a}_n)_z$  and  $\hat{a}_{n,z}$  coincide are  $z = \pm i$ .

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